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RELATIVE (p,q)-ORDER AND (p,q)-TYPE OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

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Abstract. We introduce the notions of relative (p,q)-order and relative (p,q)-type of entire functions of two complex variables. Growth properties are investigated.

1. Introduction

Let f be a non-constant entire function of two variables holomorphic in the closed poly disc $U = \{(z_1, z_2) : |z_i| \le r_i \ (i = 1, 2)\} \ (r_1, r_2 \ge 0)$. Denote $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \le r_i \ (i = 1, 2)\}$ which, by the maximum principle and Hartogs theorem [6], is an increasing function of each r_1, r_2.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, define iterations of the exponential and logarithmic functions as

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \quad \text{and} \quad \log^{[k]} x = \log\left(\log^{[k-1]} x\right),$$

with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Through out the paper we take $p, q, a \in \mathbb{N}$.

The classical order of $f(z_1, z_2)$ is defined as (see, e.g., [6], also [1])

$$\rho(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log(r_1 r_2)}.$$

The equivalent formula for $\rho(f)$ is

$$\rho(f) = \inf\left\{\mu > 0 : M_f(r_1, r_2) < \exp\left((r_1 r_2)^\mu\right), \text{ for all } r_i \ge R(\mu), \ i = 1, 2\right\},$$

which can alternatively be written as

$$\begin{aligned} \rho(f) \quad = \quad & \inf\left\{\mu > 0 : M_f(r_1, r_2) < \exp^{[2]}\left(\mu \log(r_1 r_2)\right), \right. \\ & \left. \text{for all } r_i \ge R(\mu), \ i = 1, 2\right\}. \end{aligned}$$

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Similarly the lower order $\lambda(f)$ of f is defined as

$$\begin{aligned}\lambda(f) &= \sup\Big\{\mu>0 : M_f\big(r_1,r_2\big) > \exp^{[2]}\big(\mu\log(r_1r_2)\big),\\ &\quad \text{for all } r_i \geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

The rate of growth of entire function of two variables normally depends upon the order of it. The entire function of two complex variables with higher order is of faster growth than that of lesser order. But if orders of two entire functions of two variables are the same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions of two variables called their type and lower type and thus one can define type and lower type of an entire function f of two variables denoted by $\sigma(f)$ and $\overline{\sigma}(f)$ respectively (see, e.g. [9]) as follows:

$$\overline{\sigma}(f) = \liminf_{r_1,r_2\to\infty}\frac{\log M_f\big(r_1,r_2\big)}{r_1^{\rho(f)}+r_2^{\rho(f)}} \leq \sigma(f) = \limsup_{r_1,r_2\to\infty}\frac{\log M_f\big(r_1,r_2\big)}{r_1^{\rho(f)}+r_2^{\rho(f)}},$$

where $0<\rho(f)<\infty$.

Alternatively, the above can also be written as

$$\begin{aligned}\sigma(f) &= \inf\Big\{\mu>0 : M_f(r_1,r_2) < \exp\big(\mu r_1^{\rho(f)}+\mu r_2^{\rho(f)}\big),\\ &\quad \text{for all } r_i \geq R(\mu),\ i=1,2\Big\},\\ \overline{\sigma}(f) &= \sup\Big\{\mu>0 : M_f(r_1,r_2) > \exp\big(\mu r_1^{\rho(f)}+\mu r_2^{\rho(f)}\big),\\ &\quad \text{for all } r_i \geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

Similarly one may define the following growth indicators:

$$\begin{aligned}\overline{\tau}(f) &= \sup\Big\{\mu>0 : M_f(r_1,r_2) > \exp\big(\mu r_1^{\lambda(f)}+\mu r_2^{\lambda(f)}\big),\\ &\quad \text{for all } r_i \geq R(\mu),\ i=1,2\Big\},\\ \tau(f) &= \inf\Big\{\mu>0 : M_f(r_1,r_2) < \exp\big(\mu r_1^{\lambda(f)}+\mu r_2^{\lambda(f)}\big),\\ &\quad \text{for all } r_i \geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

A generalization of the classical order and type has been studied by [11] and by Juneja, Kapoor and Bajpai [7, 8]). More precisely, for given integers p and q with $p\geq q$, the (p,q)-order is defined as

$$\rho_{pq} = \limsup_{r\to\infty}\frac{\log^{[p]} M(r)}{\log^{[q]} r} = \limsup_{t\to\infty}\frac{\log^{[p-1]} f(t)}{\log^{[q-1]} t},$$

where $M(r)=\exp[f(\log r)]$. The (p,q)-type is defined as

$$\sigma_{pq} = \limsup_{r\to\infty}\frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho_{pq}}} = \limsup_{t\to\infty}\frac{\log^{[p-2]} f(t)}{(\log^{[q-2]} t)^{\rho_{pq}}}.$$

Later on, a general relative order and type of entire functions of several variables have been investigated by Kiselman [10], where the approach of convex functions is implemented.

Extending the notion (p,q)-th order, recently Shen et al. [12] introduced the new concept of $[p,q]-\varphi$ order of an entire function of single variable where $p \geq q$. Later on, combining the definitions of (p,q)-order and $[p,q]-\varphi$ order, Biswas (see, e.g., [4]) redefined the (p,q)-order of an entire function of single variable without restriction $p \geq q$.

From all of the above, it is natural to give the (p,q)-order of entire functions of two variables in the following way.

Definition 1. *The (p,q)-order of f, denoted by $\rho^{(p,q)}(f)$, is defined by*

$$\begin{aligned}\rho^{(p,q)}(f) &= \inf\Big\{\mu>0: M_f(r_1,r_2)<\exp^{[p]}\big(\mu\log^{[q]}r_1+\mu\log^{[q]}r_2\big),\\ &\quad \textit{for all } r_i\geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

Similarly, the (p,q)-lower order of f, denoted by $\lambda^{(p,q)}(f)$, is defined as follows

$$\begin{aligned}\lambda^{(p,q)}(f) &= \sup\Big\{\mu>0: M_f(r_1,r_2)>\exp^{[p]}\big(\mu\log^{[q]}r_1+\mu\log^{[q]}r_2\big),\\ &\quad \textit{for all } r_i\geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

In this connection we give the following definition which is analogous to the definition of index-pair of an entire function of single variable introduced in [7, 8].

Definition 2. *An entire function f is said to have an index-pair (p,q) if $b<\rho^{(p,q)}(f)<\infty$ and $\rho^{(p-1,q-1)}(f)$ is a nonzero finite number, where $b=1$ if $p=q$ and $b=0$ otherwise.*

Moreover if $0<\rho^{(p,q)}(f)<\infty$, then

$$\begin{cases}\rho^{(p-a,q)}(f)=\infty, & \textit{if } a<p,\\ \rho^{(p,q-a)}(f)=0, & \textit{if } a<q,\\ \rho^{(p+a,q+a)}(f)=1, & \textit{if } a=1,2,...\end{cases}$$

Similarly, for $0<\lambda^{(p,q)}(f)<\infty$,

$$\begin{cases}\lambda^{(p-a,q)}(f)=\infty, & \textit{if } a<p,\\ \lambda(p,q-a)(f)=0, & \textit{if } a<q,\\ \lambda(p+a,q+a)(f)=1, & \textit{if } a=1,2,...\end{cases}$$

Now in order to compare the growth of entire functions having the same (p,q)-order, one may introduce the concepts of (p,q)-type and (p,q)-lower type.

Definition 3. *The (p,q)-type, $\sigma^{(p,q)}(f)$ and the (p,q)-lower type, $\overline{\sigma}^{(p,q)}(f)$ of an entire function f with $0<\rho^{(p,q)}(f)<\infty$ are defined as follows*

$$\begin{aligned}\sigma^{(p,q)}(f)=\inf\Big\{\mu>0: M_f(r_1,r_2)<\exp^{[p-1]}\Big(\mu(\log^{[q-1]}r_1)^{\rho^{(p,q)}(f)}\\ +\mu(\log^{[q-1]}r_2)^{\rho^{(p,q)}(f)}\Big),\ \textit{for all } r_i\geq R(\mu),\ i=1,2\Big\}\end{aligned}$$

and

$$\begin{aligned}\overline{\sigma}^{(p,q)}(f) &= \sup\Big\{\mu>0: M_f(r_1,r_2)>\exp^{[p-1]}\Big(\mu(\log^{[q-1]}r_1)^{\rho^{(p,q)}(f)}\\ &\quad +\mu(\log^{[q-1]}r_2)^{\rho^{(p,q)}(f)}\Big),\ \textit{for all } r_i\geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

Likewise, to compare the growth of entire functions having the same (p,q)-lower order, one can also introduce the concepts of (p,q)-weak type and (p,q)-lower weak type of an entire function f.

