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EXISTENCE RESULT FOR FRACTIONAL DIFFERENTIAL EVOLUTION EQUATION IN A HILBERT SPACE

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Abstract. In this paper we establish the existence, uniqueness and some regularity results for the solution of differential evolution equations of fractional order involving an unbounded linear maximal monotone operator in a Hilbert space. The proof relies on spectral properties of the operator. An explicit representation formula of the solution is given, allowing the computation of a sequence of approximate solutions.

1. Introduction

Fractional differential equations have a large application in a variety of fields such as physics, mathematics, electrical networks, signal and image processing, aerodynamics, economics and do so on. Hence there has been increased attention from both theoretical and the applied points, for more details see [2, 3, 4, 6, 7, 8].

The aim of this work is to prove the existence and uniqueness of a solution, as well as some regularity and approximation results, for fractional evolution equations in a Hilbert space. More precisely we consider the problem

$$ {}^cD_t^\alpha u(t) + Au(t) = f(t), \qquad t \in [0,T], \quad T > 0 \tag{1}$$

with the initial condition

$$u(0) = u_0 \tag{2}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α, A is an unbounded linear maximal monotone operator in a Hilbert space H, and $f \in L^2(0,T;H)$ is a given data. The main result relies on an appropriate decomposition of u, based on spectral properties of the operator A, leading to an explicit representation of u as

$$u(t) = \sum_{k\geq 1} v_k(t)\omega_k, \qquad t \in [0,T]$$

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where the the functions v_k are derived by solving scalar fractional differential equations and $(\omega_k)_{k\geq 1}$ is a family of eigenvectors of A.

The paper is organized as follows. In section 2 we recall some preliminaries about fractional calculus and maximal monotone operators. Then in section 3 we state our main result and give some comments, and finally section 4 is devoted to conclude with an example to illustrate the feasibility of our main result.

2. Background and basic results

In this section, we introduce notations, definitions and theorems which are used in the rest of the paper.

Let H be a real Hilbert space. We denote by $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$ its canonical inner product and norm and we let $\mathcal{L}(H)$ be the space of all bounded linear operators on H.

Definition 2.1. Let $A : D(A) \subset H \to H$ be a unbounded linear operator. The operator A is *monotone* if:

$$\big\langle Au, u\big\rangle_H \geq 0 \quad \text{ for all } \; u \in D(A).$$

The operator A is *maximal monotone*, if in addition $R(I + A) = H$, i.e. for all $f \in H$ there exists $u \in D(A)$ such that $u + Au = f$.

Definition 2.2. Let $A : D(A) \subset H \to H$ be an unbounded linear operator such that $\overline{D(A)} = H$. By identifying the space H and its dual H', the adjoint $A^* : D(A^*) \subset H \to H$ is defined by the following requirements:

- $u \in D(A^*)$ if and only if $u \in H$ and there exists $g \in H$ such that
$$\big\langle u, Av\big\rangle_H = \langle g, v\rangle_H \qquad \text{for all } v \in D(A)$$
- for $u \in D(A^*)$, set $A^*u = g$.

Definition 2.3. Let $A : D(A) \subset H \to H$ be a unbounded linear operator such that $\overline{D(A)} = H$. The operator A is symmetric if

$$\big\langle Au, v\big\rangle_H = \big\langle u, Av\big\rangle_H, \qquad \text{for all } (u, v) \in D(A) \times D(A).$$

The operator A is self-adjoint if and only if $A = A^*$.

Theorem 2.4. [9, 10] Let $A : D(A) \subset H \to H$ be an unbounded linear maximal monotone operator of the Hilbert space H. Then

- $D(A)$ is dense in H,
- A is a closed operator,
- for all $\lambda > 0$, $(I + \lambda A)$ is one-to-one from $D(A)$ into H, $(I + \lambda A)^{-1}$ is a bounded operator and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$,
- A is symmetric if and only if A is self-adjoint.

Let us introduce now some basic definitions and properties of the fractional calculus theory. For a more detailed presentation the reader is referred to [1, 5, 14] and the references therein.

Definition 2.5. The Euler-gamma function Γ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt = \int_0^\infty e^{(z-1)\ln(t)} e^{-t}\, dt.$$

This integral is convergent for all $z \in \mathbf{C}$ such that $Re(z) > 0$.

