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HANKEL DETERMINANT PROBLEMS FOR A SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH LEMNISCATE OF BERNOULLI

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ABSTRACT. In this paper, we establish the upper bounds of third and fourth Hankel determinants for a subclass of analytic functions in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$, subordinated to Lemniscate of Bernoulli. Also we extend this investigation for two-fold and three-fold symmetric functions. Some earlier known results will follow as particular cases.

1. INTRODUCTION

Let f and g be two analytic functions in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ ($\mathbb{C}$ is the complex plane). We say that f is subordinate to g (denoted as $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then the subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$.

Let $\mathcal{A}$ denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined in E and normalized by the conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$, we denote the subclass of $\mathcal{A}$ which consists of univalent functions in E.

Firstly, we shall discuss some fundamental classes of analytic functions which are very useful for better understanding of the main content.

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}, \text{ the class of starlike functions.}$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}, \text{ the class of convex functions.}$$

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Reade [28] introduced the class $\mathcal{CS}^*$ of close-to-star functions which is defined as $\mathcal{CS}^* = \left\{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{g(z)}\right) > 0, g \in \mathcal{S}^*, z \in E\right\}$. Further for $g(z) = z$, MacGregor [20] studied the following subclass of close-to-star functions:

$$\mathcal{R}^{'} = \left\{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0, z \in E\right\}.$$

MacGregor [19] established a very useful class $\mathcal{R}$ of bounded turning functions which is defined as

$$\mathcal{R} = \{f : f \in \mathcal{A}, Re(f'(z)) > 0, z \in E\}.$$

Later on, Murugusundramurthi and Magesh [22] studied the following class:

$$\mathcal{R}(\alpha) = \left\{f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, 0 \leq \alpha \leq 1, z \in E\right\}.$$

Particularly, $\mathcal{R}(1) \equiv \mathcal{R}$ and $\mathcal{R}(0) \equiv \mathcal{R}^{'}$.

Sokol and Stakiewicz [34] introduced the class $\mathcal{S}^*_{\mathcal{L}}$, consisting of functions $f \in \mathcal{A}$ such that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by right-half of the Bernoulli's lemniscate given by $|w^2 - 1| < 1$. The class $\mathcal{S}^*_{\mathcal{L}}$ can be expressed as

$$\mathcal{S}^*_{\mathcal{L}} = \left\{f : f \in \mathcal{A}, \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, z \in E\right\},$$

and in terms of subordination the class $\mathcal{S}^*_L$, can represented as

$$\mathcal{S}^*_{\mathcal{L}} = \left\{f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, z \in E\right\}.$$

Recently, Srivastava et al. [35], Rao et al. [26], Arif et al. [4] and Ullah et al. [36] studied various subclasses of analytic functions associated with right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. Getting motivated by these works, now we define the following class of analytic functions by subordinating to $\sqrt{1+z}$.

Definition 1.1 A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\alpha}_{\mathcal{L}}$ $(0 \leq \alpha \leq 1)$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \sqrt{1+z}.$$

We have the following observations:
(i) $\mathcal{R}^0_L \equiv \mathcal{R}^{'}_{\mathcal{L}}$.
(ii) $\mathcal{R}^1_L \equiv \mathcal{R}_{\mathcal{L}}$.

For $q \geq 1$ and $n \geq 1$, Pommerenke [24] introduced the q^{th} Hankel determinant as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

For specific values of q and n, the Hankel determinant $H_q(n)$ reduces to the following functionals:

(i) For $q = 2$ and $n = 1$, it redues to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional.
(ii) For $q = 2$ and $n = 2$, the Hankel determinant takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is known as Hankel determinant of second order.
(iii) For $q = 3$ and $n = 1$, the Hankel determinant reduces to $H_3(1)$, which is the Hankel determinant of third order.
(iv) For $q = 4$ and $n = 1$, $H_q(n)$ reduces to $H_4(1)$, which is the Hankel determinant of fourth order.

Ma [17] introduced the functional $J_{n,m}(f) = a_na_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2a_3 - a_4$ is a specific case of the generalized Zalcman functional. The upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions was computed by various authors. It is very useful in establishing the bounds for the third Hankel determinant.