Definition 4. *The (p,q)-weak type, $\tau^{(p,q)}(f)$ and (p,q)-lower weak type, $\overline{\tau}^{(p,q)}(f)$ of an entire function f with $0 < \lambda^{(p,q)}(f) < \infty$ are defined as follows:*

$$\begin{aligned}\tau^{(p,q)}(f) &= \inf\Big\{\mu > 0 : M_f(r_1,r_2) < \exp^{[p-1]}\Big(\mu(\log^{[q-1]} r_1)^{\lambda^{(p,q)}(f)} \\ &\quad +\mu(\log^{[q-1]} r_2)^{\lambda^{(p,q)}(f)}\Big),\ \textit{for all } r_i \geq R(\mu),\ i=1,2\Big\}\end{aligned}$$

and

$$\begin{aligned}\overline{\tau}^{(p,q)}(f) &= \sup\Big\{\mu > 0 : M_f(r_1,r_2) > \exp^{[p-1]}\Big(\mu(\log^{[q-1]} r_1)^{\lambda^{(p,q)}(f)} \\ &\quad +\mu(\log^{[q-1]} r_2)^{\lambda^{(p,q)}(f)}\Big),\ \textit{for all } r_i \geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

Remark 1. *For $p = 1$ and $q = 1$ we obtain the classical definitions of order and type above. Also for $p = k$ and $q = 1$, we get generalized order, type, and lower type $\rho^{[k]}(f)$, $\sigma^{[k]}(f)$, and $\tau^{[k]}(f)$ etc.*

However the concept of relative order of entire functions of a single variable as well as their technical advantages not comparing with the growth of $\exp z$, was first introduced by Bernal [3]. In the case of relative order, it was then natural for Banerjee and Dutta [2] to define the relative order of entire functions. Namely, the relative order of an entire function f with respect to another entire function g, denoted by $\rho_g(f)$, is defined by

$$\rho_g(f) = \inf\Big\{\mu > 0 : M_f(r_1,r_2) < M_g(r_1^{\mu}, r_2^{\mu});\ r_i \geq R(\mu),\ i=1,2\Big\}.$$

Similarly, the relative lower order of f with respect to g, denoted by $\lambda_g(f)$, is defined as follows

$$\lambda_g(f) = \sup\Big\{\mu > 0 : M_f(r_1,r_2) > M_g(r_1^{\mu}, r_2^{\mu});\ r_i \geq R(\mu),\ i=1,2\Big\}.$$

Now in order to make some progress in the study of relative order of entire functions of two variables, one may introduce the definition of relative (p,q)-order between two entire functions in the light of index-pair as follows.

Definition 5. *Let f and g be two entire functions with index-pairs (m,q) and (m,p) respectively. Then the relative (p,q)-order of f with respect to g, denoted by $\rho_g^{(p,q)}(f)$ is defined by*

$$\begin{aligned}&\rho_g^{(p,q)}(f) = \inf\Big\{\mu > 0 : M_f(r_1,r_2) \\ < \ & M_g\Big(\exp^{[p]}(\mu\log^{[q]} r_1), \exp^{[p]}(\mu\log^{[q]} r_2)\Big),\ \textit{for all } r_i \geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

Similarly, one can define the relative (p,q)-lower order of f with respect to g, denoted by $\lambda_g^{(p,q)}(f)$, is defined as follows

$$\begin{aligned}&\lambda_g^{(p,q)}(f) = \sup\Big\{\mu > 0 : M_f(r_1,r_2) \\ > \ & M_g\Big(\exp^{[p]}(\mu\log^{[q]} r_1), \exp^{[p]}(\mu\log^{[q]} r_2)\Big),\ \textit{for all } r_i \geq R(\mu),\ i=1,2\Big\}.\end{aligned}$$

Remark 2. *If f and g have the same index-pair $(p,1)$, then Definition 5 reduces to that of [2]. If f and g have index-pairs $(m,1)$ and (m,k) respectively, then we get the definition of generalized relative order (respectively generalized relative lower order). Further, if $g = \exp^{[m-1]}(z_1 z_2)$, then $\rho_g(f) = \rho^{[m]}(f)$ and $\rho_g^{(p,q)}(f) = \rho^{(m,q)}(f)$. Moreover, if f is an entire function with an index-pair $(2,1)$ and $g = \exp(z_1 z_2)$, then Definition 5 becomes the classical one.*

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative (p,q)-type and relative (p,q)-lower type between two entire functions as follows.

Definition 6. *Let f and g be two entire functions with index-pairs (m,q) and (m,p) respectively. Then the relative (p,q)-type, $\sigma_g^{(p,q)}(f)$ and the relative (p,q)-lower type, $\overline{\sigma}_g^{(p,q)}(f)$ of f with respect to g with non-zero finite relative (p,q)-order $\rho_g^{(p,q)}(f)$ are defined as*

$$\begin{aligned} & \sigma_g^{(p,q)}(f) = \inf \Big\{ \mu > 0 : M_f(r_1, r_2) \\ < \quad & M_g \Big(\exp^{[p-1]} \big(\mu (\log^{[q-1]} r_1)^{\rho_g^{(p,q)}(f)} \big), \exp^{[p-1]} \big(\mu (\log^{[q-1]} r_2)^{\rho_g^{(p,q)}(f)} \big) \Big), \\ & \text{for all } r_i \geq R(\mu),\ i = 1,2 \Big\} \end{aligned}$$

and

$$\begin{aligned} & \overline{\sigma}_g^{(p,q)}(f) = \sup \Big\{ \mu > 0 : M_f(r_1, r_2) \\ > \quad & M_g \Big(\exp^{[p-1]} \big(\mu (\log^{[q-1]} r_1)^{\rho_g^{(p,q)}(f)} \big), \exp^{[p-1]} \big(\mu (\log^{[q-1]} r_2)^{\rho_g^{(p,q)}(f)} \big) \Big) \\ & \text{for all } r_i \geq R(\mu),\ i = 1,2 \Big\}. \end{aligned}$$

Analogously, to determine the relative growth of f having same nonzero finite relative (p,q)-lower order with respect to another entire function g, one can introduce the definition of relative (p,q)-weak type $\tau_g^{(p,q)}(f)$ and relative (p,q)-lower weak type $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g of finite positive relative (p,q)-lower order $\lambda_g^{(p,q)}(f)$ in the following way.

Definition 7. *Let f and g be two entire functions with index-pairs (m,q) and (m,p) respectively. Then the relative (p,q)-weak type $\tau_g^{(p,q)}(f)$ and the relative (p,q)-lower weak type $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g with nonzero finite relative (p,q)-lower order $\lambda_g^{(p,q)}(f)$ are defined as*

$$\begin{aligned} & \tau_g^{(p,q)}(f) = \inf \Big\{ \mu > 0 : M_f(r_1, r_2) \\ < \quad & M_g \big(\exp^{[p-1]}(\mu (\log^{[q-1]} r_1)^{\lambda_g^{(p,q)}(f)}), \exp^{[p-1]}(\mu (\log^{[q-1]} r_2)^{\lambda_g^{(p,q)}(f)}) \big) \\ & \text{for all } r_i \geq R(\mu),\ i = 1,2 \Big\} \end{aligned}$$

and

$$\begin{aligned} & \overline{\tau}_g^{(p,q)}(f) = \sup \Big\{ \mu > 0 : M_f(r_1, r_2) \\ > \quad & M_g \big(\exp^{[p-1]}(\mu (\log^{[q-1]} r_1)^{\lambda_g^{(p,q)}(f)}), \exp^{[p-1]}(\mu (\log^{[q-1]} r_2)^{\lambda_g^{(p,q)}(f)}) \big) \\ & \text{for all } r_i \geq R(\mu),\ i = 1,2 \Big\}. \end{aligned}$$

Remark 3. *If f and g have the same index-pair $(p,1)$, then Definition 6 and Definition 7 reduce to the definitions of relative type $\sigma_g(f)$ (respectively relative lower type $\overline{\sigma}_g(f)$) and relative weak type $\tau_g(f)$ (respectively relative lower weak type $\overline{\tau}_g(f)$).*

If f and g have index-pair $(m,1)$ and (m,k) respectively, then we get the definitions of generalized relative type $\sigma^{[k]}(f)$ (respectively generalized relative lower type $\overline{\sigma}^{[k]}(f)$) and generalized relative weak type $\tau^{[k]}(f)$ (respectively generalized relative lower weak type $\overline{\tau}^{[k]}(f)$).

Further, if $g = \exp^{[m-1]}(z_1 + z_2)$, then $\sigma_g(f) = \sigma^{[m]}(f)$ ($\overline{\sigma}_g(f) = \overline{\sigma}^{[m]}(f)$); $\tau_g(f) = \tau^{[m]}(f)$ ($\overline{\tau}_g(f) = \overline{\tau}^{[m]}(f)$) and $\sigma_g^{(p,q)}(f) = \sigma^{(m,q)}(f)$ ($\overline{\sigma}_g(p,q)(f) = \overline{\sigma}^{(m,q)}(f)$ ($\tau_g^{(p,q)}(f) = \tau^{(m,q)}(f)$ ($\overline{\tau}_g^{(p,q)}(f) = \overline{\tau}^{(m,q)}(f)$).

Moreover if f is an entire function with an index-pair $(2,1)$ and $g = \exp(z_1+z_2)$, then Definition 6 and Definition 7 become classical ones respectively.

In this connection, we finally remind the following definitions from [5] which are needed in the sequel.

Definition 8. *1) An entire function f is said to have Property (R) if for any $\sigma > 1$ and for all sufficiently large r_1, r_2,*

$$[M_f(r_1, r_2)]^2 < M_f(r_1^\sigma, r_2^\sigma).$$

2) A pair of entire functions f and g are said to have mutually Property (X) if for all sufficiently large r_1, r_2,

$$M_{f\cdot g}(r_1, r_2) > M_f(r_1, r_2) \quad \text{and} \quad M_{f\cdot g}(r_1, r_2) > M_g(r_1, r_2)$$

hold simultaneously.

