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AN OPERATIONAL MATRIX METHOD FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON-SINGULAR KERNEL

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Abstract. In this paper, We formulate a numerical approximate solution for Caputo-Fabrizio Riccati fractional differential equations (CFRFDE) based on Euler polynomials with fractional order. The proposed technique consists of an operational matrix method along with collocation points which transform the CFRFDE into a system of algebraic equations. We compared the solutions by changing different parameters of the approximate formula considering various numerical examples and proved that the suggested technique is effective.

1. Introduction

Fractional calculus, the generalization of integer order derivatives to noninteger order, has enormous applications in distinct fields of science and engineering [1]. The theory and applications of fractional calculus have been developed intensively over the years. Numerous definitions for fractional operators were put forth to formulate specific problems. Some of them are mentioned in [2]. Riemann-Liouville and Caputo are the frequently adopted approaches in the literature. In the most recent definition, Caputo and Fabrizio replaced the singular kernel in the Caputo derivative with an exponential kernel [3]. This new operator has more supportive properties in comparison with the classical version. Caputo-Fabrizio derivatives can efficiently express the effect of memory and material heterogeneity [4, 5, 6]. It has been applied to model a wide variety of problems, including heat transfer [7], mass-spring-damper system [8], non-linear Fisher's-diffusion equation [9], the Korteweg-de Vries-Burgers functional differential equation [10], nonlinear Baggs and Freedman model [11], fractional Maxwell fluid [12], and elasticity [13]. In [14], a comparative study has performed between the Caputo and the Caputo–Fabrizio fractional models through graphical representations.

Some existing analytical and numerical approaches have been improved to solve fractional differential equations (FDE) specified in the sense of the Caputo-Fabrizio operator [16, 4, 15]. The operational matrix approach based on orthogonal polynomials such as Legendre [17], Genocchi [18] and Chebyshev [19] has been applied

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in certain works. In [20], a fractional form of the Euler polynomial $E^\alpha(x)$ is introduced to study space FDEs. Besides that, an operational matrix for integration of $E^\alpha(x)$ is presented in [21] to solve intergro-FDEs. Based on the above motivation, we propose an operational matrix of differentiation for a numerical solution to Caputo-Fabrizio Riccati Fractional differential equations (CFRFDE). The general form of the CFRFDE, with initial condition, is given below:

$$ {}_0^{CFD}D_x^v y(x) = F(x) + G(x)y(x) + H(x)y(x)^2, \qquad 0 \le v \le 1, \tag{1}$$

$$ y(0) = y_0, \tag{2}$$

where $F(x),\ G(x)$ and $H(x)$ are known functions, $y_0 \in \mathbb{R}$ and ${}_0^{CFD}D^v$ is the Caputo-Fabrizio fractional derivative of order v. The advantage of the operational matrix method is that it transforms the differential equation into a system of algebraic equations.

This article is structured as follows: In Section 2, we recall some basic definitions and properties of fractional derivatives and Euler polynomials. In Section 3, the operational matrix of the fractional differentiation is derived. In Section 4, we present the approximation method based on the operational matrix for CFRFDE. Section 5 is devoted to error analysis. Various numerical examples are illustrated in Section 6, and the paper ends with a conclusion in Section 7.

2. Preliminaries

This section recalls the definition and properties of Caputo-Fabrizio derivatives. It also provides an introduction to fractional order Euler polynomial, which is a special type of function that we are using for approximation of our numerical solution.

Definition 1 Let $f \in H^1(0,1)$ and $v \in (0,1)$. The Caputo-Fabrizio derivative (CFD) is defined as [3]

$$ {}_0^{CFD}D_x^v f(x) = \frac{M(v)}{1-v}\int_0^x f'(t)\exp(-\frac{v(x-t)^v}{1-v})dt, \tag{3}$$

where $M(v)$ is the normalization function such that $M(0) = M(1) = 1$.
The operator ${}_0^{CFD}D_x^v$ satisfies the properties:
(i) ${}_0^{CFD}D_x^v(af + bg) = a\ {}_0^{CFD}D_x^v(f) + b\ {}_0^{CFD}D_x^v(g)$, for all $a, b \in R$.
(ii) ${}_0^{CFD}D_x^v(f) = 0$, for all constant functions $f(x) = \lambda$.

Fractional order Euler polynomials. Let $E^\alpha(x) = [E_0^\alpha(x), E_1^\alpha(x), ...E_m^\alpha(x)]^T$. Here $E_j^\alpha(x)$ is the Euler polynomial with order $j\alpha$, $\alpha > 0$ which is defined as follows:[22]

$$ E_j^\alpha(x) = \frac{1}{j+1}\sum_{k=1}^{j+1} e_{(j,k)} x^{(j+1-k)\alpha},\ \ , j = 0, 1, ..., m, \tag{4}$$

where $e_{j,k} = (2 - 2^{k+1})\begin{pmatrix} j+1 \\ k \end{pmatrix} B_k$ and $B_k = B_k(0)$ is the Bernoulli number for each $k = 0, 1, ..., j$.
Let $X^\alpha = \begin{bmatrix} 1, & x^\alpha, & \dots, & x^{m\alpha} \end{bmatrix}^T$. The relation between X^α and E^α is expressed as [21]

$$ E^\alpha(x) = B^\alpha X^\alpha, \tag{5}$$

where

$$B^\alpha = \begin{bmatrix} \frac{2-2^2}{1}\begin{pmatrix}1\\1\end{pmatrix}B_1(0) & 0 & ... & 0 \\ \frac{2-2^3}{2}\begin{pmatrix}2\\2\end{pmatrix}B_2(0) & \frac{2-2^2}{2}\begin{pmatrix}2\\1\end{pmatrix}B_1(0) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2-2^{m+2}}{m+1}\begin{pmatrix}m+1\\m+1\end{pmatrix}B_{m+1}(0) & \frac{2-2^{m+1}}{m+1}\begin{pmatrix}m+1\\m\end{pmatrix}B_m(0) & ... & \frac{2-2^2}{m+1}\begin{pmatrix}m+1\\1\end{pmatrix}B_1(0) \end{bmatrix}. \tag{6}$$

