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ON ASYMPTOTICS OF SOME CONFORMABLE DIFFERENTIAL EQUATIONS

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Abstract. Two of the main concerns of researchers when considering qualitative properties of differential equations are stability and boundedness of solutions. One of the early attractive types of derivatives on this equations is the one involving conformable derivatives. In the present paper we discuss asymptotic properties for two classes of sequential conformable differential equations, namely, third order conformable differential equations. Based on reduction of order of derivation and on properties of Lyapunov functions for conformable fractional differential equations besides Gronwall inequality, we prove stability and boundedness of solutions. As an illustration of the results, a numerical example is provided.

1. Introduction

During the last decades, pure and applied branches of science and engineering are seeing an important impact of fractional calculus. Physical, Biological and other fields of the contemporary science are mathematically governed by differential equations. Over some of the last years, a specific type of differential equations has shown to be of great interest among the researchers community, namely, fractional-order differential equations [1, 2, 3, 4], this latter ones have gained considerably more attention due to their properties and their applications in this disciplines [11, 12, 15].

We turn our attention now to the main interest of this paper. We consider the following two third order differential equations with conformable derivative

$$T_\alpha T_\alpha T_\alpha x + \kappa_1 T_\alpha T_\alpha x + \kappa_2 T_\alpha x + f(x) = 0, \tag{1}$$

and

$$T_\alpha T_\alpha T_\alpha x + \kappa_1 T_\alpha T_\alpha x + \kappa_2 T_\alpha x + f(x) = p(t), \tag{2}$$

where κ_1, κ_2 are positive constants and $f(x), p(t)$ are continuous functions with $f(0) = 0$. The T_α here stands for the conformable derivative of order α.

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By a solution of (1) ((2) respectively), we mean a continuous function $x : [t_0, \infty) \to \mathbb{R}$ such that $x \in C^3_\alpha([t_0, \infty), \mathbb{R})$ and which satisfies (1) ((2) respectively) on $[t_0, \infty)$.

The reminder of this paper is divided into four sections. In the first one, we introduce some preliminary notions concerning conformable fractional calculus. Stability theory related to conformable differential equations is highlighted in the second section. In the third one, we give our main results. And, in the last one, we illustrate the findings throughout a numerical example.

2. Preliminaries

In this section, we will recall definitions and properties of conformable fractional calculus.

Definition 1 ([1]). *Let $a \in \mathbb{R}$. For a function $f : [a, \infty) \to \mathbb{R}$, the conformable fractional derivative T^a_α of f of order $\alpha \in (0, 1]$ is given by*

$$T^a_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}, \text{ for all } t > a. \tag{3}$$

If $a = 0$ then equation (3) becomes

$$T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{4}$$

If a given function f satisfies Definition 1 for all $t > a$, then f is called an α-differentiable function.

Lemma 1 ([2]). *Let $f, g : [a, \infty) \to \mathbb{R}$ be α-differentiable functions at a point $t > a$, then*

1. $T^a_\alpha(fg) = fT^a_\alpha(g) + gT^a_\alpha(f)$;
2. $T^a_\alpha(\lambda) = 0$, *for all constant functions $f(t) = \lambda$;*
3. $T^a_\alpha\left(\dfrac{f}{g}\right) = \dfrac{gT^a_\alpha(f) - fT^a_\alpha(g)}{g^2}$, *where $g \neq 0$ for all $t > a$;*
4. *If f is differentiable, then*

$$T^a_\alpha(f)(t) = (t-a)^{1-\alpha} \frac{df(t)}{dt}.$$

Remark 1. *The α-derivative is a linear operator, that is,*

$$T^a_\alpha(\lambda f + \beta g) = \lambda T^a_\alpha(f) + \beta T^a_\alpha(g).$$

Definition 2 ([2]). *The α-fractional integral for a function f , is defined as*

$$I^a_\alpha(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual improper Riemann integral and $\alpha \in (0, 1)$.