Some examples of functions with or without the Property (R) can be found in [5]. Also the functions $f(z_1, z_2) = z_1 z_2$ and $g(z_1, z_2) = (z_1 z_2)^2$ have mutually Property (X).

Our aim is to investigate several basic properties of relative (p,q)-order, relative (p,q)-type and relative (p,q)-weak type of entire functions of two variables with respect to another one under somewhat different conditions. The standard definitions and notations in the theory of entire function of several variables are available in [6]. In particular, the following result is needed in the sequel.

Lemma 1. [6] *Suppose that f is a non-constant entire function of two variables, $\alpha > 1$ and $0 < \beta < \alpha$. Then*

$$M_f(\alpha r_1, \alpha r_2) > \beta M_f(r_1, r_2) \text{ for all sufficiently large } r_1, r_2.$$

2. Relative (p,q)-order

In this section we present the main results on the relative (p,q)-order.

Theorem 1. *Let f_1, f_2 and g be three entire functions of two variables and either of f_1, f_2 is of regular relative (p,q) growth with respect to g.*
Then

$$\lambda_g^{(p,q)}(f_1 \pm f_2) \le \max\{\lambda_g^{(p,q)}(f_1), \lambda_g^{(p,q)}(f_2)\}.$$

The equality holds when any one of $\lambda_g^{(p,q)}(f_i) > \lambda_g^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g where $i, j = 1, 2$ and $i \neq j$.

Proof. If $\lambda_g^{(p,q)}(f_1 \pm f_2) = 0$ then the result is obvious. So we suppose that $\lambda_g^{(p,q)}(f_1 \pm f_2) > 0$. We can clearly assume that $\lambda_g^{(p,q)}(f_k)$ is finite for $k = 1, 2$. Further let $\max\{\lambda_g^{(p,q)}(f_1), \lambda_g^{(p,q)}(f_2)\} = \Delta$ and f_2 be of regular relative (p,q) growth with respect to g.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_g^{(p,q)}(f_1)$, we have for a sequence values of r_1, r_2 tending to infinity that

$$\begin{aligned} &M_{f_1}(r_1, r_2) \\ &\quad < M_g(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f_1)+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f_1)+\varepsilon)}) \\ &i.e.,\ M_{f_1}(r_1, r_2) \\ &\qquad < M_g(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Delta+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Delta+\varepsilon)}) . \end{aligned} \tag{1}$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\rho_g^{(p,q)}(f_2)$ $(= \lambda_g^{(p,q)}(f_2))$, we obtain for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} &M_{f_2}(r_1, r_2) \\ &\quad < M_g(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f_2)+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f_2)+\varepsilon)}) \\ &i.e.,\ M_{f_2}(r_1, r_2) \\ &\qquad < M_g(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Delta+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Delta+\varepsilon)}). \end{aligned} \tag{2}$$

Now we obtain from (1) and (2) for a sequence values of r_1, r_2 tending to infinity that

$$\begin{aligned} &M_{f_1 \pm f_2}(r_1, r_2) \\ &\qquad < 2M_g(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Delta+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Delta+\varepsilon)}). \end{aligned} \tag{3}$$

Therefore in view of Lemma 1, we obtain from (3) for a sequence values of r_1, r_2 tending to infinity that

$$M_{f_1 \pm f_2}(r_1, r_2) < M_g(3\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Delta+\varepsilon)}, 3\exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Delta+\varepsilon)})$$

$$i.e.,\ M_{f_1 \pm f_2}(r_1, r_2) < M_g(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Delta+4\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Delta+4\varepsilon)}).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_g^{(p,q)}(f_1 \pm f_2) \leq \Delta = \max\{\lambda_g^{(p,q}(f_1), \lambda_g^{(p,q}(f_2)\}.$$

Similarly, if we consider that f_1 is of regular relative (p,q) growth with respect to g or both f_1 and f_2 are of regular relative (p,q) growth with respect to g, then one can easily verify that

$$\lambda_g^{(p,q)}(f_1 \pm f_2) \leq \Delta = \max\{\lambda_g^{(p,q)}(f_1), \lambda_g^{(p,q)}(f_2)\}. \tag{4}$$

Now let $\lambda_g^{(p,q)}(f_1) > \lambda_g^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p,q) growth with respect to g. Also let $f = f_1 \pm f_2$. Then in view of (4) we get that $\lambda_g^{(p,q)}(f) \leq \lambda_g^{(p,q)}(f_1)$. As, $f_1 = (f \pm f_2)$ and in this case we obtain that $\lambda_g^{(p,q)}(f_1) \leq \max\{\lambda_g^{(p,q)}(f), \lambda_g^{(p,q)}(f_2)\}$. As we assume that $\lambda_g^{(p,q)}(f_2) < \lambda_g^{(p,q)}(f_1)$, we have $\lambda_g^{(p,q)}(f_1) \leq \lambda_g^{(p,q)}(f)$ and hence

$$\lambda_g^{(p,q)}(f_1 \pm f_2) \geq \lambda_g^{(p,q)}(f_1) = \max\{\lambda_g^{(p,q)}(f_1), \lambda_g^{(p,q)}(f_2)\}.$$

Further if we consider $\lambda_g^{(p,q)}(f_1) < \lambda_g^{(p,q)}(f_2)$ and at least f_1 is of regular relative (p,q) growth with respect to g, then one can also verify that

$$\lambda_g^{(p,q)}(f_1 \pm f_2) \geq \Delta = \max\{\lambda_g^{(p,q)}(f_1), \lambda_g^{(p,q)}(f_2)\}. \tag{5}$$

So the conclusion of the second part of the theorem follows from (4) and (5). □

Theorem 2. *Let f_1, f_2 be any two entire functions of two variables with relative order $\rho_{g_1}^{(p,q)}(f_1)$ and $\rho_{g_1}^{(p,q)}(f_2)$ with respect to another entire function g_1 of two variables. Then*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \max\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2).$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1) \neq \rho_{g_1}^{(p,q)}(f_2)$.

We omit the proof of Theorem 2 as it can easily be carried out in the line of Theorem 1.

Theorem 3. *Let f_1, g_1 and g_2 be any three entire functions of two variables such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exit. Then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}.$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_1)$.

Proof. If $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \infty$, then the result is obvious. So we suppose that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) < \infty$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1, 2$. Further let $\Psi = \min\ \{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_k}^{(p,q)}(f_1)$, we have for all sufficiently large values of r_1, r_2 that

$$M_{g_k}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda_{g_k}^{(p,q)}(f_1)-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda_{g_k}^{(p,q)}(f_1)-\varepsilon)}) < M_{f_1}(r_1, r_2)$$

where $k = 1, 2$.

Therefore, from above we get for all sufficiently large values of r_1, r_2 that

$$M_{g_k}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Psi-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Psi-\varepsilon)}) < M_{f_1}(r_1, r_2) \tag{6}$$

where $k = 1, 2$.

Now we obtain from above and Lemma 1 for all sufficiently large values of r_1, r_2 that

$$M_{g_1 \pm g_2}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Psi-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Psi-\varepsilon)}) < 2M_{f_1}(r_1, r_2)$$

$$i.e.,\ M_{g_1 \pm g_2}((\frac{1}{3}) \exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Psi-\varepsilon)}, (\frac{1}{3}) \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Psi-\varepsilon)}) < M_{f_1}(r_1, r_2)$$

$$i.e.,\ M_{g_1 \pm g_2}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Psi-4\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Psi-4\varepsilon)}) < M_{f_1}(r_1, r_2).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \Psi = \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}. \tag{7}$$

Now let $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ and $g = g_1 \pm g_2$. Then in view of (7) we get that $\lambda_{g}^{(p,q)}(f_1) \geq \lambda_{g_1}^{(p,q)}(f_1)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that

$\lambda_{g_1}^{(p,q)}(f_1) \geq \min\{\lambda_g^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we have $\lambda_{g_1}^{(p,q)}(f_1) \geq \lambda_g^{(p,q)}(f_1)$ and hence

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \leq \lambda_{g_1}^{(p,q)}(f_1) = \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}.$$

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can also derive that

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \leq \Psi = \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}. \tag{8}$$

So the conclusion of the second part of the theorem follows from (7) and (8). □

Theorem 4. *Let f_1, g_1 and g_2 be any three entire functions of two variables such that $\rho_{g_1}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_1)$ exit. Also let f_1 be of regular relative (p,q) growth with respect to at least any one of g_1 or g_2. Then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\}.$$

The equality holds when any one of $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j where $i, j = 1, 2$ and $i \neq j$.

We omit the proof of Theorem 4 as it can easily be carried out in the line of Theorem 3.

Theorem 5. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) \leq \max[\min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\}, \min\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\}]$$

when the following two conditions hold:
(i) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when $\rho_{g_1}^{(p,q)}(f_i) < \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) < \rho_{g_2}^{(p,q)}(f_j)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 2 and Theorem 4 we get that

$$\begin{aligned}
&\max[\min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\}, \min\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\}] \\
&= \max[\rho_{g_1 \pm g_2}^{(p,q)}(f_1), \rho_{g_1 \pm g_2}^{(p,q)}(f_2)] \\
&\geq \rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2).
\end{aligned} \tag{9}$$

Since $\rho_{g_1}^{(p,q)}(f_i) < \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) < \rho_{g_2}^{(p,q)}(f_j)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\} > \min\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\} \text{ or}$$

$$\min\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\} > \min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\} \text{ holds.}$$

Now in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } \rho_{g_1 \pm g_2}^{(p,q)}(f_1) > \rho_{g_1 \pm g_2}^{(p,q)}(f_2) \text{ or } \rho_{g_1 \pm g_2}^{(p,q)}(f_2) > \rho_{g_1 \pm g_2}^{(p,q)}(f_1)$$

which is the condition for holding equality in (9).