Further, a function $f(x) \in L^2[0,1]$, can be approximated as [21]

$$f(x) = \sum_{i=0}^{\infty} c_i E_i^\alpha. \tag{7}$$

Here $C = [c_0, c_1, ..., c_m]$ is evaluated from the following relation,

$$C^T = A^T D^{-1}, \tag{8}$$

$$A^T = [a_0, a_1, a, ..., a_m], \quad a_j = \int_0^1 E_j^\alpha(x) f(x) x^{1-\alpha},$$

$$D = [d_{ij}^\alpha], \quad d_{ij} = \int_0^1 E_i^\alpha(x) E_j^\alpha(x) x^{1-\alpha} dx.$$

3. Operational Matrix

In this section, we approximate ${}_0^{CFD}D^v$ with an $(m+1)\times(m+1)$ operational matrix $O^{(v,m\alpha)}$ for $E^\alpha(x)$. For, we prove the following theorem.

Theorem 1 Let $r \in Z^+$ and $v, \alpha \in (0,1)$. Then

$${}_0^{CFD}D_x^v\ (x^{r\alpha}) = \frac{M(v)}{1-v}\Gamma(r\alpha+1)\sum_{j=0}^{\infty}\frac{(-p)^j}{\Gamma(r\alpha+j+1)}x^{r\alpha+j}, \tag{9}$$

where $p = \frac{\alpha}{1-\alpha}$.

Proof.

$$\begin{aligned} {}_0^{CFD}D_x^v\ (x^{r\alpha}) &= \frac{M(v)}{1-v}\int_0^x r\alpha t^{r\alpha-1}\exp(-p(x-t))dt \\ &= \frac{M(v)}{1-v} r\alpha \int_0^x t^{r\alpha-1}\sum_{j=0}^{\infty}\frac{(-p(x-t))^j}{j!}dt \\ &= \frac{M(v)}{1-v} r\alpha \sum_{j=0}^{\infty}\frac{(-p)^j}{j!}\int_0^x t^{r\alpha-1}(x-t)^j dt \\ &= \frac{M(v)}{1-v} r\alpha \sum_{j=0}^{\infty}\frac{(-p)^j}{j!}\frac{\Gamma(r\alpha)\Gamma(j+1)}{\Gamma(r\alpha+j+1)}x^{r\alpha+j} \\ &= \frac{M(v)}{1-v}\sum_{j=0}^{\infty}(-p)^j\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+j+1)}x^{r\alpha+j} \\ &= \frac{M(v)}{1-v}\Gamma(r\alpha+1)\sum_{j=0}^{\infty}\frac{(-p)^j}{\Gamma(r\alpha+j+1)}x^{r\alpha+j}. \end{aligned}$$

Now applying equation (9) and equation (7), we have, for $r = 0, 1, ...m$

$$ {}_{0}^{CFD}D_x^v \; x^{r\alpha} = \frac{M(v)}{1-v}\Gamma(r\alpha+1)\sum_{j=0}^{\infty}\frac{(-p)^j}{\Gamma(r\alpha+j+1)}x^{r\alpha+j} = N_r E^{\alpha}(x). $$

By using equations (7) and (8), we get

$$ N_r = \overline{a}_r D^{-1};\; \overline{a}_r = [\; a_{r0}, a_{r1}, ..., a_{rm}] \;, \qquad D = [d_{ij}], $$

$$ a_{ri} = \frac{M(v)}{1-v}\Gamma(r\alpha+1)\sum_{j=0}^{r-1}\frac{(-p)^j}{\Gamma(r\alpha+j+1)}\int_0^1 E_i^{\alpha}(x)x^{r\alpha+j}dx, \qquad i = 0, 1, ...m, $$

$$ d_{ij} = \int_0^1 E_i^{\alpha}(x)E_j^{\alpha}(x)x^{1-\alpha}dx, \qquad i = 0, 1, ...m, \qquad j = 0, 1, ...m. $$

Therefore

$$ {}_{0}^{CFD}D_x^v \; X^{\alpha}(x) = [N_0, N_1, ..., N_m]E^{\alpha}(x). \tag{10} $$

Using equations (5) and (10) and the properties of the fractional operator ${}_{0}^{CFD}D_x^v$,

$$ \begin{aligned} {}_{0}^{CFD}D_x^v \; E^{\alpha}(x) &= {}_{0}^{CFD} \; D_x^v \; B^{\alpha}X^{\alpha}(x) = B^{\alpha} \; {}_{0}^{CFD}D_x^v \; X^{\alpha}(x) \\ &= B^{\alpha}[N_0, N_1, ..., N_m]E^{\alpha}(x) = J^{(v,\alpha)}E^{\alpha}(x). \end{aligned} $$

Hence $O^{(v,m\alpha)} = B^{\alpha}[N_0, N_1, ..., N_m]$ is the operational matrix of $E^{\alpha}(x)$.