Proposition 2.6. [15] The Euler-gamma function satisfies the following reduction formula

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{for all } z \in \mathbf{C} \text{ such that } Re(z) > 0.$$

In particular, if $z = n \in \mathbf{N}_0$ then

$$\Gamma(n+1) = n! \qquad \text{for all } n \in \mathbf{N}_0$$

with (as usual) $0! = 1$.

Definition 2.7. Let $\alpha > 0$ and $f : \mathbf{R}_+ \to H$ be in $L^1(\mathbf{R}_+, H)$. Then the Riemann-Liouville integral $I_t^\alpha f$ is given by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \qquad t > 0.$$

Definition 2.8. The Caputo derivative of order α of a function $f : \mathbf{R}_+ \to H$ can be written as

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0,\ n-1 \le \alpha < n.$$

If $0 < \alpha < 1$, then

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

Obviously, the Caputo derivative of a constant is equal to zero.

Definition 2.9. The Mittag-Leffler function $E_{\alpha,\beta}$, with $\alpha > 0$ and $\beta \in \mathbf{R}$, is given by

$$E_{\alpha,\beta}(z) = \sum_{k \ge 0} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbf{C}.$$

The Mittag-Leffler functions are entire functions and we have

Theorem 2.10. [12] Let $0 < \alpha < 2$, β be an arbitrary real number and μ be such that $\frac{\pi\alpha}{2} < \mu < \min(\pi, \pi\alpha)$. Then there exists $C > 0$ such that

$$\left|E_{\alpha,\beta}(z)\right| \le \frac{C}{1+|z|}$$

for all $z \in \mathbf{C}$ such that $\mu \le |\arg(z)| \le \pi$.

Finally let us recall some important properties of Laplace transform with respect to fractional calculus.

Proposition 2.11. [11] The following properties hold:

$$\mathcal{L}\left({}^cD^\alpha y(t)\right)(s) = s^\alpha \mathcal{L}\big(y(t)\big)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} D^k y(0), \qquad n-1 < \alpha \leq n, \quad n \in \mathbf{N}.$$

In particular if $0 < \alpha \leq 1$ then:

$$\mathcal{L}\left({}^cD^\alpha y(t)\right)(s) = s^\alpha \mathcal{L}\big(y(t)\big)(s) - s^{\alpha-1} y(0).$$

Moreover

$$\mathcal{L}\left(t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha)\right)(s) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \qquad \alpha > 0, \quad (\beta, \lambda) \in \mathbf{R}^2.$$

3. Existence of the solutions

Let H and V be two real Hilbert spaces of infinite dimension such that $V \subset H$ and $A : D(A) \subset H \to H$ be an unbounded linear self-adjoint maximal monotone operator of H.

We assume that

(A1) the domain of the operator A is included into V and the injection of V into H is continuous and compact,

(A2) there exists a bilinear symmetric continuous and coercive form $a : V \times V \to \mathbf{R}$ such that

$$a(u, v) = \langle Au, v\rangle_H \qquad \text{for all } u \in D(A) \text{ and } v \in V.$$

With assumption (A1) we obtain that the triplet (V, H, V') is a Gelfand triplet

and with assumption (A2) we infer that a defines an inner product on V and the corresponding norm $\|\cdot\|_{V,a}$, given by

$$\|u\|_{V,a} = \sqrt{a(u,u)} \qquad \text{for all } u \in V$$

is equivalent to the canonical norm of V. Moreover there exists an increasing sequence of positive real numbers $(\lambda_k)_{k\geq 1}$ and a Hilbertian basis $(\omega_k)_{k\geq 1}$ of H such that

$$a(\omega_k, v) = \lambda_k \big\langle \omega_k, v\big\rangle_H \qquad \text{for all } v \in V, \text{ for all } k \in \mathbf{N}_0$$

and $\lim_{k\to+\infty} \lambda_k = +\infty$. Furthermore the sequence $(\lambda_k^{-1/2}\omega_k)_{k\geq 1}$ is a Hilbertian basis of V endowed with the inner product $a(\cdot,\cdot)$. By using classical properties of