Hence the theorem follows. □

Theorem 6. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Then*

$$\lambda_{g_1\pm g_2}^{(p,q)}(f_1\pm f_2)\geq \min[\max\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_1}^{(p,q)}(f_2)\},\max\{\lambda_{g_2}^{(p,q)}(f_1),\lambda_{g_2}^{(p,q)}(f_2)\}]$$

when the following two conditions hold:
(i) $\lambda_{g_1}^{(p,q)}(f_i)>\lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for $i=1,2$, $j=1,2$ and $i\neq j$; and
(ii) $\lambda_{g_2}^{(p,q)}(f_i)>\lambda_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_2 for $i=1,2$, $j=1,2$ and $i\neq j$.
The equality holds when $\lambda_{g_i}^{(p,q)}(f_1)<\lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2)<\lambda_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i=1,2$; $j=1,2$ and $i\neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 1 and Theorem 3, we obtain that

$$\begin{aligned}&\min[\max\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_1}^{(p,q)}(f_2)\},\max\{\lambda_{g_2}^{(p,q)}(f_1),\lambda_{g_2}^{(p,q)}(f_2)\}]\\&=\min[\lambda_{g_1}^{(p,q)}(f_1\pm f_2),\lambda_{g_2}^{(p,q)}(f_1\pm f_2)]\\&\leq \lambda_{g_1\pm g_2}^{(p,q)}(f_1\pm f_2).\end{aligned}\tag{10}$$

Since $\lambda_{g_i}^{(p,q)}(f_1)<\lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2)<\lambda_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i=1,2$; $j=1,2$ and $i\neq j$, we get that

$$\text{either }\max\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_1}^{(p,q)}(f_2)\}<\max\{\lambda_{g_2}^{(p,q)}(f_1),\lambda_{g_2}^{(p,q)}(f_2)\}\text{ or}$$

$$\max\{\lambda_{g_2}^{(p,q)}(f_1),\lambda_{g_2}^{(p,q)}(f_2)\}<\max\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_1}^{(p,q)}(f_2)\}\text{ holds.}$$

Since conditions (i) and (ii) of the theorem hold, it follows from above that

$$\text{either }\lambda_{g_1}^{(p,q)}(f_1\pm f_2)<\lambda_{g_2}^{(p,q)}(f_1\pm f_2)\text{ or }\lambda_{g_2}^{(p,q)}(f_1\pm f_2)<\lambda_{g_1}^{(p,q)}(f_1\pm f_2)$$

which is the condition for holding equality in (10).

Hence the theorem follows. □

Theorem 7. *Let f_1, f_2 and g_1 be any three entire functions of two variables. Also let at least f_1 or f_2 be of regular relative (p,q) growth with respect to g_1. Then*

$$\lambda_{g_1}^{(p,q)}(f_1\cdot f_2)\leq \max\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_1}^{(p,q)}(f_2)\}$$

provided g_1 has the Property (R). The equality holds when f_1 and f_2 satisfy Property (X).

Proof. Suppose that $\lambda_{g_1}^{(p,q)}(f_1\cdot f_2)>0$. Otherwise if $\lambda_{g_1}^{(p,q)}(f_1\cdot f_2)=0$ then the result is obvious. Let us consider that f_2 is of regular relative (p,q) growth with respect to g_1. Also suppose that max $\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_1}^{(p,q)}(f_2)\}=\Delta$. We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k)$ is finite for $k=1,2$. Now we have from (1), (2) for a sequence values of r_1,r_2 tending to infinity that

$$\begin{aligned}&M_{f_1\cdot f_2}(r_1,r_2)\\&<[M_{g_1}(\exp^{[p-1]}(\log^{[q-1]}r_1)^{(\Delta+\varepsilon)},\exp^{[p-1]}(\log^{[q-1]}r_2)^{(\Delta+\varepsilon)}]^2.\end{aligned}$$

Also in view of Definition 8, we obtain from above for any $\delta>1$ and for a sequence values of r_1,r_2 tending to infinity that

$$\begin{aligned}&M_{f_1\cdot f_2}(r_1,r_2)\\&<M_{g_1}(\exp^{[p-1]}(\log^{[q-1]}r_1)^{\delta(\Delta+\varepsilon)},\exp^{[p-1]}(\log^{[q-1]}r_2)^{\delta(\Delta+\varepsilon)}),\end{aligned}$$

since g_1 has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \to 1^+$, we get from above that

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}.$$

Similarly, if we consider that f_1 is of regular relative (p, q) growth with respect to g_1 or both f_1 and f_2 are of regular relative (p, q) growth with respect to g_1, then also one can easily verify that

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}. \tag{11}$$

Now let f_1 and f_2 satisfy Property (X), then of course we have $M_{f_1 \cdot f_2}(r_1, r_2) > M_{f_1}(r_1, r_2)$ and $M_{f_1 \cdot f_2}(r_1, r_2) > M_{f_2}(r_1, r_2)$ for all sufficiently large values of r_1, r_2. Therefore from the definition of relative (p, q)-th lower order, we get for a sequence values of r_1, r_2 tending to infinity that

$$\begin{aligned} &M_{f_1}(r_1, r_2) < M_{f_1 \cdot f_2}(r_1, r_2) \\ &< M_{g_1}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2)+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2)+\varepsilon)}). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \geq \lambda_{g_1}^{(p,q)}(f_1)$. Similarly $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \geq \lambda_{g_1}^{(p,q)}(f_2)$ and therefore

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \geq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}. \tag{12}$$

Hence the theorem follows from (11) and (12). □

Theorem 8. *Let f_1, f_2 be any two entire functions of two variables with relative order $\rho_{g_1}^{(p,q)}(f_1)$ and $\rho_{g_1}^{(p,q)}(f_2)$ with respect to another entire function g_1 of two variables. Then*

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\}$$

provided g_1 has the Property (R). The equality holds when f_1 and f_2 satisfy Property (X).

We omit the proof of Theorem 8 as it can easily be carried out in the line of Theorem 7.

Theorem 9. *Let f_1, g_1 and g_2 be any three entire functions of two variables. Also let $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exist. Then*

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \min\{\lambda_{g_1}^{(p,q)}(f_1),\ \lambda_{g_2}^{(p,q)}(f_1)\}$$

provided $g_1 \cdot g_2$ has the Property (R). The equality holds when g_1 and g_2 satisfy Property (X).

Proof. Suppose that $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) < \infty$. Otherwise if $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) = \infty$ then the result is obvious. Also suppose that $\min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\} = \Psi$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1, 2$.

Now we get in view of (6) for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} &M_{g_1 \cdot g_2}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Psi-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Psi-\varepsilon)}) \\ &\qquad < [M_{f_1}(r_1, r_2)]^2 \\ &i.e.,\ [M_{g_1 \cdot g_2}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\Psi-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\Psi-\varepsilon)})]^{\frac{1}{2}} \\ &\qquad < M_{f_1}(r_1, r_2). \end{aligned}$$

Now in view of Definition 8 we obtain from above for any $\delta > 1$ and for all sufficiently large values of r_1, r_2 that

$$M_{g_1\cdot g_2}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{\frac{(\Psi-\varepsilon)}{\delta}}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{\frac{(\Psi-\varepsilon)}{\delta}}) < M_{f_1}(r_1, r_2)$$

since $g_1 \cdot g_2$ has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \to 1^+$, we obtain from above that

$$\lambda^{(p,q)}_{g_1\cdot g_2}(f_1) \geq \Psi = \min\{\lambda^{(p,q)}_{g_1}(f_1),\ \lambda^{(p,q)}_{g_2}(f_1)\}. \quad (13)$$

Now let g_1 and g_2 satisfy Property (X), then of course we have $M_{g_1\cdot g_2}(r_1, r_2) > M_{g_1}(r_1, r_2)$ and $M_{g_1\cdot g_2}(r_1, r_2) > M_{g_2}(r_1, r_2)$ for all sufficiently large values of r_1, r_2. Therefore from the definition of relative (p,q)-th lower order, we get for all sufficiently large values of r_1, r_2 that

$$M_{g_1}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda^{(p,q)}_{g_1\cdot g_2}(f_1)-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda^{(p,q)}_{g_1\cdot g_2}(f_1)-\varepsilon)})$$

$$< M_{g_1\cdot g_2}(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda^{(p,q)}_{g_1\cdot g_2}(f_1)-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda^{(p,q)}_{g_1\cdot g_2}(f_1)-\varepsilon)})$$

$$< M_{f_1}(r_1, r_2).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that $\lambda^{(p,q)}_{g_1}(f_1) \geq \lambda^{(p,q)}_{g_1\cdot g_2}(f_1)$. Similarly $\lambda^{(p,q)}_{g_2}(f_1) \geq \lambda^{(p,q)}_{g_1\cdot g_2}(f_1)$ and therefore

$$\lambda^{(p,q)}_{g_1\cdot g_2}(f_1) \leq \Psi = \min\{\lambda^{(p,q)}_{g_1}(f_1),\ \lambda^{(p,q)}_{g_2}(f_1)\}. \quad (14)$$

Hence the theorem follows from (13) and (14). □

Theorem 10. *Let f_1, g_1 and g_2 be any three entire functions of two variables. Also let f_1 be of regular relative (p,q) growth with respect to at least any one of g_1 or g_2. Then*

$$\rho^{(p,q)}_{g_1\cdot g_2}(f_1) \geq \min\{\rho^{(p,q)}_{g_1}(f_1),\ \rho^{(p,q)}_{g_2}(f_1)\}$$

provided $g_1 \cdot g_2$ has the Property (R). The equality holds when g_1 and g_2 satisfy Property (X).