4. Operational Matrix Method

In this part, we develop a numerical solution for the Caputo-Fabrizio Riccati differential equation of the form,

$$ {}_{0}^{CFD}D_x^v y(x) = F(x) + G(x)y(x) + H(x)y(x)^2, \qquad 0 \le v \le 1, \tag{11} $$

$$ y(0) = y_0, \tag{12} $$

where y_0 is a given real number. Let $y_{\alpha}(x) \approx C^T E^{\alpha}(x)$ be the approximate solution for the above mentioned initial value problem. To determine C^T, we approximate $F(x)$, $G(x)$ and $H(x)$ as follows:

$$ \begin{aligned} F(x) &\approx F^T E^{\alpha}(x), \qquad (13) \\ G(x) &\approx G^T E^{\alpha}(x), \\ H(x) &\approx H^T E^{\alpha}(x), \end{aligned} $$

where

$F^T = [\; p_0 \quad p_1 \quad ... \quad p_m \;]$, $\quad G^T = [\; g_0 \quad g_1 \quad ... \quad g_m \;]$ and $H^T = [\; h_0 \quad h_1 \quad ... \quad h_m \;]$.

Then

$$ {}_{0}^{CFD}D_x^v y_{\alpha}(x) = {}_{0}^{CFD} \; D_x^v(C^T E^{\alpha}(x)) = C^T({}_{0}^{CFD}D_x^v E^{\alpha}(x)) = C^T(O^{(v,m\alpha)}E^{\alpha}(x)). \tag{14} $$

By applying equations 13 and 14 into equations 11 and 12, we get

$$ \begin{aligned} C^T(O^{(v,m\alpha)}E^{\alpha}(x)) - F^T E^{\alpha}(x) - G^T E^{\alpha}(x)(C^T E^{\alpha}(x)) \qquad (15) \\ -H^T E^{\alpha}(x)(C^T E^{\alpha}(x)(E^{v}(\alpha))^T C) = 0 \end{aligned} $$

and

$$ [C^T E^{\alpha}(x)]_{x=0} = y_0. \tag{16} $$

By using equation 16, express one of the $m+1$ unknown elements of $C^T = \begin{bmatrix} c_0 & c_1 & \dots & c_m \end{bmatrix}$ in terms of the others and substituting it in equation 15, we get

$$\begin{aligned} C^T(O^{(v,m\alpha)}E^\alpha(x)) - F^T E^\alpha(x) - G^T E^\alpha(x)(C^T E^\alpha(x)) \\ -H^T E^\alpha(x)(C^T E^\alpha(x)(E^\alpha(x))^T C) = 0. \end{aligned} \tag{17}$$

Now we substitute m different values for x, $\{\frac{1}{m}, \frac{2}{m}, \dots, \frac{m}{m}\}$ in equation 17, we obtain a system of algebraic equations from which the unknown elements of the vector C can be determined.

5. Error Analysis

This section provides an error bound for the approximation of ${}_0^{CFD}D_x^v f(x)$ with $O^{(v,m\alpha)}$.

Theorem 1 The error in approximating ${}_0^{CFD}D_x^v f(x)$ by using operational matrix $O^{(v,m\alpha)} = [\delta_{i,j}^\alpha]$ for $E^\alpha(x)$ is bounded as follows:

$$|{}_0^{CFD}D_x^v f(x) - C^T O^{(v,m\alpha)} E^{(\alpha)}| \leq \sum_{i=m+1}^{\infty} |c_i| \sum_{j=0}^{m} |\delta_{i,j}^\alpha| \sum_{k=1}^{j+1} |e_{j,k}|. \tag{18}$$

Proof. Let $f(x) \in L^2[0,1]$. Then from the equation (7)

$$f(x) = \sum_{i=0}^{\infty} c_i E_i^{(\alpha)}(x).$$

We apply the CFD operator on both sides of the above equation to get,

$${}_0^{CFD}D_x^v f(x) = \sum_{i=0}^{\infty} c_i({}_0^{CFD}D_x^v E_i^{(\alpha)}(x)) = \sum_{i=0}^{\infty} c_i \sum_{j=0}^{m} \delta_{i,j}^\alpha E_j^\alpha(x). \tag{19}$$

If we consider the first $m+1$ terms, then

$${}_0^{CFD}D_x^v f(x) - \sum_{i=0}^{m} c_i \sum_{j=0}^{m} \delta_{i,j}^\alpha E_j^\alpha(x) = \sum_{i=m+1}^{\infty} c_i \sum_{j=0}^{m} \delta_{i,j}^\alpha E_j^\alpha(x). \tag{20}$$

Then,

$$\begin{aligned} |{}_0^{CFD}D_x^v f(x) - C^T O^{(v,m\alpha)} E^\alpha(x)| &= |\sum_{i=m+1}^{\infty} c_i \sum_{j=0}^{m} \delta_{i,j}^\alpha E_j^\alpha(x)| \\ &\leq \sum_{i=m+1}^{\infty} |c_i| \sum_{j=0}^{m} |\delta_{i,j}^\alpha| \quad |E_j^\alpha(x)|. \end{aligned} \tag{21}$$

From equation (4),

$$|E_j^\alpha(x)| = \frac{1}{j+1} \sum_{k=1}^{j+1} |e_{(j,k)} x^{(j+1-k)\alpha}| \leq \frac{1}{j+1} \sum_{k=1}^{j+1} |e_{(j,k)}|. \tag{22}$$

Now by substituting equation(22) in equation (21), we have

$$|{}_0^{CFD}D_x^v f(x) - C^T O^{(v,m\alpha)} E^{(\alpha)}(x)| \leq \sum_{i=m+1}^{\infty} |c_i| \sum_{j=0}^{m} |\delta_{i,j}^\alpha| \sum_{k=1}^{j+1} |e_{(j,k)}|.$$

Hence the proof.