Hilbertian bases we obtain that, for any $u \in H$, the sequences $\Big(\sum_{k=1}^{m}\big\langle u,\omega_k\big\rangle_H\omega_k\Big)_{m\geq 1}$ and $\Big(\sum_{k=1}^{m}\big\langle u,\omega_k\big\rangle_H^2\Big)_{m\geq 1}$ are convergent in H and $\mathbf{R}$ respectively and we have

$$u=\sum_{k\geq 1}\big\langle u,\omega_k\big\rangle_H\omega_k \quad \text{and} \quad \|u\|_H^2=\sum_{k\geq 1}\big\langle u,\omega_k\big\rangle_H^2 \qquad \text{for all } u\in H. \tag{3}$$

Similarly, for any $u \in V$, we have

$$u=\sum_{k\geq 1}\frac{1}{\lambda_k}a(u,\omega_k)\omega_k=\sum_{k\geq 1}\big\langle u,\omega_k\big\rangle_H\omega_k \quad \text{and} \quad \|u\|_{V,a}^2=\sum_{k\geq 1}\lambda_k\big\langle u,\omega_k\big\rangle_H^2.$$

Remark 3.1. Let us observe that, for all $k \in \mathbf{N}_0$, ω_k is an eigenvector of the operator A associated with the eigenvalue λ_k. Indeed, reminding that $\omega_k \in V$ and $D(A) \subset V$, we have

$$\langle\omega_k, Av\rangle_H=\langle Av,\omega_k\rangle_H=a(v,\omega_k)=a(\omega_k,v)=\langle\lambda_k\omega_k,v\rangle_H \quad \text{for all } v\in D(A).$$

Hence there exists $g=\lambda_k\omega_k \in H$ such that

$$\langle\omega_k, Av\rangle_H=\langle g,v\rangle_H \quad \text{for all } v\in D(A)$$

and we obtain that $\omega_k \in D(A^*)=D(A)$ and $A^*\omega_k=g=\lambda_k\omega_k=A\omega_k$.

Moreover the operator A is strongly monotone. Indeed, for all $u \in D(A)$, we have

$$\langle Au,u\rangle_H=a(u,u)\geq\gamma\|u\|_V^2$$

where $\gamma>0$ is the coercivity constant of the bilinear form a on V and, recalling that the injection of V into H is continuous, we infer that

$$\langle Au,u\rangle_H=a(u,u)\geq\gamma'\|u\|_H^2 \qquad \text{for all } u\in D(A)$$

with $\gamma'>0$.

Let us state now our main result.

Theorem 3.2. Let H and V be two real Hilbert spaces of infinite dimension such that $V \subset H$ and $A : D(A) \subset H \to H$ be an unbounded linear self-adjoint maximal monotone operator of H such that assumptions (A1) and (A2) hold. Let $\alpha \in (1/2, 1)$.Then, for any $u_0 \in H$ and $f \in L^2(0,T;H)$ (with $T>0$) the fractional differential problem (1)-(2) admits an unique solution $u \in C^0\big([0,T];H\big)\cap L^2(0,T;V)$. Moreover u is given by the following representation formula

$$\begin{aligned} u(t) \;&=\; \sum_{k\geq 1}E_{\alpha,1}(-\lambda_k t^\alpha)\langle u_0,\omega_k\rangle_H\omega_k \\ &+\; \sum_{k\geq 1}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-s)^\alpha\big)\big\langle f(s),\omega_k\big\rangle_H\omega_k\,ds, \qquad t\in[0,T]. \end{aligned}$$

The next corollary will be crucial to the proof of Theorem 3.2.

Corollary 3.3. Let H and V be two real Hilbert spaces of infinite dimension such that $V \subset H$ and $A : D(A) \subset H \to H$ be an unbounded linear self-adjoint maximal monotone operator of H such that assumptions (A1) and (A2) hold. For all $m \in \mathbf{N}_0$ let us define V^m as the finite dimensional subspace of V generated by the m first eigenvectors of A, i.e. $V^m = \text{Vect}\{\omega_1, \dots, \omega_m\}$. Let $\alpha \in (1/2, 1)$. Then, for any $u_0 \in H$ and $f \in L^2(0, T; H)$ (with $T > 0$), the Galerkin approximation of problem (1)-(2) on V^m admits an unique solution $u_m \in C^0\big([0, T]; V\big)$ given by

$$\begin{aligned} u_m(t) &= \sum_{k=1}^{m} E_{\alpha,1}(-\lambda_k t^\alpha)\langle u_0, \omega_k\rangle_H \omega_k \\ &+ \sum_{k=1}^{m} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big)\big\langle f(s), \omega_k\big\rangle_H \omega_k \, ds, \qquad t \in [0, T] \end{aligned}$$

and the sequence $(u_m)_{m\geq 1}$ converges strongly to the unique solution u of problem (1)-(2) in the Banach spaces $C^0\big([0, T]; H\big)$ and $L^2(0, T; V)$.