On expanding, the third Hankel determinant can be expressed as

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and after applying the triangle inequality, it yields

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1}$$

Also the expansion of fourth Hankel determinant can be expressed as

$$H_4(1) = a_7H_3(1) - 2a_4a_6(a_2a_4 - a_3^2) - 2a_5a_6(a_2a_3 - a_4) - a_6^2(a_3 - a_2^2)$$
$$+ a_5^2(a_2a_4 - a_3^2) + a_5^2(a_2a_4 + 2a_3^2) - a_5^3 + a_4^4 - 3a_3a_4^2a_5. \tag{2}$$

Extensive work has been done on the estimation of second Hankel determinant by various authors including Noor [23], Ehrenborg [11], Layman [14], Singh [30], Mehrok and Singh [21] and Janteng et al. [13]. The estimation of third Hankel determinant is little bit complicated. Babalola [5] was the first researcher who successfully obtained the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Further a few researchers including Shanmugam et al. [29], Bucur et al. [7], Altinkaya and Yalcin [1], Singh and Singh [31] have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions. Now a days, the study of fourth Hankel determinant for various subclasses of analytic functions, is an active topic of research. A few authors including Arif et al. [3], Singh et al. [32, 33] and Zhang and Tang [37] established the bounds of fourth Hankel determinant for certain subclasses of $\mathcal{A}$.

This paper is concerned with the establishment of the upper bounds of the third and fourth Hankel determinants for the class $\mathcal{R}_{\mathcal{L}}^{\alpha}$. Also various known results follow as particular cases.

Let $\mathcal{P}$ denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

In order to prove our main results, the following lemmas have been used:
Lemma 1 If $p \in \mathcal{P}$, then

$$|p_k| \leq 2, k \in \mathbb{N}.$$

The above well known result is due to Carathéodory [8, 9].
Further Hayami and Owa [12], established the following result:

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1.$$

Also Ma and Minda [18] proved that if ρ is any complex number, then

$$|p_2 - \rho p_1^2| \leq 2max\{1, |2\rho - 1|\}.$$

Lemma 2 [2] Let $p \in \mathcal{P}$, then

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [25] that

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

Lemma 3 [15, 16] If $p \in \mathcal{P}$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

Lemma 4 [27] Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and

$$8r(1-r)\left[(mn - 2l)^2 + (m(r + m) - n)^2\right] + m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2r(1-r).$$

If $p \in \mathcal{P}$, then

$$\left|lp_1^4 + rp_2^2 + 2mp_1p_3 - \frac{3}{2}np_1^2p_2 - p_4\right| \leq 2.$$

2. Results for the class $\mathcal{R}_{\mathcal{L}}^{\alpha}$

Theorem 2.1 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, then

$$|a_2| \leq \frac{1}{2(1 + \alpha)}, \tag{3}$$

$$|a_3| \leq \frac{1}{2(1 + 2\alpha)}, \tag{4}$$

$$|a_4| \leq \frac{1}{2(1 + 3\alpha)}, \tag{5}$$

and

$$|a_5| \leq \frac{1}{2(1 + 4\alpha)}. \tag{6}$$

Proof. Since $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, by the principle of subordination, we have

$$(1 - \alpha)\frac{f(z)}{z} + \alpha f'(z) = \sqrt{1 + w(z)}. \tag{7}$$

Define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1+p_1z+p_2z^2+p_3z^3+...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$.
On expanding, we have

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z) = 1+(1+\alpha)a_2z+(1+2\alpha)a_3z^2+(1+3\alpha)a_4z^3+(1+4\alpha)a_5z^4+... \tag{8}$$

Also

$$\sqrt{1+w(z)} = 1+\frac{1}{4}p_1z+\left(\frac{p_2}{4}-\frac{5p_1^2}{32}\right)z^2 + \left(\frac{13p_1^3}{128}-\frac{5p_1p_2}{16}+\frac{p_3}{4}\right)z^3+\left(-\frac{141p_1^4}{2048}+\frac{39p_1^2p_2}{128}-\frac{5p_3p_1}{16}-\frac{5p_2^2}{32}+\frac{p_4}{4}\right)z^4+... \tag{9}$$