Lemma 2 ([1]). *Assume that $f : [a, \infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$T^a_\alpha I^a_\alpha(f)(t) = f(t),$$

and

$$I^a_\alpha T^a_\alpha(f)(t) = f(t) - f(a).$$

Proposition 1 ([1])**.** *Let $f : [a, \infty) \to \infty$ be twice differentiable on (a, ∞) and $0 < \alpha, \beta \leq 1$ such that $1 < \alpha + \beta \leq 2$. Then*

$$(T_\alpha^a T_\beta^a f)(t) = T_{\alpha+\beta}^a f(t) + (1 - \beta)(t - a)^{-\beta} T_\alpha^a f(t). \tag{5}$$

Remark 2. *Note that in equation (5) if $\alpha, \beta \to 1$, then we have*

$$(T_\alpha^a T_\beta^a f)(t) = T_2^a f(t) = f''(t).$$

Lemma 3 ([4])**.** *Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function such that $T_\alpha^a f(t)$ exists on (a, ∞), if $T_\alpha^a f(t) \geq 0$ (respectively $T_\alpha^a f(t) \leq 0$), for all $t \in (a, \infty)$, then the graph of f is increasing (respectively decreasing).*

Remark 3 ([1])**.** *Let $f : [a, \infty) \to \mathbb{R}$ such that $T_\alpha^a f(t)$ exists on (a, ∞). Then $T_\alpha^a f^2(t)$ exists on (a, ∞) and*

$$T_\alpha^a f^2(t) = 2f(t) T_\alpha^a f(t), \quad \forall t > a.$$

Lemma 4 ([1])**.** *Assume $f, g : (a, \infty) \to \mathbb{R}$ be α-differentiable functions, where $0 < \alpha \leq 1$. Let $h(t) = f(g(t))$. Then $h(t)$ is α-differentiable and for all t with $t \neq a$ and $g(t) \neq 0$ we have*

$$T_\alpha^a h(t) = (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}.$$

The following theorem states the fractional version of Gronwall inequality which is useful in studying stability of (conformable) fractional systems.

Theorem 1 ([1])**.** *Let r be a continuous, nonnegative function on an interval $J = [a, b]$ and δ and k be nonnegative constants such that*

$$r(t) \leq \delta + \int_a^t k r(s)(s - a)^{\alpha-1} ds \, (t \in J).$$

Then, for all $t \in J$

$$r(t) \leq \delta e^{k \frac{(t-a)^\alpha}{\alpha}}.$$

A generalization of this theorem was proved in [3] and is stated as follows

Lemma 5 ([3])**.** *Let f and g be continuous, nonnegative functions on $[a, b]$ and λ a nonnegative constant such that*

$$f(t) \leq \lambda + I_\alpha^a (fg)(t), \quad \text{for } t \in [a, b],$$

then

$$f(t) \leq \lambda e^{I_\alpha^a (g)(t)}, \quad \text{for } t \in [a, b].$$

Remark 4. *From now on, we consider $a = t_0 \geq 0$ and instead of T_α^a (respectively I_α^a) we use $T_\alpha^{t_0}$ (respectively $I_\alpha^{t_0}$).*

3. Stability Theory

In this section, we recall stability theory in the sense of Lyapunov for systems of conformable fractional differential equations having the form

$$T_\alpha^{t_0} x(t) = f(t, x), \, t \geq t_0, \tag{6}$$

$$x(t_0) = x_0. \tag{7}$$

where $x \in \mathbb{R}^n$, $0 < \alpha \leq 1$ and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying $f(t, 0) = 0$, for every $t \geq t_0$.

By a solution of (6)-(7), we mean a continuous function $x(t;t_0,x_0) \in C_\alpha([t_0,\infty),\mathbb{R}^n)$ satisfying (6)-(7) on $[t_0,\infty)$.

Definition 3 ([1]). *The fractional conformable exponential function is defined for every $s \geq 0$ by*

$$E_\alpha(\lambda, s) = \exp\left(\lambda \frac{s^\alpha}{\alpha}\right),$$

where $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$.

Definition 4 (Fractional Exponential Stability [4]). *The origin of system (6)-(7) is said to be fractional exponentially stable if*

$$\|x(t)\| \leq K \|x_0\| E_\alpha(-\lambda, t-t_0), t \geq t_0,$$

where $\lambda, K > 0$.