6. Numerical Examples

In this section, we numerically solve some examples for FDEs containing CFD by applying our proposed technique.

Example 6.1. Consider the following Caputo-Fabrizio fractional differential equation:

$$ {}_{a}^{CFD}D^{\frac{4}{5}}y(x) = \frac{5}{8}e^{-4x} - \frac{5}{8} + \frac{5}{2}x, \tag{23} $$

$$ y(0) = 0. \tag{24} $$

The exact solution to this equation is $y(x) = x^2$. The above fractional differential equation of order $v = 4/5$ can be solved by choosing different values of α. α is the parameter regarding Fractional order Euler functions $E^{\alpha}(x)$, with which approximate solution is calculated. The numerical solutions for different values of α have been determined and the corresponding absolute errors are presented in Table 1.

Table 1. Absolute errors of the Example 6.1 for different values of α

x	$\alpha = 0.99999$	$\alpha = 0.999999$	1.00001	$\alpha = 1.000001$	$\alpha = 1$
0.0000	1.65452E-18	8.45437E-19	0	1.32946E-17	0
0.2000	3.525261E-6	3.525293E-7	3.525333E-6	3.525301E-7	0
0.4000	4.972136E-6	4.972218E-7	4.972318E-6	4.972236E-7	0
0.6000	5.687067E-6	5.687151E-7	5.687254E-6	5.687170E-7	0
0.8000	6.429392E-6	6.429426E-7	6.429466E-6	6.429433E-7	0
1.0000	7.756693E-6	7.756627E-7	7.756546E-6	7.756612E-7	0

Figure 1(a) and Figure 1(b) compare the behaviour of the approximate solutions when α is taken close to 1 from the right and left sides respectively. In both cases, the solution is compared with the exact solution and the error function is depicted in Figure 2.

Figure 1. Comparison of results with exact solution of Example 6.1 for different values of α

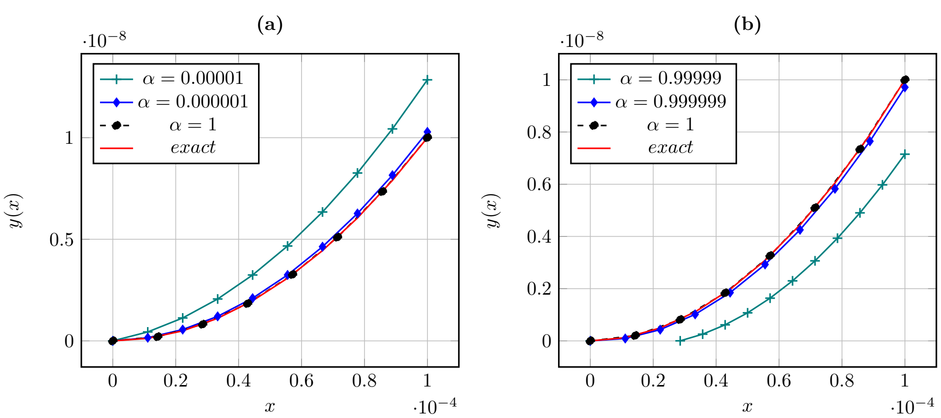
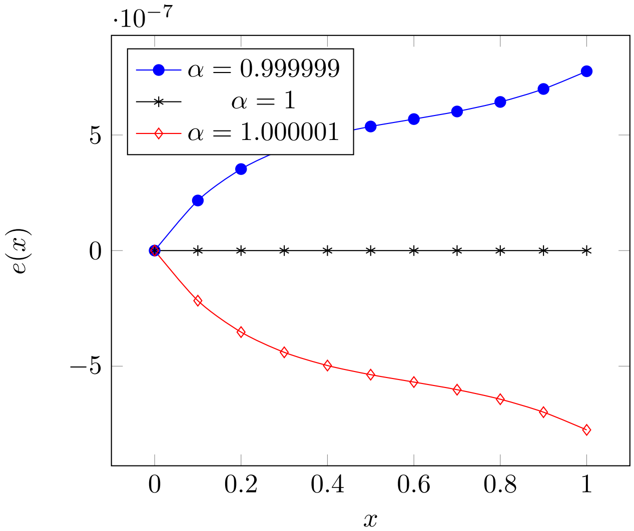


FIGURE 2. Error function for approximate solution of Example 6.1



It shows that the current method provides greater accuracy as the parameter α is chosen very close to 1. There for, in all the subsequent numerical examples of Riccati differential equations, $\alpha = 1$ is taken as constant and the parameters v (order of fractional derivative) and m (number of polynomials in $E^{\alpha}(x)$) are changed to compare the results.

Example 6.2. Consider the following Caputo-Fabrizio fractional differential equation:

$$ {}_{a}^{CFD}D^{v}y(x) + ve^{x}(-1 + e^{\frac{x}{-1+v}}) = y(x) \qquad 0 \leq v \leq 1 \tag{25}$$
$$ y(0) = 0 \tag{26}$$

The exact solution to the above problem is given by

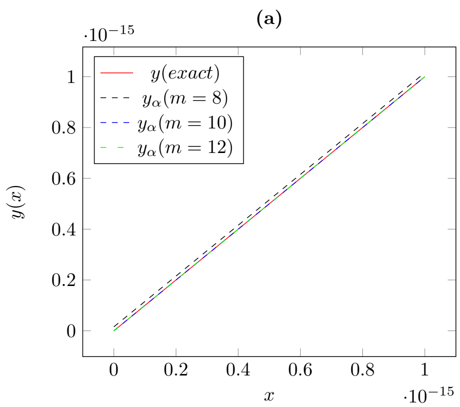
$$y(x) = xe^{x}$$

The approximate solution for Example 6.2 is obtained for $m = 8, 10, 12$ and $v = 0.5, 0.9$. The results are summarized in Table 2. Table 2 clearly indicates the efficiency of the proposed technique when compared with the methods in [23] and [17] through their maximum absolute errors. A graphical representation of the numerical solutions and the corresponding absolute errors is shown in Figures 3 and 4 respectively.