Using (8) and (9), (7) yields

$$1+(1+\alpha)a_2z+(1+2\alpha)a_3z^2+(1+3\alpha)a_4z^3+(1+4\alpha)a_5z^4+... = 1+\frac{1}{4}p_1z+\left(\frac{p_2}{4}-\frac{5p_1^2}{32}\right)z^2+\left(\frac{13p_1^3}{128}-\frac{5p_1p_2}{16}+\frac{p_3}{4}\right)z^3+\left(-\frac{141p_1^4}{2048}+\frac{39p_1^2p_2}{128}-\frac{5p_3p_1}{16}-\frac{5p_2^2}{32}+\frac{p_4}{4}\right)z^4+... \tag{10}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (10) and on simplification, we obtain

$$a_2 = \frac{1}{4(1+\alpha)}p_1, \tag{11}$$

$$a_3 = \frac{1}{4(1+2\alpha)}\left[p_2-\frac{5p_1^2}{8}\right], \tag{12}$$

$$a_4 = \frac{1}{128(1+3\alpha)}\left[13p_1^3-40p_1p_2+32p_3\right], \tag{13}$$

and

$$a_5 = \frac{1}{4(1+4\alpha)}\left[\frac{141p_1^4}{512}+\frac{320p_2^2}{512}+\frac{640p_3p_1}{512}-\frac{624p_1^2p_2}{512}-p_4\right]. \tag{14}$$

Using first inequality of Lemma 1 in (11), the result (3) is obvious.
From (12), we have

$$|a_3| = \frac{1}{4(1+2\alpha)}\left|p_2-\frac{5}{8}p_1^2\right|. \tag{15}$$

Using third inequality of Lemma 1 in (15), the result (4) can be easily obtained.
(13) can be expressed as

$$|a_4| = \frac{1}{128(1+3\alpha)}\left|13p_1^3-40p_1p_2+32p_3\right|. \tag{16}$$

On applying Lemma 2 in (16), the result (5) is obvious.
Further, on using Lemma 4 in (14), the result (6) is obvious.

On putting $\alpha = 0$, Theorem 2.1 yields the following result:
Corollary 2.1 If $f \in \mathcal{R}'_{\mathcal{L}}$, then

$$|a_2| \le \frac{1}{2}, |a_3| \le \frac{1}{2}, |a_4| \le \frac{1}{2}, |a_5| \le \frac{1}{2}.$$

For $\alpha = 1$, Theorem 2.1 gives the following result due to Ullah et al. [36]:
Corollary 2.2 If $f \in \mathcal{R}_{\mathcal{L}}$, then

$$|a_2| \le \frac{1}{4}, |a_3| \le \frac{1}{6}, |a_4| \le \frac{1}{8}, |a_5| \le \frac{1}{10}.$$

Conjecture If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, then

$$|a_n| \leq \frac{1}{2(1+(n-1)\alpha)}, n \geq 2.$$

Theorem 2.2 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$ and μ is any complex number, then

$$|a_3 - a_2^2| \leq \frac{1}{2(1+2\alpha)}. \tag{17}$$

Proof. From (11) and (12), we obtain

$$|a_3 - a_2^2| = \frac{1}{4(1+2\alpha)} \left| p_2 - \frac{5(1+\alpha)^2 + 2(1+2\alpha)}{8(1+\alpha)^2} p_1^2 \right|. \tag{18}$$

Using third inequality of Lemma 1, (18) takes the form

$$|a_3 - a_2^2| \leq \frac{1}{2(1+2\alpha)} max \left\{ 1, \frac{(1+\alpha)^2 + 2(1+2\alpha)}{4(1+\alpha)^2} \right\}. \tag{19}$$

Substituting for $\alpha = 0$, Theorem 2.2 yields the following result:
Corollary 2.3 If $f \in \mathcal{R}_{\mathcal{L}}^{'}$, then

$$|a_3 - a_2^2| \leq \frac{1}{2}.$$

Putting $\alpha = 1$, Theorem 2.2 yields the following result:
Corollary 2.4 If $f \in \mathcal{R}_{\mathcal{L}}$, then

$$|a_3 - a_2^2| \leq \frac{1}{6}.$$

Theorem 2.3 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{2(1+3\alpha)}. \tag{20}$$

Proof. Using (11), (12), (13) and after simplification, we have
$|a_2 a_3 - a_4|$

$$= \frac{1}{128(1+\alpha)(1+2\alpha)(1+3\alpha)} \left| (18 + 54\alpha + 26\alpha^2) p_1^3 - (48 + 144\alpha + 80\alpha^2) p_1 p_2 + 32(1+\alpha)(1+2\alpha) p_3 \right|. \tag{21}$$

On applying Lemma 2 in (21), it yields (20).