Definition 5 ([4]). *The origin of system (6)-(7) is said to be*

(i) *stable, if for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta := \delta(\varepsilon, t_0)$ such that for any $x_0 \in \mathbb{R}^n$, the inequality $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$ for $t \geq t_0$.*

(ii) *attractive, if for any $t_0 \geq 0$ there exists $c := c(t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$, the inequality $\|x_0\| < c$ implies $\lim_{t\to\infty} x(t) = 0$.*

(iii) *globally attractive, if for any initial condition $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} x(t) = 0$.*

(iv) *asymptotically stable, if it is stable and attractive.*

(v) *globally asymptotically stable, if it is stable and globally attractive.*

Definition 6 ([4]). *A continuous function $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is to belong to class $\mathcal{K}_\infty$ if $\lim_{s\to+\infty} \alpha(s) = +\infty$.*

Lemma 6 ([4]). *Let $0 < \alpha < 1$ and $g : [t_0,\infty) \to \mathbb{R}^+$ be a continuous function and α-differentiable on (t_0,∞), such that*

$$T_\alpha^{t_0} g(t) \leq -\lambda g(t),$$

where λ is a positive constant. Then

$$g(t) \leq E_\alpha(-\lambda, t-t_0)\, g(t_0).$$

Theorem 2 ([4]). *Let $x = 0$ be an equilibrium point for system (6)-(7). Assume that there exist a continuous function $V(t,x)$ and class $\mathcal{K}$ function α satisfying*

(A1) $\alpha(\|x\|) \leq V(t,x), V(t,0) = 0,$

(A2) *$V(t,x(t))$ has a fractional derivative of order α for all $t > t_0 \geq 0$,*

(A3) $T_\alpha^{t_0} V(t,x(t)) \leq 0.$

Then the origin of system (6)-(7) is stable.

Theorem 3 ([4]). *Let $x = 0$ be an equilibrium point for system (6)-(7). Assume that there exist a continuous function $V(t,x)$ and class $\mathcal{K}$ functions $\alpha_i, (i = 1,2,3)$ satisfying*

(J1) $\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|),$

(J2) *$V(t,x(t))$ has a fractional derivative of order α for all $t > t_0 \geq 0$,*

(J3) $T_\alpha^{t_0} V(t,x(t)) \leq -\alpha_3(\|x\|).$

Then the origin of system (6)-(7) is asymptotically stable. Moreover, if $\alpha_i \in \mathcal{K}_\infty, (i = 1,2,3)$, then the origin of system (6)-(7) is globally asymptotically stable.

Theorem 4 ([4]). *Let $x = 0$ be an equilibrium point for system (6)-(7) and let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be continuous. Suppose that a, b, c_1, c_2, c_3 are arbitrary positive constants. If the following conditions are satisfied*

(H1) $c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$,
(H2) $V(t,x(t))$ *has a fractional derivative of order* α *for all* $t > t_0 \geq 0$,
(H3) $T_\alpha^{t_0} V(t,x(t)) \leq -c_3\|x\|^2$.

then the origin of system (6)-(7) is fractional exponentially stable.

4. Main results

Before proceeding further, we start by making the following change of variables :

$$y = T_\alpha x, \quad z = T_\alpha T_\alpha x. \tag{8}$$

Therefor, we can write

$$\begin{cases} T_\alpha x = y, \\ T_\alpha y = z, \\ T_\alpha z = -\kappa_1 z - \kappa_2 z - f(x) + p(t), \end{cases}$$

with $p(t) = 0$ for equation (1) and $p(t) \neq 0$ for equation (2).

4.1. STABILITY RESULT.

Our interest in this subsection is to give stability results for equation (1). We start by writing it as the following equivalent system :

$$\begin{cases} T_\alpha x = y, \\ T_\alpha y = z, \\ T_\alpha z = -\kappa_1 z - \kappa_2 y - f(x). \end{cases} \tag{9}$$

Before proceeding further, we make some assumptions. Suppose there exist non-negative constants f_0, f_1 and μ, such that the following conditions are satisfied :

i) $\dfrac{f(x)}{x} \geq f_0 > 0$, for $x \neq 0$, and $|T_\alpha(f)(x)| \leq f_1$, for all x.
ii) $\kappa_1\kappa_2 > f_0$.
iii) $\mu\kappa_2 > 2f_1$.

We are now ready to state the main result of this part.