We omit the proof of Theorem 10 as it can easily be carried out in the line of Theorem 9.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of Theorem 8, Theorem 7, Theorem 9 and Theorem 10 and in the line of Theorem 5 and Theorem 6 respectively.

Theorem 11. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Also let $g_1 \cdot g_2$ satisfy the Property (R). Then,*

$$\rho^{(p,q)}_{g_1\cdot g_2}(f_1\cdot f_2) = \max[\min\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\}, \min\{\rho^{(p,q)}_{g_1}(f_2), \rho^{(p,q)}_{g_2}(f_2)\}],$$

when the following four conditions hold:
(i) f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(ii) f_2 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(iii) f_1 and f_2 satisfy Property (X); and
(iv) g_1 and g_2 satisfy Property (X).

Theorem 12. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Also let $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (R). Then,*

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \min[\max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}, \max\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\}],$$

when the following four conditions hold:
(i) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1;
(ii) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_2;
(iii) f_1 and f_2 satisfy Property (X); and
(iv) g_1 and g_2 satisfy Property (X).

Theorem 13. *Let f, g and h be any three entire functions of two variables such that $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$. Then*

$$\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} .$$

Proof. From the definition of $\rho_g^{(p,q)}(f)$ it follows for all sufficiently large values of r_1, r_2 that

$$M_f(r_1, r_2) < M_g\big(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\rho_g^{(p,q)}(f)+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\rho_g^{(p,q)}(f)+\varepsilon)}\big), \tag{15}$$

and for a sequence values of r_1, r_2 tending to infinity we obtain that

$$M_f(r_1, r_2) > M_g\big(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\rho_g^{(p,q)}(f)-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\rho_g^{(p,q)}(f)-\varepsilon)}\big). \tag{16}$$

Similarly from the definition of $\lambda_g^{(p,q)}(f)$, we have for all sufficiently large values of r_1, r_2 that

$$M_f(r_1, r_2) > M_g\big(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f)-\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f)-\varepsilon)}\big) \tag{17}$$

and also for a sequence values of r_1, r_2 tending to infinity that

$$M_f(r_1, r_2) < M_g\big(\exp^{[p-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f)+\varepsilon)}, \exp^{[p-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f)+\varepsilon)}\big). \tag{18}$$

Further from the definition of $\rho_h^{(m,p)}(g)$ it follows for all sufficiently large values of r_1, r_2 that

$$M_g(r_1, r_2) < M_h\big(\exp^{[m-1]}(\log^{[p-1]} r_1)^{(\rho_h^{(m,p)}(g)+\varepsilon)}, \exp^{[m-1]}(\log^{[p-1]} r_2)^{(\rho_h^{(m,p)}(g)+\varepsilon)}\big) \tag{19}$$

and for a sequence values of r_1, r_2 tending to infinity we obtain that

$$M_g(r_1, r_2) > M_h\big(\exp^{[m-1]}(\log^{[p-1]} r_1)^{(\rho_h^{(m,p)}(g)-\varepsilon)}, \exp^{[m-1]}(\log^{[p-1]} r_2)^{(\rho_h^{(m,p)}(g)-\varepsilon)}\big). \tag{20}$$

Likewise from the definition of $\lambda_h^{(m,p)}(g)$ it follows for all sufficiently large values of r_1, r_2 that

$$M_g(r_1,r_2) > M_h\big(\exp^{[m-1]}(\log^{[p-1]} r_1)^{(\lambda_h^{(m,p)}(g)-\varepsilon)}, \exp^{[m-1]}(\log^{[p-1]} r_2)^{(\lambda_h^{(m,p)}(g)-\varepsilon)}\big) \tag{21}$$

and for a sequence values of r_1, r_2 tending to infinity we obtain that

$$M_g(r_1,r_2) < M_h\big(\exp^{[m-1]}(\log^{[p-1]} r_1)^{(\lambda_h^{(m,p)}(g)+\varepsilon)}, \exp^{[m-1]}(\log^{[p-1]} r_2)^{(\lambda_h^{(m,p)}(g)+\varepsilon)}\big). \tag{22}$$

Now from (15) and in view of (19) we get for all sufficiently large values of r_1, r_2 that

$$M_f(r_1,r_2) < M_h\big(\exp^{[m-1]}(\log^{[q-1]} r_1)^{(\rho_g^{(p,q)}(f)+\varepsilon)(\rho_h^{(m,p)}(g)+\varepsilon)}, \exp^{[m-1]}(\log^{[q-1]} r_2)^{(\rho_g^{(p,q)}(f)+\varepsilon)(\rho_h^{(m,p)}(g)+\varepsilon)}\big).$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\rho_h^{(m,q)}(f) \le \rho_g^{(p,q)}(f) \cdot \rho_h^{(m,p)}(g)$$

$$i.e.,\ \rho_g^{(p,q)}(f) \ge \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\ . \tag{23}$$

Similarly from (18) and in view of (19), for a sequence values of r_1, r_2 tending to infinity, we get that

$$M_f(r_1,r_2) < M_h\big(\exp^{[m-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f)+\varepsilon)(\rho_h^{(m,p)}(g)+\varepsilon)}, \exp^{[m-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f)+\varepsilon)(\rho_h^{(m,p)}(g)+\varepsilon)}\big).$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \le \lambda_g^{(p,q)}(f). \tag{24}$$

Analogously from (15) and (22), we get that

$$\rho_g^{(p,q)}(f) \ge \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}. \tag{25}$$

Likewise from (17) and (21), it follows for all sufficiently large values of r_1, r_2 that

$$M_f(r_1,r_2) > M_h\big(\exp^{[m-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f)-\varepsilon)(\lambda_h^{(m,p)}(g)-\varepsilon)}, \exp^{[m-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f)-\varepsilon)(\lambda_h^{(m,p)}(g)-\varepsilon)}\big).$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lambda_h^{(m,q)}(f) \ge \lambda_g^{(p,q)}(f) \cdot \lambda_h^{(m,p)}(g)$$

$$i.e.,\ \lambda_g^{(p,q)}(f) \le \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}. \tag{26}$$

Moreover from (17) and in view of (20) it follows for a sequence values of r_1, r_2 tending to infinity that

$$M_f(r_1,r_2) > M_h\big(\exp^{[m-1]}(\log^{[q-1]} r_1)^{(\lambda_g^{(p,q)}(f)-\varepsilon)(\rho_h^{(m,p)}(g)-\varepsilon)}, \exp^{[m-1]}(\log^{[q-1]} r_2)^{(\lambda_g^{(p,q)}(f)-\varepsilon)(\rho_h^{(m,p)}(g)-\varepsilon)}\big).$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lambda_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}. \tag{27}$$

Similarly from (16) and (21), we get that

$$\rho_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}. \tag{28}$$

Hence the theorem follows from (23), (24), (25), (26), (27) and (28). □

Remark 4. *From the conclusion of Theorem 13, one may write* $\rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$. *Similarly* $\rho_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,q)}(f) = \rho_h^{(m,q)}(f)$.

3. Relative (p,q)-type

Next we intend to find out some theorems of relating to relative (p,q)-th type and relative (p,q)-th weak type of entire functions of two variables with respect to another one taking into consideration of the above theorems.

Theorem 14. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Also let $\rho_{g_1}^{(p,q)}(f_1)$, $\rho_{g_1}^{(p,q)}(f_2)$, $\rho_{g_2}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_2)$ be all non zero and finite.*
(A)*If $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ for i, $j = 1,2$ and $i \neq j$, then*

$$\sigma_{g_1}^{(p,q)}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q)}(f_i) \text{ and } \overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_i).$$

(B) *If $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for i, $j = 1,2$ and $i \neq j$, then*

$$\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) = \sigma_{g_i}^{(p,q)}(f_1) \text{ and } \overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_i}^{(p,q)}(f_1).$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(ii) $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) growth with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(iii) $\rho_{g_1}^{(p,q)}(f_i) < \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) < \rho_{g_2}^{(p,q)}(f_j)$ hold simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(iv)$\rho_{g_m}^{(p,q)}(f_l) = \max[\min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\}, \min\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\}] \mid l, m = 1,2$;
then we have

$$\sigma_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \sigma_{g_m}^{(p,q)}(f_l) \text{ and } \overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \overline{\sigma}_{g_m}^{(p,q)}(f_l).$$