For $\alpha = 0$, the following result is a consequence of Theorem 2.3:
Corollary 2.6 If $f \in \mathcal{R}_{\mathcal{L}}^{'}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{2}.$$

On putting $\alpha = 1$ in Theorem 2.3, we can obtain the following result:
Corollary 2.7 If $f \in \mathcal{R}_{\mathcal{L}}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{8}.$$

Theorem 2.4 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{4(1+2\alpha)^2}. \tag{22}$$

Proof. Using (11), (12) and (13), we have
$|a_2a_4 - a_3^2|$

$$= \frac{1}{1024(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \left|64(1+2\alpha)^2 p_1p_3 - 80\alpha^2 p_1^2 p_2 + (1+4\alpha+29\alpha^2)p_1^4 - 64(1+\alpha)(1+3\alpha)p_2^2\right|.$$

Substituting for p_2 and p_3 from Lemma 3 and letting $p_1 = p$, we get

$$|a_2a_4 - a_3^2| = \frac{1}{1024(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \Big|(45\alpha^2+4\alpha+1)p^4 - 8\alpha^2p^2(4-p^2)x$$

$$-16(1+2\alpha)^2p^2(4-p^2)x^2 - 16(1+\alpha)(1+3\alpha)(4-p^2)^2x^2 + 32(1+2\alpha)^2p(4-p^2)(1-|x|^2)z\Big|.$$

Since $|p| = |p_1| \leq 2$, we may assume that $p \in [0,2]$. By using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0,1]$, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{1024(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \Big[(45\alpha^2+4\alpha+1)p^4 + 8\alpha^2p^2(4-p^2)x$$

$$+16(1+2\alpha)^2p^2(4-p^2)t^2 + 16(1+\alpha)(1+3\alpha)(4-p^2)^2t^2 + 32(1+2\alpha)^2p(4-p^2) - 32(1+2\alpha)^2p(4-p^2)t^2\Big] = F(p,t).$$

$$\frac{\partial F}{\partial t} = \frac{1}{1024(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[8\alpha^2p^2(4-p^2) + 32(4-p^2)(2-p)t\{(6-p)\alpha^2+8\alpha+2\}\Big] \geq 0,$$

and so $F(p,t)$ is an increasing function of t.

Therefore, $max\{F(p,t)\} = F(p,1) = \frac{1}{1024(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[(45\alpha^2+4\alpha+1)p^4$

$$+8\alpha^2p^2(4-p^2) + 16(1+2\alpha)^2p^2(4-p^2) + 16(1+\alpha)(1+3\alpha)(4-p^2)^2\Big] = H(p).$$

$H'(p) = 0$ gives $p = 0$. Also $H''(p) < 0$ for $p = 0$.

This implies $max\{H(p)\} = H(0) = \frac{1}{4(1+2\alpha)^2}$, which proves (22).

Putting $\alpha = 0$, Theorem 2.4 gives the following result:

Corollary 2.8 If $f \in \mathcal{R}_{\mathcal{L}}^{'}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}.$$

Substituting for $\alpha = 1$ in Theorem 2.4, the following result is obvious:

Corollary 2.9 If $f \in \mathcal{R}_{\mathcal{L}}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{36}.$$

Theorem 2.5 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, then

$$|H_3(1)| \leq \frac{1}{4(1+2\alpha)}\left[\frac{1}{2(1+2\alpha)^2} + \frac{1}{1+4\alpha}\right] + \frac{1}{4(1+3\alpha)^2}. \tag{23}$$

Proof. By using (4), (5), (6), (20), (22) and Theorem 2.2 in (1), the result (23) can be easily obtained.

For $\alpha = 0$, Theorem 2.5 yields the following result:

Corollary 2.10 If $f \in \mathcal{R}_{\mathcal{L}}^{'}$, then

$$|H_3(1)| \leq \frac{5}{8}.$$

For $\alpha = 1$, Theorem 2.5 yields the following result:
Corollary 2.11 If $f \in \mathcal{R}_{\mathcal{L}}$, then

$$|H_3(1)| \leq \frac{319}{8640}.$$

Theorem 2.6 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha}$, then

$$|H_4(1)| \leq \frac{1}{(1+2\alpha)^2(1+4\alpha)}\left[\frac{1+8\alpha+13\alpha^2}{4(1+2\alpha)(1+4\alpha)(1+6\alpha)} + \frac{1+4\alpha+2\alpha^2}{4(1+4\alpha)^2} + \frac{3+12\alpha+8\alpha^2}{8(1+3\alpha)(1+5\alpha)}\right]$$

$$+\frac{1}{16(1+3\alpha)^2(1+6\alpha)}\left[\frac{3+18\alpha+18\alpha^2}{(1+3\alpha)^2} + \frac{5+30\alpha+18\alpha^2}{(1+2\alpha)(1+4\alpha)}\right]+\frac{1}{8(1+2\alpha)(1+5\alpha)^2}.$$

Proof. We have

$$|a_2a_4 + 2a_3^2| \leq |a_2a_4 - a_3^2| + 3|a_3|^2.$$

Applying the triangle inequality in (2) and using the above inequality along with Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5, the proof of the Theorem 2.6 is obvious.