Theorem 5. *Suppose conditions (i)-(iii) being satisfied, then the zero solution of (9) is stable.*

Proof. For the sake of brevity, let

$$F(x) = I_\alpha^0(f)(x) = I_\alpha(f)(x), \tag{10}$$

and

$$\mu = \frac{\kappa_1\kappa_2 + f_0}{4\kappa_2}. \tag{11}$$

Recall that

$$T_\alpha(f^2)(t) = 2fT_\alpha(f)(t) \implies I_\alpha(T_\alpha(f^2))(t) = f^2(t) - f^2(0) = f^2(t) = 2I_\alpha(fT_\alpha(f))(t).$$

Define a Lyapunov function by

$$V(t,x,y,z) = V_1(t) + V_2(t), \tag{12}$$

with

$$V_1(t) = \mu F(x) + f(x)y + \frac{\kappa_2}{2}y^2, \tag{13}$$

and

$$V_2(t) = \frac{\mu\kappa_1}{2}y^2 + \mu yz + \frac{1}{2}z^2. \tag{14}$$

The first step is to show that V defined by (12) is positive definite. It is easy to see that $V(t,0) = 0$. It can be easily seen that equation (13) is equivalent to

$$\begin{aligned} V_1(t) &= \mu I_\alpha(f)(x) + \frac{1}{\kappa_2}\left(f(x) + \frac{\kappa_2}{2}y\right)^2 - \frac{1}{\kappa_2}f^2(x) + \frac{\kappa_2}{4}y^2 \\ &\geq \mu I_\alpha(f)(x) - \frac{2}{\kappa_2}I_\alpha(fT_\alpha f)(x) + \frac{\kappa_2}{4}y^2 \\ &\geq \left(\mu - \frac{2f_1}{\kappa_2}\right)I_\alpha(f)(x) + \frac{\kappa_2}{4}y^2. \end{aligned} \tag{15}$$

Observe from condition (i), the following estimate

$$F(x) \geq \frac{1}{2f_1}f^2(x) \geq \frac{f_0^2}{2f_1}x^2. \tag{16}$$

Consequently, in view of condition (iii), there exists a positive constant κ_3 such that

$$V_1(t) \geq \kappa_3(x^2 + y^2). \tag{17}$$

In the same fashion, equation (14) can be written as

$$\begin{aligned} V_2(t) &= \frac{1}{4}z^2 + \frac{1}{8}(z + 2\mu y)^2 + \frac{1}{4}\mu\kappa_1\left(\frac{1}{\kappa_1}z + y\right)^2 + (\kappa_1 - 2\mu)\left(\frac{1}{4}\mu y^2 + \frac{1}{8\kappa_1}z^2\right) \\ &\geq \frac{1}{4}z^2 + (\kappa_1 - 2\mu)\left(\frac{1}{4}\mu y^2 + \frac{1}{8\kappa_1}z^2\right). \end{aligned} \tag{18}$$

From the definition (11) of μ besides condition (ii) of Theorem 5, we have

$$\kappa_1 - 2\mu = \frac{\kappa_1\kappa_2 - f_0}{2\kappa_2} > 0,$$

hence, there exists a positive constant κ_4, such that

$$V_2(t) \geq \kappa_4(y^2 + z^2). \tag{19}$$

It is clear that

$$\bar{V}(x,y,z) = k_0(x^2 + y^2 + z^2) = 0 \quad \text{iff} \quad x = y = z = 0, \tag{20}$$

and

$$V(t,x,y,z) \geq k_0(x^2 + y^2 + z^2) = \bar{V}(x,y,z) > 0 \quad \text{if} \quad (x,y,z) \neq (0,0,0). \tag{21}$$

Therefor, we conclude that

$$V \geq k_0(x^2 + y^2 + z^2), \tag{22}$$

where $k_0 = \dfrac{1}{2}\min\left\{\dfrac{f_0^2}{f_1}, \dfrac{(\kappa_1 - 2\mu)}{2}\left(\mu + \dfrac{1}{2\kappa_1}\right)\right\}$.
This shows that V is positive definite.

The next step is to show that the α-derivative (conformable derivative) of (12) is negative definite. Differentiating (12) along trajectories of (9) is equivalent to

$$T_\alpha V_{(9)}(t) = T_\alpha V_{1_{(9)}}(t) + T_\alpha V_{2_{(9)}}(t). \tag{23}$$