Proof. From the definition of relative (p,q)-th type and relative (p,q)-th lower type, we have for all sufficiently large values of r_1, r_2 that

$$M_{f_k}(r_1,r_2) < M_{g_l}\big(\exp^{[p-1]}((\sigma_{g_l}^{(p,q)}(f_k)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_l}^{(p,q)}(f_k)}), \exp^{[p-1]}((\sigma_{g_l}^{(p,q)}(f_k)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_l}^{(p,q)}(f_k)})\big), \quad (29)$$

$$M_{f_k}(r_1,r_2) > M_{g_l}\big(\exp^{[p-1]}((\overline{\sigma}_{g_l}^{(p,q)}(f_k)-\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_l}^{(p,q)}(f_k)}), \exp^{[p-1]}((\overline{\sigma}_{g_l}^{(p,q)}(f_k)-\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_l}^{(p,q)}(f_k)})\big), \quad (30)$$

$$i.e., M_{g_l}(r_1,r_2) < M_{f_k}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\overline{\sigma}_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}, \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\overline{\sigma}_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\Big), \quad (31)$$

and for a sequence of values of r_1, r_2 tending to infinity, we obtain

$$M_{f_k}(r_1,r_2) > M_{g_l}\big(\exp^{[p-1]}((\sigma_{g_l}^{(p,q)}(f_k)-\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_l}^{(p,q)}(f_k)}), \exp^{[p-1]}((\sigma_{g_l}^{(p,q)}(f_k)-\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_l}^{(p,q)}(f_k)})\big), \quad (32)$$

$$i.e., M_{g_l}(r_1,r_2) < M_{f_k}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\sigma_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}, \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\sigma_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\Big), \quad (33)$$

and

$$M_{f_k}(r_1,r_2) < M_{g_l}\big(\exp^{[p-1]}((\overline{\sigma}_{g_l}^{(p,q)}(f_k)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_l}^{(p,q)}(f_k)}), \exp^{[p-1]}((\overline{\sigma}_{g_l}^{(p,q)}(f_k)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_l}^{(p,q)}(f_k)})\big), \quad (34)$$

where $\varepsilon > 0$ is any arbitrary positive number, $k = 1, 2$ and $l = 1, 2$.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ holds. Also let $\varepsilon(> 0)$ be arbitrary. Now in view of (29), we get for all sufficiently large values of r_1, r_2 that

$$M_{f_1 \pm f_2}(r_1,r_2) < M_{g_1}\big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_1}^{(p,q)}(f_1)}), \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_1}^{(p,q)}(f_1)})\big) \times (1+A), \quad (35)$$

where $A = \frac{M_{g_1}(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_2)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_1}^{(p,q)}(f_2)}), \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_2)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_1}^{(p,q)}(f_2)}))}{M_{g_1}(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_1}^{(p,q)}(f_1)}), \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_1}^{(p,q)}(f_1)}))}$, and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, and for all sufficiently large values of r_1, r_2, we can make the term A sufficiently small. Hence for any $\alpha = 1 + \varepsilon_1$, where $\varepsilon_1 = A$, we get in view of Theorem 2, $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ and from (35) for all sufficiently large values of r_1, r_2 that

$$M_{f_1 \pm f_2}(r_1,r_2) < M_{g_1}\big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)}), \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)})\big) \times \alpha.$$

Hence making $\alpha \to 1+$, we obtain that

$$\sigma_{g_1}^{(p,q)}(f_1 \pm f_2) \le \sigma_{g_1}^{(p,q)}(f_1). \tag{36}$$

Now let $f = f_1 \pm f_2 \Rightarrow f_1 = (f \pm f_2)$. Since $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, from above $\sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2) \le \sigma_{g_1}^{(p,q)}(f_1)$. Also in view of Theorem 2 and $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we obtain that $\rho_{g_1}^{(p,q)}(f) > \rho_{g_1}^{(p,q)}(f_2)$. Hence in view of (36) $\sigma_{g_1}^{(p,q)}(f_1) \le \sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1) \Rightarrow \sigma_{g_1}^{(p,q)}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\sigma_{g_1}^{(p,q)}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q)}(f_2)$.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ holds. Also let $\varepsilon(> 0)$ be arbitrary. Now there exists a nondecreasing sequence r_{ip}, $r_{ip} \to \infty$; $i = 1, 2$ as $p \to \infty$ such that from (29) and (34) we get

$$\begin{aligned} M_{f_1 \pm f_2}(r_{1p}, r_{2p}) \quad < \quad & M_{g_1}\Big(\exp^{[p-1]}((\overline{\sigma}_{g_1}^{(p,q)}(f_1) + \varepsilon)(\log^{[q-1]} r_{1p})^{\rho_{g_1}^{(p,q)}(f_1)}), \\ & \exp^{[p-1]}((\overline{\sigma}_{g_1}^{(p,q)}(f_1) + \varepsilon)(\log^{[q-1]} r_{2p})^{\rho_{g_1}^{(p,q)}(f_1)})\Big) \times (1 + B), \end{aligned} \tag{37}$$

where $B = \frac{M_{g_1}(\exp^{[p-1]}(((\sigma_{g_1}^{(p,q)}(f_2)+\varepsilon)(\log^{[q-1]} r_{1p})^{\rho_{g_1}^{(p,q)}(f_2)}),\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_2)+\varepsilon)(\log^{[q-1]} r_{2p})^{\rho_{g_1}^{(p,q)}(f_2)}))}{M_{g_1}(\exp^{[p-1]}((\overline{\sigma}_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_{1p})^{\rho_{g_1}^{(p,q)}(f_1)}),\exp^{[p-1]}((\overline{\sigma}_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]} r_{2p})^{\rho_{g_1}^{(p,q)}(f_1)}))}$, and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we can make the term B sufficiently small by taking p sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (37) that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ when $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_2)$. Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ where at least f_1 is of regular relative (p, q) growth with respect to g_2. Therefore there exists a nondecreasing sequence $\{r_{ip}\}$, $r_{ip} \to \infty$;$i = 1, 2$ as $p \to \infty$ such that in view of (31) and (33), we obtain that

$$\begin{aligned} M_{g_1 \pm g_2}(r_{1p}, r_{2p}) \quad < \quad & M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_{1p}}{(\sigma_{g_1}^{(p,q)}(f_1) - \varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}, \\ & \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_{2p}}{(\sigma_{g_1}^{(p,q)}(f_1) - \varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big) \times (1 + C), \end{aligned} \tag{38}$$

where $C = \frac{M_{f_1}\left(\exp^{[q-1]}\left(\frac{\log^{[p-1]} r_{1p}}{(\overline{\sigma}_{g_2}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}},\exp^{[q-1]}\left(\frac{\log^{[p-1]} r_{2p}}{(\overline{\sigma}_{g_2}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)}{M_{f_1}\left(\exp^{[q-1]}\left(\frac{\log^{[p-1]} r_{1p}}{(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}},\exp^{[q-1]}\left(\frac{\log^{[p-1]} r_{2p}}{(\sigma_{g_1}^{(p,q)}(f_1-\varepsilon)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)}$, and since $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term C sufficiently small by taking p sufficiently large. Hence for any $\alpha = 1 + \varepsilon_1$, where $\varepsilon_1 = C$, we get from (38) and

Theorem 4, for a nondecreasing sequence r_{ip}, $r_{ip} \to \infty$; $i = 1,2$ as $p \to \infty$ that

$$M_{g_1 \pm g_2}(r_{1p}, r_{2p}) < M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_{1p}}{(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1 \pm g_2}^{(p,q)}(f_1)}}, \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_{2p}}{(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1 \pm g_2}^{(p,q)}(f_1)}}\Big) \times \alpha.$$

Hence, making $\alpha \to 1+$, we obtain that

$$\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \sigma_{g_1}^{(p,q)}(f_1). \tag{39}$$

Now let $g = g_1 \pm g_2 \Rightarrow g_1 = (g \pm g_2)$. Since $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least f_1 is of regular relative (p,q) growth with respect to g_2, from above $\sigma_g^{(p,q)}(f_1) = \sigma_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \sigma_{g_1}^{(p,q)}(f_1)$. Therefore in view of Theorem 4 and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we obtain that $\rho_g^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ as at least f_1 is of regular relative (p,q) growth with respect to g_2. Hence in view of (39), $\sigma_{g_1}^{(p,q)}(f_1) \geq \sigma_g^{(p,q)}(f_1) = \sigma_{g_1 \pm g_2}^{(p,q)}(f_1)$. Therefore $\sigma_g^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1) \Rightarrow \sigma_{g_1 \pm g_2}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1)$.

Similarly if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then $\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) = \sigma_{g_2}^{(p,q)}(f_1)$.

Case IV. In this case suppose that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ where at least f_1 is of regular relative (p,q) growth with respect to g_2. Hence from (31), we get for all sufficiently large values of r_1, r_2 that

$$M_{g_1 \pm g_2}(r_{1p}, r_{2p}) < M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_{1p}}{(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}, \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_{2p}}{(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big) \times (1+D), \tag{40}$$

where $D = \dfrac{M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\overline{\sigma}_{g_2}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}, \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\overline{\sigma}_{g_2}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\Big)}{M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}, \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term D sufficiently small by taking r_1, r_2 sufficiently large and therefore using the similar technique for as executed in the proof of Case III we get from (40) that $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ where $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least f_1 is of regular relative (p,q) growth with respect to g_2. Likewise if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_2}^{(p,q)}(f_1)$. Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 5 and the first part and second part of the theorem. Hence its proof is omitted. □

Theorem 15. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Also let $\lambda_{g_1}^{(p,q)}(f_1)$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ be all non zero and finite.*
(A) *If $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for i, $j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_i) \text{ and } \overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_i).$$

(B) *If* $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ *for* $i = j = 1, 2$ *and* $i \neq j$, *then*

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \text{ and } \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1).$$

(C) *Assume the functions* f_1, f_2, g_1 *and* g_2 *satisfy the following conditions:*
(i) $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ *with at least* f_j *is of regular relative* (p, q) *growth with respect to* g_1 *for* i,$j = 1, 2$ *and* $i \neq j$;
(ii) $\lambda_{g_2}^{(p,q)}(f_i) > \lambda_{g_2}^{(p,q)}(f_j)$ *with at least* f_j *is of regular relative* (p, q) *growth with respect to* g_2 *for* i, $j = 1, 2$ *and* $i \neq j$;
(iii) $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ *and* $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ *hold simultaneously for* i, $j = 1, 2$ *and* $i \neq j$;
(iv)$\lambda_{g_m}^{(p,q)}(f_l) = \min[\max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}, \max\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\}] \mid l, m = 1, 2$;
then we have

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \tau_{g_m}^{(p,q)}(f_l) \mid l, m = 1, 2$$

and

$$\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_m}^{(p,q)}(f_l) \mid l, m = 1, 2.$$

We omit the proof of Theorem 15 as it can easily be carried out in the line of Theorem 14.