For $\alpha = 0$, Theorem 2.6 yields the following result:
Corollary 2.12 If $f \in \mathcal{R}_{\mathcal{L}}^{'}$, then

$$|H_4(1)| \leq \frac{3}{2}.$$

For $\alpha = 1$, Theorem 2.6 yields the following result:
Corollary 2.13 If $f \in \mathcal{R}_{\mathcal{L}}$, then

$$|H_4(1)| \leq 0.0101.$$

3. Bounds of $|H_3(1)|$ for Two-fold and Three-fold symmetric functions

A function f is said to be n-fold symmetric if is satisfy the following condition:

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.
By $S^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class S.
The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{24}$$

An analytic function f of the form (24) belongs to the family $\mathcal{R}_{\mathcal{L}}^{\alpha(n)}$ if and only if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \sqrt{\frac{2p(z)}{p(z)+1}}, p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^n = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E\right\}. \tag{25}$$

Theorem 3.1 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha(2)}$, then

$$|H_3(1)| \leq \frac{1}{4(1+2\alpha)(1+4\alpha)}. \tag{26}$$

Proof. If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha(2)}$, so there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \sqrt{\frac{2p(z)}{p(z)+1}}. \tag{27}$$

Using (24) and (25) for $n = 2$, (27) yields

$$a_3 = \frac{1}{4(1+2\alpha)} p_2, \tag{28}$$

$$a_5 = \frac{1}{4(1+4\alpha)} \left(p_4 - \frac{5}{8} p_2^2 \right). \tag{29}$$

Also

$$H_3(1) = a_3 a_5 - a_3^3. \tag{30}$$

Using (28) and (29) in (30), it yields

$$H_3(1) = \frac{1}{16(1+2\alpha)(1+4\alpha)} p_2 \left[p_4 - \frac{5(1+2\alpha)^2 + 2(1+4\alpha)}{8(1+2\alpha)^2} p_2^2 \right]. \tag{31}$$

On applying triangle inequality in (31) and using second inequality of Lemma 1, we can easily get the result (26).

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 3.1:

Corollary 3.1 If $f \in \mathcal{R}_{\mathcal{L}}^{'(2)}$, then

$$|H_3(1)| \leq \frac{1}{4}.$$

For $\alpha = 1$, Theorem 3.1 agrees with the following result:

Corollary 3.2 If $f \in \mathcal{R}_{\mathcal{L}}^{(2)}$, then

$$|H_3(1)| \leq \frac{1}{60}.$$

Theorem 3.2 If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha(3)}$, then

$$|H_3(1)| \leq \frac{1}{4(1+3\alpha)^2}. \tag{32}$$

Proof. If $f \in \mathcal{R}_{\mathcal{L}}^{\alpha(3)}$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \sqrt{\frac{2p(z)}{p(z)+1}}. \tag{33}$$

Using (24) and (25) for $n = 3$, (33) gives

$$a_4 = \frac{1}{4(1+3\alpha)} p_3. \tag{34}$$

Also

$$H_3(1) = -a_4^2. \tag{35}$$

Using (34) in (35), it yields

$$H_3(1) = -\frac{1}{16(1+3\alpha)^2} p_3^2. \tag{36}$$

On applying triangle inequality and using first inequality of Lemma 1, (32) can be easily obtained.

For $\alpha = 0$, Theorem 3.2 yields the following result:
Corollary 3.3 If $f \in \mathcal{R}_{\mathcal{L}}^{'(3)}$, then

$$|H_3(1)| \leq \frac{1}{4}.$$

For $\alpha = 1$, Theorem 3.2 yields the following result:
Corollary 3.4 If $f \in \mathcal{R}_{\mathcal{L}}^{(3)}$, then

$$|H_3(1)| \leq \frac{1}{64}.$$

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