Proof. For all $m \in \mathbf{N}_0$, let $u_m \in C^0\big([0, T]; V\big)$ be given by

$$u_m(t) = \sum_{k=1}^{m} v_k(t)\omega_k, \qquad t \in [0, T].$$

Recalling that $(\omega_k)_{k\geq 1}$ is a Hilbertian basis of H we infer that

$$\langle \omega_k, \omega_j\rangle_H = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } (i, j) \in \mathbf{N}_0^2.$$

Thus, for all $m \in \mathbf{N}_0$ and for all $j \in \{1, \dots, m\}$, we have

$$\begin{aligned} &\big\langle {}^c D_t^\alpha u_m(t), \omega_j\big\rangle_H + \big\langle A u_m(t), \omega_j\big\rangle_H \\ &= \sum_{k=1}^{m} {}^c D_t^\alpha v_k(t)\langle \omega_k, \omega_j\rangle_H + \sum_{k=1}^{m} \lambda_k v_k(t)\langle \omega_k, \omega_j\rangle_H \\ &= f_j(t) = \big\langle f(t), \omega_j\big\rangle_H, \qquad t \in [0, T]. \end{aligned}$$

Moreover

$$u_m(0) = \sum_{k=1}^{m} v_k(0)\omega_k = \sum_{k=1}^{m} \langle u_0, \omega_k\rangle_H \omega_k = \mathbf{P}_H(u_0, V^m)$$

where $\mathbf{P}_H(\cdot, V^m)$ is the projection operator on V^m relatively to the inner product of H.

It follows that u_m is the unique solution of the Galerkin approximation of problem (1)-(2) on V^m.

Let us prove that $(u_m)_{m\geq 1}$ is a Cauchy sequence in $L^2(0,T;V)$. Let m and r be two natural numbers such that $r > m$. Then we have

$$
\begin{aligned}
&a\big(u_r(t)-u_m(t),u_r(t)-u_m(t)\big) = \sum_{k=m+1}^{r} \lambda_k \big(v_k(t)\big)^2 \\
&= \sum_{k=m+1}^{r} \lambda_k \left\{ E_{\alpha,1}(-\lambda_k t^\alpha) v_k(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_k (t-s)^\alpha\right) f_k(s)\, ds \right\}^2 \\
&\leq 2 \sum_{k=m+1}^{r} \lambda_k \Big(E_{\alpha,1}(-\lambda_k t^\alpha) v_k(0) \Big)^2 \\
&+2 \sum_{k=m+1}^{r} \lambda_k \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big) f_k(s)\, ds \right)^2 \quad \text{for all } t \in [0,T].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_0^T a\big(u_r(t)-u_m(t),u_r(t)-u_m(t)\big)\, dt \\
&\leq 2 \sum_{k=m+1}^{r} \lambda_k \int_0^T \Big(E_{\alpha,1}(-\lambda_k t^\alpha) v_k(0) \Big)^2 dt \\
&+2 \sum_{k=m+1}^{r} \lambda_k \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big) f_k(s)\, ds \right)^2 dt.
\end{aligned}
\tag{4}
$$

By using Theorem 2.10 we may estimate the first sum in the right hand side of (4) as follows

$$
\begin{aligned}
&\sum_{k=m+1}^{r} \lambda_k \int_0^T \Big(E_{\alpha,1}(-\lambda_k t^\alpha) v_k(0) \Big)^2 dt \leq \sum_{k=m+1}^{r} \lambda_k \big(v_k(0)\big)^2 \int_0^T \frac{C^2}{(1+\lambda_k t^\alpha)^2}\, dt \\
&= \sum_{k=m+1}^{r} C^2 \big(v_k(0)\big)^2 \int_0^T \left(\frac{\lambda_k}{1+\lambda_k t^\alpha} \right) \left(\frac{1}{1+\lambda_k t^\alpha} \right) dt \\
&\leq \sum_{k=m+1}^{r} C^2 \big(v_k(0)\big)^2 \int_0^T t^{-\alpha}\, dt \\
&= \frac{C^2 T^{1-\alpha}}{1-\alpha} \sum_{k=m+1}^{r} \big(v_k(0)\big)^2 = \frac{C^2 T^{1-\alpha}}{1-\alpha} \sum_{k=m+1}^{r} \Big(\langle u_0, \omega_k \rangle_H \Big)^2.
\end{aligned}
$$