TABLE 2. Maximum absolute errors of Example 6.2 for different values of v and m.

v	Method	$m=8$	$m=10$	$m=12$
0.5	Ref.[23]	1.134E-4	1.049E-5	9.921E-7
	Ref.[17]	5.112E-8	4.94E-11	3.06E-14
	Current	3.41E-10	2.78E-13	1.49E-12
0.9	Ref.[23]	3.007E-4	1.847E-5	2.559E-6
	Ref.[17]	3.880E-8	3.97E-11	2.55E-14
	Current	7.57E-10	6.67E-13	6.66E-16

FIGURE 3. Exact and approximated solutions for Example 6.3 : **(a)** $v=0.5$ **(b)** $v=0.9$



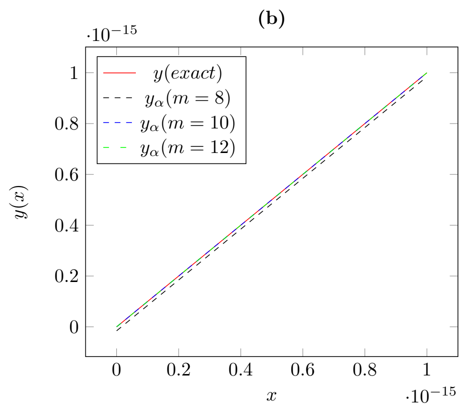
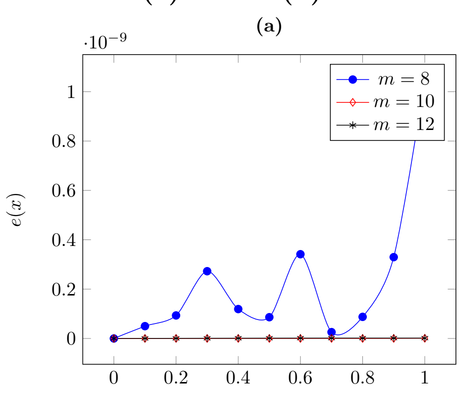
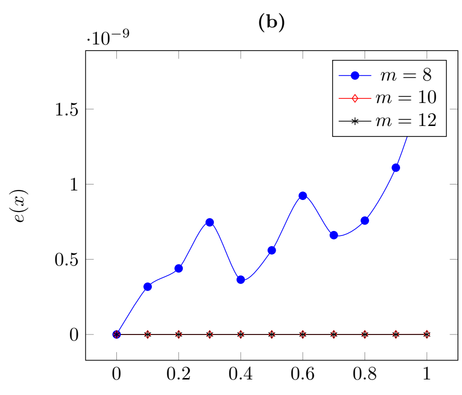


FIGURE 4. Absolute errors of approximate solutions for Example 6.2: **(a)** $v=0.5$ **(b)** $v=0.9$





Example 6.3. Consider the following Caputo-Fabrizio fractional differential equation:

$$ {}^{CFD}_{a}D^{v}y(x) - 2y(x) - h(x) = 0 \qquad 0 \leq v \leq 1 \tag{27} $$

$$ y(0) = 1 \tag{28} $$

where

$$ h(x) = \frac{(\alpha-\alpha^2-1)e^{\frac{\alpha x}{\alpha-1}} - (1+2\alpha(-1+\alpha))(-1+2\alpha x) - \alpha((1-3\alpha+4\alpha^2)\cos x + \alpha \sin x)}{\alpha+2\alpha+2\alpha^2(\alpha-1)} $$

The exact solution to this problem is given by

$$ y(x) = x + \cos x $$

TABLE 3. Maximum absolute errors of Example 6.3 for different values of v and m.

v	Method	$m = 8$	$m = 10$	$m = 12$
0.5	Ref.[23]	9.778E-4	1.062E-5	1.141E-6
	Ref.[17]	1.490E-8	1.415E-11	8.681E-15
	Current	8.850E-12	5.55E-15	4.440E-16
0.9	Ref.[23]	3.163E-4	1.929E-5	1.262E-6
	Ref.[17]	2.264E-9	1.906E-12	1.043E-15
	Current	1.022E-10	4.929E-14	2.220E-16

The numerical solution to Example 6.3 is obtained for $m = 8$, 10, 12 and $v = 0.5$, 0.9. The results are summarized in Table 3. Table 3 clearly indicates the efficiency of the proposed technique when compared with the methods in [17] and [23] through their maximum absolute errors. A comparison of numerical solution with the exact solution for $v = 0.5$ and $v = 0.9$ is shown graphically in Figure 5 whereas the corresponding absolute errors are presented in Figure 6.