Recall from Lemma 1, Lemma 4 and Remark 3 that

$$T_\alpha^a(gh)(t) = hT_\alpha^a(g)(t) + gT_\alpha^a(h)(t),$$
$$T_\alpha^a h^2(t) = 2h(t)T_\alpha^a h(t),$$

and

$$(h(t) = f(g(y))), \qquad T_\alpha^a h(t) = (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}.$$

for sufficiently good functions f, g and h with $0 < \alpha \leq 1$.
Straight forward computations give

$$T_\alpha V_{1_{(9)}}(t) = \mu f(x)y + y^2 T_\alpha(f)(x) + f(x)z + \kappa_2 yz. \tag{24}$$

In the same way,

$$T_\alpha V_{2_{(9)}}(t) = \mu\kappa_1 yz + \mu z^2 - \mu\kappa_1 yz - \mu\kappa_2 y^2 - \mu f(x)y - \kappa_1 z^2 - \kappa_2 yz - f(x)z. \tag{25}$$

Summing up

$$\begin{aligned} T_\alpha\left(V_{1_{(9)}} + V_{2_{(9)}}\right)(t) &= y^2 T_\alpha(f)(x) + \mu z^2 - \mu\kappa_2 y^2 - \kappa_1 z^2 \\ &= -(\kappa_1 - \mu)z^2 + \left(T_\alpha(f)(x) - \mu\kappa_2\right) y^2. \end{aligned} \tag{26}$$

In virtue of condition (iii), there exists a positive constant k_1 such that

$$T_\alpha V_{(9)}(t) = T_\alpha V_{1_{(9)}}(t) + T_\alpha V_{2_{(9)}}(t) \leq -k_1(y^2 + z^2), \tag{27}$$

where $k_1 = \max\{\kappa_1 - \mu, \mu\kappa_2 - f_1\}$.
In view of Theorem 3, the proof is complete. □

Remark 5. *We could have drawn the fact that the zero solution of (1) is exponential stable. This can be seen by applying Lemma 6 to equation (27); this will lead to the desired result.*

4.2. **BOUNDEDNESS RESULT.** We turn our focus in this subsection to boundedness results for equation (2). We start by writing it as the following equivalent system :

$$\begin{cases} T_\alpha x = y, \\ T_\alpha y = z, \\ T_\alpha z = -\kappa_1 z - \kappa_2 y - f(x) + p(t). \end{cases} \tag{28}$$

The main result of this part is as follows

Lemma 7. *Besides conditions of Theorem 5 being satisfied, suppose there exists a positive constant p_1 such that $I_\alpha(|p|)(t) < p_1$, hold. Then, every solution of (28) satisfies*

$$|x(t)| \leq N, \quad |y(t)| \leq N, \quad \textit{and} \quad |z(t)| \leq N, \tag{29}$$

where N is a positive constant.

Proof. Over each solution $(x(t), y(t), z(t))$ of (28), we have

$$T_\alpha\left(V_{(28)}\right)(t) = T_\alpha\left(V_{(28)}\right)(t) + (\mu y + z)p(t).$$

From (27), we get

$$T_\alpha\left(V_{(28)}\right)(t) \leq N_1(|y| + |z|)|p(t)|,$$

where $N_1 = \max\{1, \mu\}$.
In view of inequality (22) together with the fact $|u| \leq u^2 + 1$, we obtain

$$\begin{aligned} T_\alpha \left(V_{(28)}\right)(t) &\leq N_1(y^2 + z^2 + 2)|p(t)| \\ &\leq N_2|p(t)|V(t) + N_2|p(t)|, \end{aligned} \tag{30}$$

where $N_2 = N_1 \max\left\{2, \dfrac{1}{k_0}\right\}$.
α-Integrate from t_1 to t to arrive at

$$V(t) - V(t_1) \leq N_2 I_\alpha(|p|V)(t) + N_2 I_\alpha(|p|)(t).$$

Thus

$$V(t) \leq V(t_1) + p_1 N_2 + N_2 I_\alpha(|p|V)(t).$$

Now, Gronwall inequality (see Lemma 5) leads to

$$V(t) \leq (V(t_1) + p_1 N_2) \exp\left(N_2 I_\alpha(|p|)(t)\right) \leq N, \tag{31}$$

where $N = (V(t_1) + p_1 N_2) \exp(p_1 N_2)$, and p_1 is as defined in Lemma 7. This last inequality implies

$$|x(t)| \leq N, \quad |y(t)| \leq N, \quad \text{and} \quad |z(t)| \leq N, \tag{32}$$

therefore, the proof is complete. □

5. Example

In this section an example is dealt with to illustrate our findings. Start by remark that if f is differentiable, then

$$T_\alpha^a(f)(t) = (t-a)^{(1-\alpha)} f'(t).$$

Therefore, we have the following equivalence (with $a = 0$):

$$\begin{cases} T_\alpha x = y, \\ T_\alpha y = z, \\ T_\alpha z = -\kappa_1 z - \kappa_2 y - f(x) + p(t). \end{cases} \iff \begin{cases} t^{1-\alpha} x = y, \\ t^{1-\alpha} y = z, \\ t^{1-\alpha} z = -\kappa_1 z - \kappa_2 y - f(x) + p(t). \end{cases}$$