Let us estimate now the second sum in the right hand side of (4). By using Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&\sum_{k=m+1}^{r} \lambda_k \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big) f_k(s)\, ds \right)^2 dt \\
&\leq \sum_{k=m+1}^{r} \lambda_k \int_0^T \left(\int_0^t (t-s)^{\alpha-1} \Big(E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big) \Big)^2 ds \right) \\
&\qquad \times \left(\int_0^t (t-s)^{\alpha-1} \big(f_k(s)\big)^2 ds \right) dt.
\end{aligned}
$$

By using once again Theorem 2.10 we obtain

$$\int_0^t \lambda_k (t-s)^{\alpha-1}\Big(E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big)\Big)^2\, ds \le C^2 \int_0^t \frac{\lambda_k (t-s)^{\alpha-1}}{\big(1+\lambda_k (t-s)^\alpha\big)^2}\, ds$$
$$= \frac{C^2}{\alpha}\left(1-\frac{1}{1+\lambda_k t^\alpha}\right) \le \frac{C^2}{\alpha} \quad \text{for all } t\in[0,T].$$

Hence

$$\sum_{k=m+1}^{r} \lambda_k \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big) f_k(s)\, ds\right)^2 dt$$
$$\le \sum_{k=m+1}^{r} \frac{C^2}{\alpha} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} \big(f_k(s)\big)^2\, ds\right) dt.$$

Let us introduce the mappings h_α and g_k given by

$$h_\alpha : \begin{cases} \mathbf{R}\to\mathbf{R} \\ \sigma \mapsto \begin{cases} \sigma^{\alpha-1} & \text{if } \sigma\in(0,T) \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

and

$$g_k : \begin{cases} \mathbf{R}\to\mathbf{R} \\ \sigma \mapsto \begin{cases} \big(f_k(\sigma)\big)^2 & \text{if } \sigma\in(0,T) \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

for all $k\in\mathbf{N}_0$. Since f belongs to $L^2(0,T;H)$ the mappings f_k belong to $L^2(0,T;\mathbf{R})$ for all $k\in\mathbf{N}_0$ and since $\alpha\in(0,1)$ we have $h_\alpha\in L^1(\mathbf{R};\mathbf{R})$. Thus the convolution product of h_α and g_k is defined and belongs to $L^1(\mathbf{R};\mathbf{R})$. It follows that

$$\sum_{k=m+1}^{r} \lambda_k \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_k (t-s)^\alpha\big) f_k(s)\, ds\right)^2 dt$$
$$\le \sum_{k=m+1}^{r} \frac{C^2}{\alpha} \int_0^T (h_\alpha * g_k)(t)\, dt \le \sum_{k=m+1}^{r} \frac{C^2}{\alpha} \|h_\alpha * g_k\|_{L^1(\mathbf{R};\mathbf{R})}$$
$$\le \sum_{k=m+1}^{r} \frac{C^2}{\alpha} \|h_\alpha\|_{L^1(\mathbf{R};\mathbf{R})} \|g_k\|_{L^1(\mathbf{R};\mathbf{R})} \le \frac{C^2 T^\alpha}{\alpha^2} \sum_{k=m+1}^{r} \|f_k\|^2_{L^2(0,T;\mathbf{R})}.$$

Finally we obtain an estimate of $u_r - u_m$ in $L^2(0,T;V)$ i.e.