FIGURE 5. Exact and approximated solutions for Example 6.3 : **(a)** $v = 0.5$ **(b)** $v = 0.9$

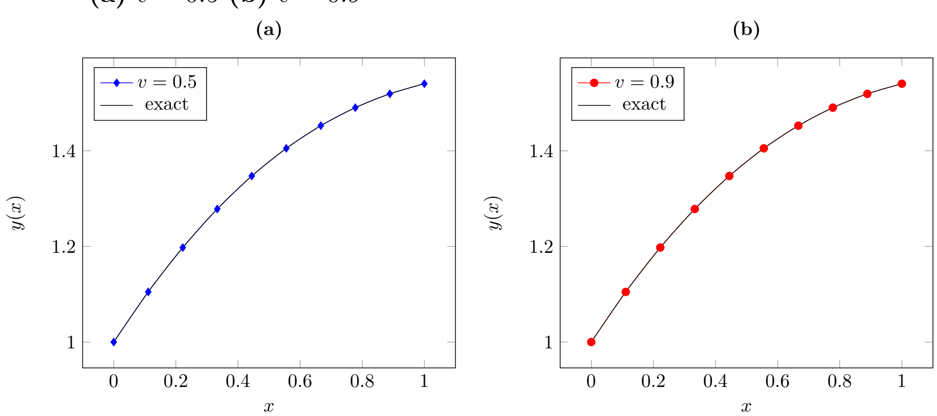
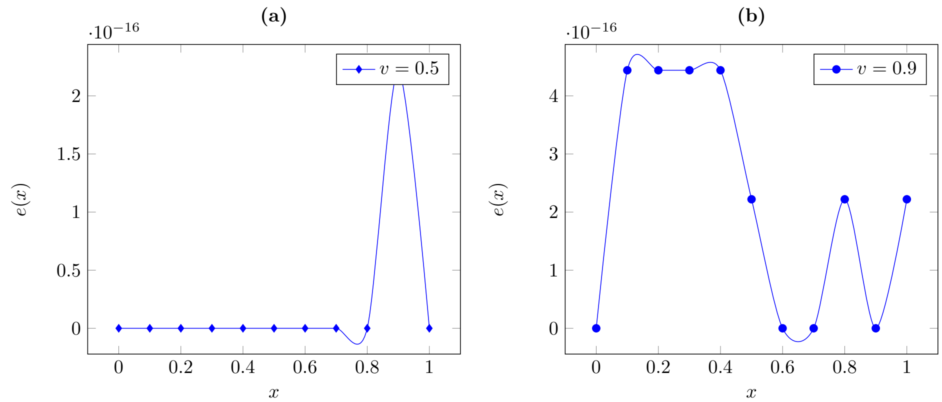


FIGURE 6. Absolute errors of approximate solutions for Example 6.3



Example 6.4. Consider the following Caputo-Fabrizio fractional differential equation:

$$ {}^{CFD}_{a}D^{v}y(x) = y^2(x) + 1 \qquad 0 \leq v \leq 1 \tag{29}$$

$$ y(0) = 0 \tag{30}$$

For $v = 1$, the exact solution is given by

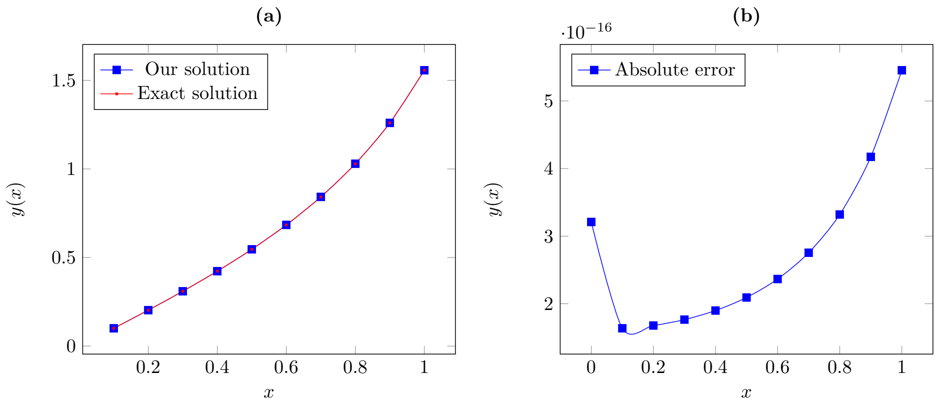
$$ y(x) = \tan x $$

TABLE 4. Numerical results of the Example 6.4 for $v = 1$

x	Exact	Ref.[24]	Ref.[25]	Our result
0.1	0.100331	0.100335	0.100342	0.100330
0.2	0.202710	0.20271	0.202726	0.202708
0.3	0.309336	0.309336	0.309372	0.309334
0.4	0.422793	0.422793	0.422832	0.422791
0.5	0.546302	0.546302	0.546363	0.546300
0.6	0.684136	0.684131	0.684251	0.684134
0.7	0.842288	0.842245	0.842411	0.842285
0.8	1.029639	1.02937	1.029849	1.029635
0.9	1.260158	1.2588	1.260573	1.260154
1.0	1.557408	1.55137	1.557938	1.557402

The numerical solution for Example 6.4 is obtained for $v = 1$. The solution is compared with the existing results in [25] and [24]. It can be seen from Table 4 that our numerical approach gives better accuracy than the results reported in [25] and [24]. The numerical solution and the exact solution are depicted in Figure 7(a) and the corresponding absolute error function is presented in Figure 7(b).