This way, numerical simulations are easily done. Moreover, we will give the simulations for multiple values of the order of derivation so we can show that not only the results are valid for one value but for multiple ones. Let us consider now our simple example.
Consider equation

$$T_\alpha T_\alpha T_\alpha x + 2T_\alpha T_\alpha x + 2.5T_\alpha x + \left(0.5x + \frac{x}{10 + |x|}\right) = \frac{1}{1+t^2}. \tag{33}$$

An observation of (33) shows :

$$\begin{aligned} \kappa_1 &= 2, & \kappa_2 &= 2.5, \\ f_0 &= 0.5, & f_1 &= 0.6, \\ f(x) &= \left(0.5x + \frac{x}{10+|x|}\right), & p(t) &= \frac{1}{1+t^2}. \end{aligned}$$

The conditions are :

$$\kappa_1\kappa_2 = 5 > 0.5 = f_0, \qquad \mu\kappa_2 = 0.55 > 1.2 = 2f_1.$$

All conditions of Theorem 5 (Lemma 5 respectively) are satisfied, hence, stability (Boundedness) of the solutions.

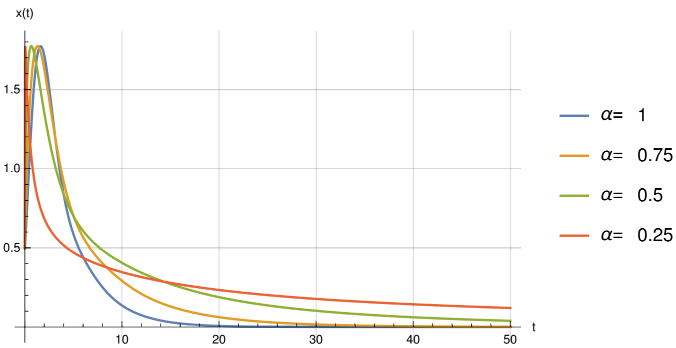

Figure 1. Numerical simulation of $x(t)$.

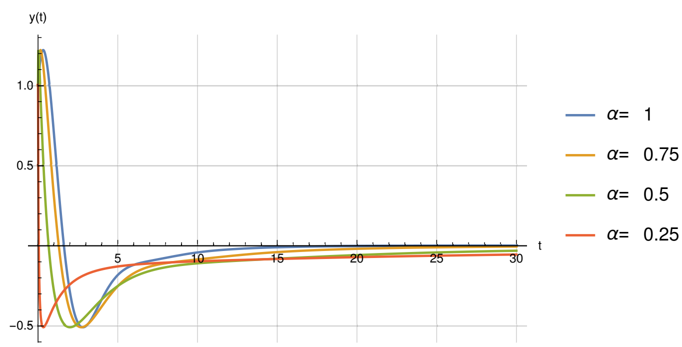

Figure 2. Numerical simulation of $y(t)$.

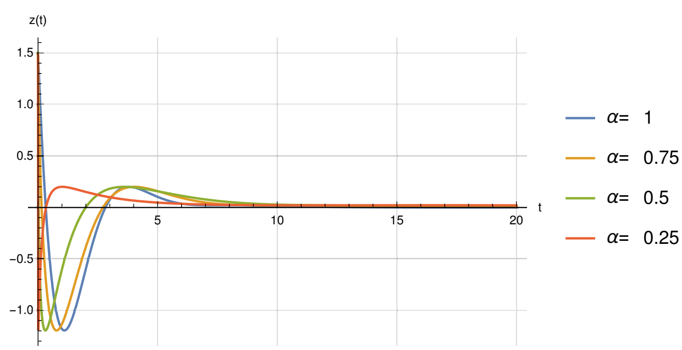

Figure 3. Numerical simulation of $z(t)$.

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