Theorem 16. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions of two variables.*
(A) *The following condition is assumed to be satisfied:*
(i) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$ *or* $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$ *holds, then*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2).$$

(B) *The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$ *or* $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$ *holds;*
(ii) f_1 *is of regular relative* (p, q) *growth with respect to at least any one of* g_1 *or* g_2, *then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1).$$

Proof. **Case I.** Suppose that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2) < \infty)$. Now in view of Theorem 2 it is easy to see that $\rho_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. If possible let

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) < \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2). \tag{41}$$

Let $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$. Then in view of the first part of Theorem 14 and (41) we obtain that $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2) = \sigma_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. Similarly, with the help of the first part of Theorem 14, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1) < \infty)$ and f_1 is of regular relative (p, q) growth with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$. Therefore in view of Theorem 4, it follows that $\rho_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1). \tag{42}$$

Let us consider that $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$. Then in view of the proof of the second part of Theorem 14 and (42) we obtain that $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1) = \sigma_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\rho_{g_1 \pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$. Also in view of the proof of second part of Theorem 14 one can derive the same conclusion for the condition $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established. □

Theorem 17. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables.*
(A) *The following conditions are assumed to be satisfied:*
(i) $(f_1 \pm f_2)$ is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(ii) Either $\sigma_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \sigma_{g_2}^{(p,q)}(f_1 \pm f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1 \pm f_2)$;
(iii) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$;
(iv) Either $\sigma_{g_2}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_2}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1 and f_2 are of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(ii) Either $\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \sigma_{g_1 \pm g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_2)$;
(iii) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$;
(iv) Either $\sigma_{g_1}^{(p,q)}(f_2) \neq \sigma_{g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_2) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2).$$

We omit the proof of Theorem 17 as it is a natural consequence of Theorem 16.

Theorem 18. *Let f_1,f_2, g_1 and g_2 be any four entire functions of two variables.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular relative (p,q) growth with respect to g_1;
(ii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$ holds, then

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1, g_1 and g_2 be any three entire functions such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists;
(ii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ holds, then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) \ .$$

Proof. **Case I.** Let $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2) < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ be of regular relative (p,q) growth with respect to g_1. Now, in view of Theorem 1, it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) < \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \ . \tag{43}$$

Let $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$. Then in view of the proof of the first part of Theorem 15 and (43 we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2) = \tau_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. Similarly in view of the proof of the first part of Theorem 15, one can establish the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1) < \infty)$. Therefore in view of Theorem 3, it follows that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) . \tag{44}$$

Suppose $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$. Then in view of the second part of Theorem 15 and (44), we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$. Analogously with the help of the second part of Theorem 15, the same conclusion can also be derived under the condition $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established. □

Theorem 19. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 and g_2;
(ii) Either $\tau_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \tau_{g_2}^{(p,q)}(f_1 \pm f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1 \pm f_2)$;
(iii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$;
(iv) Either $\tau_{g_2}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$; then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2) .$$

(B) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 are of regular relative (p,q) growth with respect to $g_1 \pm g_2$;
(ii) Either $\tau_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \tau_{g_1 \pm g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_2)$ holds;
(iii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ holds;
(iv) Either $\tau_{g_1}^{(p,q)}(f_2) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$ holds, then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2) .$$

We omit the proof of Theorem 19 as it is a natural consequence of Theorem 18.

Theorem 20. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Also let $\rho_{g_1}^{(p,q)}(f_1)$, $\rho_{g_1}^{(p,q)}(f_2)$, $\rho_{g_2}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_2)$ be all non zero.*
(A) *Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*
(i) g_1 satisfies the Property (R) and
(ii) f_1 and f_2 satisfy Property (X); then

$$\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) = \sigma_{g_1}^{(p,q)}(f_i) \text{ and } \overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_i).$$

(B) *Assume the functions g_1, g_2 and f_1 satisfy the following conditions:*
(i) f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2

and f_1 satisfy the Property (R) and
(ii) g_1 and g_2 satisfy Property (X); then

$$\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)=\sigma_{g_i}^{(p,q)}(f_1)\ \textit{and}\ \ \overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1)=\overline{\sigma}_{g_i}^{(p,q)}(f_1).$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $g_1\cdot g_2$, f_1 and f_2 satisfy the Property (R);
(ii) f_1 and f_2 satisfy Property (X);
(iii) g_1 and g_2 satisfy Property (X);
(iv) f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(v) f_2 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(vi) $\rho_{g_m}^{(p,q)}(f_l)=\max[\min\{\rho_{g_1}^{(p,q)}(f_1),\rho_{g_2}^{(p,q)}(f_1)\},\min\{\rho_{g_1}^{(p,q)}(f_2),\rho_{g_2}^{(p,q)}(f_2)\}]\mid l,m=1,2$; then

$$\sigma_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2)=\sigma_{g_m}^{(p,q)}(f_l)\ \textit{and}\ \overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2)=\overline{\sigma}_{g_m}^{(p,q)}(f_1).$$

Proof. **Case I.** Suppose that $\rho_{g_1}^{(p,q)}(f_1)>\rho_{g_1}^{(p,q)}(f_2)$. Also let g_1 satisfies the Property (R). Since $M_{f_1\cdot f_2}(r_1,r_2)\le M_{f_1}(r_1,r_2)\cdot M_{f_2}(r_1,r_2)$, we have from (29) for all sufficiently large values of r_1,r_2 and for any arbitrary $\varepsilon>0$ that

$$\begin{aligned} M_{f_1\cdot f_2}(r_1,r_2) &< \Big(M_{g_1}\Big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})(\log^{[q-1]}r_1)^{\rho_{g_1}^{(p,q)}(f_1)}),\\ &\quad \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})(\log^{[q-1]}r_2)^{\rho_{g_1}^{(p,q)}(f_1)})\Big)\Big)^2. \end{aligned} \tag{45}$$

Let us observe that $\delta_1:=\frac{\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon}{\sigma_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}}>1$ which implies

$$\frac{\exp^{[p-2]}(\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)[\log^{[q-1]}r_i]^{\rho_{g_1}^{(p,q)}(f_1)}}{\exp^{[p-2]}(\sigma_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})[\log^{[q-1]}r_i]^{\rho_{g_1}^{(p,q)}(f_1)}}=\delta(\text{say})>1, \tag{46}$$

where $i=1,2$.

Since g_1 satisfies the Property (R), in view of Definition 8, Theorem 8, and (46) we obtain from (45) for all sufficiently large values of r_1,r_2 that

$$\begin{aligned} M_{f_1\cdot f_2}(r_1,r_2) &< M_{g_1}\Big(\Big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})(\log^{[q-1]}r_1)^{\rho_{g_1}^{(p,q)}(f_1)})\Big)^{\delta},\\ &\quad \Big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})(\log^{[q-1]}r_2)^{\rho_{g_1}^{(p,q)}(f_1)})\Big)^{\delta}\Big) \end{aligned}$$

$$\begin{aligned} \text{i.e., } M_{f_1\cdot f_2}(r_1,r_2) &< M_{g_1}\Big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]}r_1)^{\rho_{g_1}^{(p,q)}(f_1\cdot f_2)}),\\ &\quad \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon)(\log^{[q-1]}r_2)^{\rho_{g_1}^{(p,q)}(f_1\cdot f_2)})\Big). \end{aligned}$$

Hence we obtain from above that

$$\sigma_{g_1}^{(p,q)}(f_1\cdot f_2)\le\sigma_{g_1}^{(p,q)}(f_1). \tag{47}$$

Now we establish the equality of (47). Since f_1 and f_2 satisfy Property (X), of course we have $M_{f_1\cdot f_2}(r_1,r_2)>M_{f_1}(r_1,r_2)$ for all sufficiently large values of r_1,r_2. Therefore from the definition of relative (p,q)-th type, we get for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} M_{f_1}(r_1,r_2) &< M_{f_1\cdot f_2}(r_1,r_2) \\ &< M_{g_1}\Big(\exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1\cdot f_2)+\varepsilon)(\log^{[q-1]} r_1)^{\rho_{g_1}^{(p,q)}(f_1)}), \\ &\quad \exp^{[p-1]}((\sigma_{g_1}^{(p,q)}(f_1\cdot f_2)+\varepsilon)(\log^{[q-1]} r_2)^{\rho_{g_1}^{(p,q)}(f_1)})\Big). \end{aligned}$$

So $\sigma_{g_1}^{(p,q)}(f_1\cdot f_2) \geq \sigma_{g_1}^{(p,q)}(f_1)$. Hence $\sigma_{g_1}^{(p,q)}(f_1\cdot f_2) = \sigma_{g_1}^{(p,q)}(f_1)$. Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can verify that $\sigma_{g_1}^{(p,q)}(f_1\cdot f_2) = \sigma_{g_1}^{(p,q)}(f_2)$.