$$\|u_r-u_m\|^2_{L^2(0,T;V)} \le \frac{1}{\gamma}\int_0^T a\big(u_r(t)-u_m(t), u_r(t)-u_m(t)\big)\, dt$$
$$\le \frac{2}{\gamma}\left(\frac{C^2 T^{1-\alpha}}{1-\alpha} \sum_{k=m+1}^{r} \Big(\langle u_0,\omega_k\rangle_H\Big)^2 + \frac{C^2 T^\alpha}{\alpha^2}\sum_{k=m+1}^{r} \|f_k\|^2_{L^2(0,T;\mathbf{R})}\right)$$

where $\gamma>0$ is the coercivity constant of the bilinear form a on V. Since $u_0\in H$ and $f\in L^2(0,T;H)$ we infer from (3) that

$$\lim_{m\to+\infty} \sum_{k\ge m+1} \Big(\langle u_0,\omega_k\rangle_H\Big)^2 = 0, \quad \lim_{m\to+\infty} \sum_{k\ge m+1} \int_0^T \big(f_k(t)\big)^2\, dt = 0$$

and we may conclude that $(u_m)_{m\ge1}$ is a Cauchy sequence in $L^2(0,T;V)$. Let us prove now that $(u_m)_{m\geq 1}$ is also a Cauchy sequence in $C^0\big([0,T];H\big)$. Let m and r be two natural numbers such that $r>m$. For all $t\in[0,T]$ we have

$$\begin{aligned}
&\big\|u_r(t)-u_m(t)\big\|_H^2=\sum_{k=m+1}^{r}\big(v_k(t)\big)^2\\
&=\sum_{k=m+1}^{r}\left\{E_{\alpha,1}(-\lambda_k t^\alpha)v_k(0)+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_k(t-s)^\alpha\right)f_k(s)\,ds\right\}^2\\
&\leq 2\sum_{k=m+1}^{r}\Big(E_{\alpha,1}(-\lambda_k t^\alpha)v_k(0)\Big)^2\\
&+2\sum_{k=m+1}^{r}\left(\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-s)^\alpha\big)f_k(s)\,ds\right)^2\quad\text{for all } t\in[0,T].
\end{aligned}\tag{5}$$

Once again we apply Theorem 2.10 to get an estimate of the first sum of the right hand side of (5), i.e.

$$\begin{aligned}
&\sum_{k=m+1}^{r}\Big(E_{\alpha,1}(-\lambda_k t^\alpha)v_k(0)\Big)^2\leq\sum_{k=m+1}^{r}\frac{C^2}{(1+\lambda_k t^\alpha)^2}\big(v_k(0)\big)^2\\
&\leq C^2\sum_{k=m+1}^{r}\big(v_k(0)\big)^2\leq C^2\sum_{k=m+1}^{r}\Big(\langle u_0,\omega_k\rangle_H\Big)^2.
\end{aligned}$$

Next we estimate the second sum of the right hand side of (5) as follows.

$$\begin{aligned}
&\sum_{k=m+1}^{r}\left(\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-s)^\alpha\big)f_k(s)\,ds\right)^2\\
&\leq\sum_{k=m+1}^{r}\left(\int_0^t\Big((t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-s)^\alpha\big)\Big)^2ds\right)\left(\int_0^t\big(f_k(s)\big)^2ds\right)\\
&\leq\sum_{k=m+1}^{r}\left(\int_0^t C^2\frac{(t-s)^{2\alpha-2}}{\big(1+\lambda_k(t-s)^\alpha\big)^2}\,ds\right)\left(\int_0^t\big(f_k(s)\big)^2ds\right).
\end{aligned}$$

Since $\alpha\in(1/2,1)$ the first integral can be estimated by

$$\int_0^t C^2\frac{(t-s)^{2\alpha-2}}{\big(1+\lambda_k(t-s)^\alpha\big)^2}\,ds\leq\int_0^t C^2\sigma^{2\alpha-2}\,d\sigma=\frac{C^2}{2\alpha-1}t^{2\alpha-1}\leq\frac{C^2T^{2\alpha-1}}{2\alpha-1}$$

and we get

$$\begin{aligned}
&\sum_{k=m+1}^{r}\left(\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-s)^\alpha\big)f_k(s)\,ds\right)^2\\
&\leq\frac{C^2T^{2\alpha-1}}{2\alpha-1}\sum_{k=m+1}^{r}\int_0^t\big(f_k(s)\big)^2ds\leq\frac{C^2T^{2\alpha-1}}{2\alpha-1}\sum_{k=m+1}^{r}\int_0^T\big(f_k(s)\big)^2ds.
\end{aligned}$$