FIGURE 7. Comparison of exact and approximate solutions of Example 6.4 for $v = 1$



Example 6.5. Consider the following Caputo-Fabrizio fractional differential equation:

$$ {}_{a}^{CFD}D^{v}y(x) = 1 + 2y(x) - y^{2}(x) \qquad 0 \leq v \leq 1 \tag{31} $$

$$ y(0) = 0 \tag{32} $$

The exact solution of this equation for $v = 1$ is given by

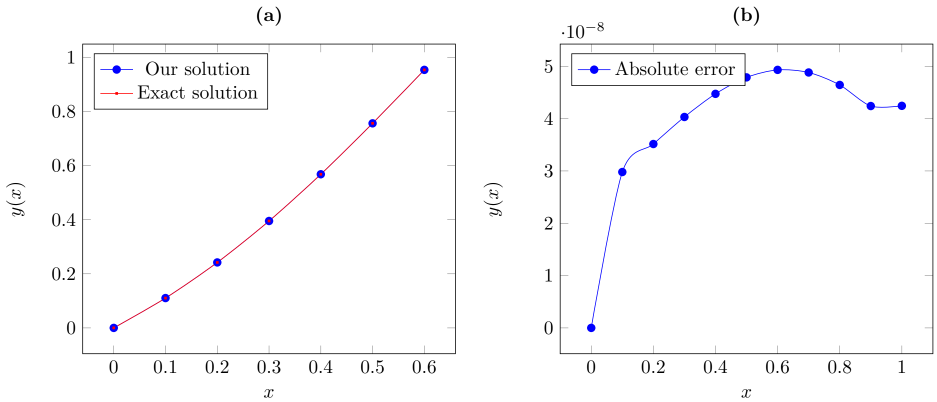
$$ y(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) $$

TABLE 5. Numerical solutions of Example 6.4 for $v = 1$

x	Exact	Ref.[26]	Our result
0.0	0.0000000	0.462E-12	8.533E-17
0.2	0.2419768	0.2419760	0.2419767
0.4	0.5678121	0.5678141	0.5678121
0.6	0.9535662	0.9535635	0.9535661
0.8	1.3463637	1.3463659	1.3463636
1.0	1.6894984	1.6894983	1.6894983

The approximate solution for Example 6.5 is obtained for $v = 1$. The results are summarized in Table 5. Table 5 indicates the efficiency of the proposed technique when compared with the method in [26]. A graphical representation of the numerical solution and absolute errors obtained is shown in Figures 8(a) and 8(b) respectively.

FIGURE 8. Numerical results of Example 6.5 for $v = 1$



Example 6.6. Consider the following Caputo-Fabrizio fractional differential equation:

$$ {}^{CFD}_{a}D^{v}y(x) = 1 - y^2(x) \qquad 0 \le v \le 1 \tag{33} $$

$$ y(0) = 0 \tag{34} $$

The exact solution of this equation for $v = 1$ is

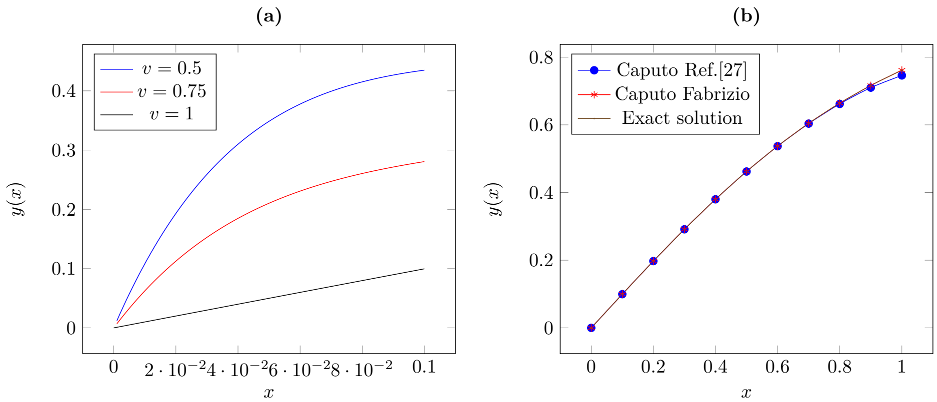
$$ y(x) = \frac{e^{2x} - 1}{e^{2x} + 1} $$

TABLE 6. Comparison between solutions under Caputo [27] and Caputo-Fabrizio derivatives

x	Caputo [27]			Caputo-Fabrizio			Exact solution
	$v = 0.5$	$v = 0.75$	$v = 1$	$v = 0.5$	$v = 0.75$	$v = 1$	for $v = 1$
0.2	0.454125	0.313795	0.197375	0.462676	0.339769	0.197374	0.197375
0.4	0.644422	0.492889	0.379944	0.513784	0.445527	0.379949	0.379949
0.6	0.671987	0.597393	0.535867	0.560021	0.536619	0.537049	0.537049
0.8	0.613306	0.660412	0.661706	0.601888	0.614322	0.664037	0.664037
1.0	0.558557	0.718260	0.746032	0.640024	0.680510	0.761594	0.761594

The numerical solutions of Example 6.6 is determined for $v = 0.5, 0.75, 1$. Table 6 shows the comparison between the numerical results under Caputo fractional derivative solved in [27] and the values obtained under Caputo-Fabrizio fractional derivative. It can be seen from Table 6 that the approximate solutions are in well agreement with the exact solutions, when $v = 1$. For all other values of v, the solution depends on the definition of fractional derivative used in the modeling of the differential equation.
The numerical solutions under the Caputo-Fabrizio derivative for different values of v are plotted in Figure 9(a) and for $v = 1$, the comparison of numerical solutions considering different fractional derivatives with the exact solution is shown in Figure 9(b).

FIGURE 9. Numerical results of Example 6.6 for:**(a)** different values of v **(b)** different fractional derivatives for $v = 1$



7. Conclusion

In this paper, a numerical solution by an operational matrix, combined with the collocation method, has been developed for Riccati differential equations involving the Caputo-Fabrizio derivative. The solutions have been compared by changing different parameters such as α, m, and v. Besides that, the numerical results are also compared with other approximation methods to show the accuracy and efficiency of the proposed method.

References

[1] L. Debnath, Recent applications of fractional calculus to science and engineering, International Journal of Mathematics and Mathematical Sciences, Vol. 2003, No. 54, 3413-3442, 2003.

[2] A. Kilbas, H. Srivastava and J. Trujillo, Theory and applications of fractional differential equations, Vol. 204, elsevier, 2006.

[3] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. App., Vol. l, No. 2, 1-13, 2015.

[4] A. Atangana and J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel, Advances in Mechanical Engineering, Vol. 7, No. 10, 2015.

[5] J. Gomez-Aguilar, M. Lopez-Lopez, V. Alvarado-Martinez, J. Reyes-Reyes and M. Adam-Medina, Modeling diffusive transport with a fractional derivative without singular kernel, Physica A: Statistical Mechanics and its Applications, Vol. 447, 467-481, 2016.

[6] J. Gomez-Aguilar, H. Yépez-Martinez, C. Calderon-Ramon, I. Cruz-Orduna, R. Escobar-Jimenez and V. Olivares-Peregrino, Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel, Entropy, Vol. 17, No. 9, 6289-6303, 2015.

[7] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, arXiv preprint arXiv:1602.03408, 2016.

[8] N. Al-Salti, E .Karimov and K. Sadarangani, On a differential equation with Caputo-Fabrizio fractional derivative of order $1 < \beta \leq 2$ and application to mass-spring-damper system, arXiv preprint arXiv:1605.07381, 2016.

[9] A. Atangana, On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation, Applied Mathematics and Computation, Vol. 273, 948-956, 2016.

[10] E. Goufo, Application of the Caputo-Fabrizio Fractional Derivative without Singular Kernel to Korteweg-de Vries-Burgers Equation∗, Mathematical Modelling and Analysis, Vol. 21, No. 2, 188-198, 2016.

[11] A. Atangana and I. Koca, On the new fractional derivative and application to nonlinear Baggs and Freedman model, J. Nonlinear Sci. Appl, Vol. 9, No.5, 2467-2480, 2016.

[12] N. Raza and M. Ullah, A comparative study of heat transfer analysis of fractional Maxwell fluid by using Caputo and Caputo–Fabrizio derivatives, Canadian Journal of Physics, Vol. 98, No. 1, 89-101, 2020.
[13] M. Caputo and M. Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, Progr. Fract. Differ. Appl, Vol. 2, No. 2, 1-11, 2016.
[14] M. A. Imran, S. Sarwar, M. Abdullah and I. Khan, An analysis of the semi-analytic solutions of a viscous fluid with old and new definitions of fractional derivatives, Chinese journal of physics, vol. 56, No. 5, 1853-1871, 2018.
[15] B. Alkahtani and A. Atangana, Controlling the wave movement on the surface of shallow water with the Caputo–Fabrizio derivative with fractional order, Chaos, Solitons & Fractals, Vol. 89, 539-546, 2016.
[16] A. Atangana and R. Alqahtani, Numerical approximation of the space-time Caputo-Fabrizio fractional derivative and application to groundwater pollution equation, Advances in Difference Equations, Vol. 2016, No. 1, 1-13, 2016.
[17] J. Loh, A. Isah, C. Phang and Y. Toh, On the new properties of Caputo–Fabrizio operator and its application in deriving shifted Legendre operational matrix, Applied Numerical Mathematics, Vol. 132, 138-153, 2018.
[18] S. Roshan, H. Jafari and D. Baleanu, Solving FDEs with Caputo-Fabrizio derivative by operational matrix based on Genocchi polynomials, Mathematical Methods in the Applied Sciences, Vol. 41, 2018.
[19] S. Kumar, J. Gómez Aguilar and P. Pandey, Numerical solutions for the reaction–diffusion, diffusion-wave, and Cattaneo equations using a new operational matrix for the Caputo–Fabrizio derivative, Mathematical Methods in the Applied Sciences, Vol. 43, No. 15, 8595-8607, 2020.
[20] K. Krishnarajulu, R. Sevugan and G. Sivaramakrishnan, A new approach to space fractional differential equations based on fractional order Euler polynomials, Publications de l'Institut Mathematique, Vol. 104, No. 118, 157-168, 2018.
[21] Y. Wang, L. Zhu and Z. Wang, Fractional-order Euler functions for solving fractional integro-differential equations with weakly singular kernel, Advances in Difference Equations, Vol. 2018, No. 1, 1-13, 2018.
[22] G. S. Cheon, A note on the Bernoulli and Euler polynomials, Applied Mathematics Letters, Vol. 16, No. 3, 365-368, 2003.
[23] B. F. Kazemi and H. Jafari, Error estimate of the MQ-RBF collocation method for fractional differential equations with Caputo-Fabrizio derivative, Mathematical Sciences, Vol. 11, No. 4, 297-305, 2017.
[24] N. T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, Applied Mathematics and Computation, Vol. 131, 517-529, 2002.
[25] Y. Li, N. Sun, B. Zheng, Q. Wang and Y. Zhang, Wavelet operational matrix method for solving the Riccati differential equation, Communications in Nonlinear Science and Numerical Simulation, Vol. 19, No. 3, 483-493, 2014.
[26] Y. Ozturk and M. Gulsu, Numerical solution of Riccati equation using operational matrix method with Chebyshev polynomials, Asian-European Journal of Mathematics, Vol. 8, No. 2, 1550020, 2015.
[27] Z. Odibat and S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos, Solitons & Fractals, Vol. 36, No. 1, 167-174, 2008.

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