Finally

$$\big\|u_r(t)-u_m(t)\big\|_H^2\leq 2C^2\sum_{k=m+1}^{r}\Big(\langle u_0,\omega_k\rangle_H\Big)^2+2C^2\frac{T^{2\alpha-1}}{2\alpha-1}\sum_{k=m+1}^{r}\int_0^T\big(f_k(s)\big)^2ds$$

for all $t \in [0,T]$, which yields

$$\|u_r - u_m\|_{C^0([0,T];H)} \leq 2C^2 \sum_{k=m+1}^{r} \Big(\langle u_0, \omega_k\rangle_H\Big)^2 + 2C^2 \frac{T^{2\alpha-1}}{2\alpha-1} \sum_{k=m+1}^{r} \int_0^T \big(f_k(s)\big)^2 \, ds$$

and we may conclude that $(u_m)_{m\geq 1}$ is a Cauchy sequence in $C^0\big([0,T];H\big)$. Therefore $(u_m)_{m\geq 1}$ converges strongly to u both in $L^2(0,T;V)$ and in $C^0\big([0,T];H\big)$. Hence

$$u_m(0) = \sum_{k=1}^{m} \langle u_0, \omega_k\rangle_H \omega_k \longrightarrow u(0) = \sum_{k\geq 1} \langle u_0, \omega_k\rangle_H \omega_k = u_0$$

and, for all $k \in \mathbf{N}_0$,

$$\begin{aligned} &\big\langle {}^cD_t^\alpha u(t), \omega_k\big\rangle_H + \big\langle Au(t), \omega_k\big\rangle_H = {}^cD_t^\alpha v_k(t) + \lambda_k v_k(t) \\ &= f_k(t) = \big\langle f(t), \omega_k\big\rangle_H, \qquad t \in [0,T]. \end{aligned}$$

It follows that, for all $v \in V^m$ and for all $m \in \mathbf{N}_0$, we have

$$\big\langle {}^cD_t^\alpha u(t), v\big\rangle_H + \big\langle Au(t), v\big\rangle_H = \big\langle f(t), v\big\rangle_H, \qquad t \in [0,T].$$

□

We deal now with the proof of Theorem 3.2

Proof. We investigate the existence and uniqueness of a solution to problem (1)-(2) by using the Hilbertian basis given by the eigenvectors $(\omega_k)_{k\geq 1}$ of the operator A. Then $u : [0,T] \to H$ admits the following decomposition

$$u(t) = \sum_{k\geq 1} v_k(t)\omega_k, \qquad t \in [0,T]$$

where $v_k : [0,T] \to \mathbf{R}$ is given by

$$v_k(t) = \big\langle u(t), \omega_k\big\rangle_H, \qquad t \in [0,T].$$

Starting from (1) we obtain

$$\big\langle {}^cD_t^\alpha u(t), \omega_k\big\rangle_H + \big\langle Au(t), \omega_k\big\rangle_H = \big\langle f(t), \omega_k\big\rangle_H, \qquad t \in [0,T]$$

for all $k \in \mathbf{N}_0$. By observing that

$$\big\langle {}^cD_t^\alpha u(t), \omega_k\big\rangle_H = {}^c D_t^\alpha \big\langle u(t), \omega_k\big\rangle_H, \quad \big\langle Au(t), \omega_k\big\rangle_H = a\big(u(t), \omega_k\big) = \lambda_k \big\langle u(t), \omega_k\big\rangle_H$$

we get

$${}^cD_t^\alpha v_k(t) + \lambda_k v_k(t) = f_k(t), \qquad t \in [0,T]$$

with $f_k(t) = \big\langle f(t), \omega_k\big\rangle_H$. Hence we consider the scalar fractional differential equations given by

$${}^cD_t^\alpha v_k(t) + \lambda_k v_k(t) = f_k(t), \qquad k \in \mathbf{N}_0 \tag{6}$$

with the initial condition

$$v_k(0) = \big\langle u(0), \omega_k\big\rangle_H = \big\langle u_0, \omega_k\big\rangle_H. \tag{7}$$

By applying Laplace transform to (6) and taking into account the initial condition (7), we obtain

$$s^\alpha\big(\mathcal{L}\big(v_k(t)\big)\big)(s) - s^{\alpha-1} v_k(0) + \lambda_k\big(\mathcal{L}\big(v_k(t)\big)\big)(s) = \big(\mathcal{L}\big(f_k(t)\big)\big)(s)$$

which can be rewritten as

$$\begin{aligned}\big(\mathcal{L}\big(v_k(t)\big)\big)(s) &= \frac{1}{s^\alpha+\lambda_k}\big(\mathcal{L}\big(f_k(t)\big)\big)(s)+\frac{s^{\alpha-1}}{s^\alpha+\lambda_k}v_k(0)\\ &= \big(\mathcal{L}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha))\big)(s)\big(\mathcal{L}\big(f_k(t)\big)\big)(s)\\ &\quad+\big(\mathcal{L}(E_{\alpha,1}(-\lambda_k t^\alpha))\big)(s)v_k(0)\\ &= \mathcal{L}\Big(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)*f_k(t)+E_{\alpha,1}(-\lambda_k t^\alpha)v_k(0)\Big)(s).\end{aligned}$$

Hence

$$v_k(t)=E_{\alpha,1}(-\lambda_k t^\alpha)\big\langle u_0,\omega_k\big\rangle_H+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-s)^\alpha\big)\big\langle f(s),\omega_k\big\rangle_H\,ds$$

which proves the uniqueness of the solution to problem (1)-(2). For the existence, using corollary 3.3 and by recalling that $\bigcup_{m\geq 1} V^m$ is dense in V, we may conclude that u is the unique solution of problem (1)-(2). Hence theorem 3.2 is completely proved. □

Remark 3.4. When $\alpha=1$ the Caputo derivative of order α corresponds to the classical first order derivative. Hence Theorem 3.2 and Corollary 3.3 extend to differential evolution equation of fractional order $\alpha\in(1/2,1)$ the well-known existence and uniqueness result for parabolic problems ([13]).

3.1. **Conclusion.** We deal in this paper with the existence of solution for some fractional evolution equation using a linear maximal monotone in Hilbert space not necessarily bounded, we project in the future, to replace the Hilbert space by a Banach space X with a nonlinear m-accretive operator and an upper semi-inner product on X defined by

$$\langle x,y\rangle_+=\sup\{x^*(y),\ x^*\in J(x)\}$$

where, $J: X\to 2^{X^*}$ is the duality mapping, given by

$$J(x)=\{x^*\in X^*, x^*(x)=\|x\|^2=\|x*\|_*^2\},\ \forall\, x\in X,$$

to study the existence of solutions sets and its topological structure.

4. Example

We may illustrate this result by an example. Let Ω be an open bounded domain of $\mathbf{R}^d$, with $d\geq 1$, of class C^2 such that $\partial\Omega$ is bounded. We let $H=L^2(\Omega)$ and $V=H_0^1(\Omega)$. We consider a symmetric second order uniformly elliptic operator A defined by

$$\begin{cases} D(A)=H^2(\Omega)\cap H_0^1(\Omega)\\ Au=-\displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\end{cases}$$

with $a_{ij}\in C^1(\overline{\Omega})$ for all $i,j\in\{1,\dots,d\}$ and (i) $a_{ij}(x)=a_{ji}(x)$ for all $x\in\overline{\Omega}$, (ii) there exists $C_a > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq C_a \sum_{i=1}^{d} \xi_i^2 \qquad \text{for all } (\xi_i)_{1\leq i\leq d} \in \mathbf{R}^d \text{ and for all } x \in \overline{\Omega}.$$

By using Green's formula we obtain

$$\langle Au, v\rangle_H = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \Big)\, dx \qquad \text{for all } u \in D(A) \text{ and } v \in V$$

and we may define $a : V \times V \to \mathbf{R}$ by

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \Big)\, dx \qquad \text{for all } u \in V,\, v \in V.$$

It follows that A is a linear self-adjoint maximal monotone operator (see [9]). Obviously a is bilinear symmetric and continuous on V and, by using Poincaré's inequality, we obtain that

$$a(u,u) \geq C_a \int_\Omega \big\|\nabla u(x)\big\|^2\, dx \geq \frac{C_a}{C_P^2 + 1} \|u\|_V^2 \qquad \text{for all } u \in V$$

where $C_P > 0$ denotes Poincaré's constant on Ω. It follows that a is coercive on V. Finally, with Rellich theorem, we infer that the injection of V into H is compact Hence assumptions (A1) and (A2) are satisfied. By applying Theorem 3.2 we obtain the existence and uniqueness of a solution u to the non-classical diffusion problem with Caputo fractional derivative in time and homogeneous Dirichlet boundary conditions, with the following regularity property $u \in C^0\big([0,T]; L^2(\Omega)\big) \cap L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big)$.

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