Case II. Let $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ and g_1 satisfies the Property (R). Now for any arbitrary $\varepsilon > 0$, like Case I, we have from (29) and (34) for a sequence of values of r_1, r_2 tending to infinity that

$$\begin{aligned} M_{f_1\cdot f_2}(r_1,r_2) &< \Big(M_{g_1}\Big(\exp^{[p-1]}((\overline{\sigma}_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})(\log^{[q-1]} r_1)^{\rho_{g_1}^{(p,q)}(f_1)}), \\ &\quad \exp^{[p-1]}((\overline{\sigma}_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2})(\log^{[q-1]} r_2)^{\rho_{g_1}^{(p,q)}(f_1)})\Big)\Big)^2. \end{aligned}$$

Now using the similar technique for a sequence of values of r_1, r_2 tending to infinity as explored in the proof of Case I, one can easily verify from above that $\overline{\sigma}_{g_1}^{(p,q)}(f_1\cdot f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ under the conditions specified in the theorem. Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can also verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1\cdot f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_2)$. Therefore the first part of theorem follows from Case I and Case II.

Case III. Let f_1 satisfies the Property (R) and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2. Since $M_{g_1\cdot g_2}(r_1,r_2) \leq M_{g_1}(r_1,r_2)\cdot M_{g_2}(r_1,r_2)$, we have in view of (31) and (33) for a sequence of values of r_1, r_2 tending to infinity that

$$\begin{aligned} M_{g_1\cdot g_2}(r_1,r_2) &< \Big(M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\sigma_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2})}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}, \\ &\quad \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\sigma_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2})}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big)\Big)^2. \end{aligned} \tag{48}$$

Now we observe that $\delta_1 := \frac{\sigma_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}}{\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon} > 1$, which implies

$$\frac{\exp^{[q-2]}\Big(\Big(\frac{\log^{[p-1]} r_i}{\left(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big)}{\exp^{[q-2]}\Big(\Big(\frac{\log^{[p-1]} r_i}{\left(\sigma_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big)} = \delta(\text{say}) > 1, \tag{49}$$

where $i = 1, 2$.

Since f_1 satisfies the Property (R), in view of Definition 8, Theorem 10 and (49) we obtain from (48) for a sequence of values of r_1, r_2 tending to infinity that

$$\begin{aligned} M_{g_1\cdot g_2}(r_1,r_2) &< M_{f_1}\Big(\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\sigma_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2})}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big)^{\delta}, \\ &\quad \Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\sigma_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2})}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big)^{\delta}\Big) \end{aligned}$$

$$\begin{aligned} i.e., M_{g_1\cdot g_2}(r_1,r_2) &< M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1\cdot g_2}^{(p,q)}(f_1)}}, \\ &\quad \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1\cdot g_2}^{(p,q)}(f_1)}}\Big). \end{aligned}$$

Since $\varepsilon>0$ is arbitrary, it follows from above that

$$\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)\geq\sigma_{g_1}^{(p,q)}(f_1). \tag{50}$$

Now we establish the equality of (50). Since g_1 and g_2 satisfy Property (X), of course we have $M_{g_1\cdot g_2}(r_1,r_2)>M_{g_1}(r_1,r_2)$ for all sufficiently large values of r_1,r_2. Therefore from the definition of relative (p,q)-th type we get for a sequence of values of r_1,r_2 tending to infinity that

$$\begin{aligned} M_{g_1}(r_1,r_2) &< M_{g_1\cdot g_2}(r_1,r_2) \\ &< M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}, \\ &\quad \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Big). \end{aligned}$$

So $\sigma_{g_1}^{(p,q)}(f_1)\geq\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)$. Therefore $\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)=\sigma_{g_1}^{(p,q)}(f_1)$. Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1)>\rho_{g_2}^{(p,q)}(f_1)$, then one can verify that $\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)=\sigma_{g_2}^{(p,q)}(f_1)$.

Case IV. Suppose f_1 satisfies the Property (R). Also let $\rho_{g_1}^{(p,q)}(f_1)<\rho_{g_2}^{(p,q)}(f_1)$ with f_1 be of regular relative (p,q) growth with respect to at least any one of g_1 or g_2. Therefore like Case I and in view of (31), we obtain for all sufficiently large values of r_1,r_2 that

$$\begin{aligned} M_{g_1\cdot g_2}(r_1,r_2) &< \Big(M_{f_1}\Big(\exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_1}{(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2})}\Big)^{\frac{1}{\rho_{g_1}^{p,q)}(f_1)}}, \\ &\quad \exp^{[q-1]}\Big(\frac{\log^{[p-1]} r_2}{(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2})}\Big)^{\frac{1}{\rho_{g_1}^{p,q)}(f_1)}}\Big)\Big)^2. \end{aligned}$$

Now using the similar technique for all sufficiently large values of r_1,r_2 as explored in the proof of Case III, one can easily verify that $\overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1)=\overline{\sigma}_{g_1}^{(p,q)}(f_1)$ under the conditions specified in the theorem. Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1)>\rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then one can verify that $\overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1)=\overline{\sigma}_{g_2}^{(p,q)}(f_1)$. Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 11 and the above cases. □

Theorem 21. *Let f_1, f_2, g_1 and g_2 be any four entire functions of two variables. Also let $\lambda_{g_1}^{(p,q)}(f_1)$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ be all non zero and finite.*
(A) *Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*
(i) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 and g_1 satisfy the Property (R) and
(ii) f_1 and f_2 satisfy Property (X); then

$$\tau_{g_1}^{(p,q)}(f_1\cdot f_2)=\tau_{g_1}^{(p,q)}(f_i) \text{ and } \overline{\tau}_{g_1}^{(p,q)}(f_1\cdot f_2)=\overline{\tau}_{g_1}^{(p,q)}(f_i).$$

(B) *Assume the functions g_1, g_2 and f_1 satisfy the following conditions:*
(i) f_1 satisfies the Property (R) and
(ii) g_1 and g_2 satisfy Property (X); then

$$\tau_{g_1\cdot g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \text{ and } \overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1).$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $g_1 \cdot g_2$, f_1 and f_2 are satisfy the Property (R);
(ii) f_1 and f_2 satisfy Property (X);
(iii) g_1 and g_2 satisfy Property (X);
(iv) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 for $i = 1$, 2, $j = 1, 2$ and $i \neq j$;
(v) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_2 for $i = 1$, 2, $j = 1, 2$ and $i \neq j$;
(vi) $\lambda_{g_m}^{(p,q)}(f_l) = \min[\max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}, \max\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\}] \mid l, m = 1, 2$; then

$$\tau_{g_1\cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \tau_{g_m}^{(p,q)}(f_l) \text{ and } \overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \overline{\tau}_{g_m}^{(p,q)}(f_l).$$

We omit the proof of Theorem 21 as it is a natural consequence of Theorem 20.

Theorem 22. *Let f, g and h be any three entire functions of two variables such that $0 < \rho_h^{(m,q)}(f) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$. Then*

$$\begin{aligned} &\max\Big\{\Big(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \Big(\frac{\sigma_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\Big\} \\ \leq \quad & \sigma_g^{(p,q)}(f) \leq \Big(\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}}. \end{aligned}$$

Proof. Let us consider that $\varepsilon(> 0)$ is an arbitrary number. Now from the definitions of $\sigma_g^{(p,q)}(f)$ and $\overline{\sigma}_g^{(p,q)}(f)$, we have for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} M_f(r_1, r_2) \quad ¡ \quad & M_g\big(\exp^{[p-1]}((\sigma_g^{(p,q)}(f) + \varepsilon)(\log^{[q-1]} r_1)^{\rho_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\sigma_g^{(p,q)}(f) + \varepsilon)(\log^{[q-1]} r_2)^{\rho_g^{(p,q)}(f)})\big), \end{aligned} \tag{51}$$

$$\begin{aligned} M_f(r_1, r_2) \quad ¿ \quad & M_g\big(\exp^{[p-1]}((\overline{\sigma}_g^{(p,q)}(f) - \varepsilon)(\log^{[q-1]} r_1)^{\rho_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\overline{\sigma}_g^{(p,q)}(f) - \varepsilon)(\log^{[q-1]} r_2)^{\rho_g^{(p,q)}(f)})\big), \end{aligned} \tag{52}$$

and also for a sequence of values of r_1, r_2 tending to infinity, we get that

$$\begin{aligned} M_f(r_1, r_2) \quad ¿ \quad & M_g\big(\exp^{[p-1]}((\sigma_g^{(p,q)}(f) - \varepsilon)(\log^{[q-1]} r_1)^{\rho_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\sigma_g^{(p,q)}(f) - \varepsilon)(\log^{[q-1]} r_2)^{\rho_g^{(p,q)}(f)})\big), \end{aligned} \tag{53}$$

$$\begin{aligned} M_f(r_1, r_2) \quad ¡ \quad & M_g\big(\exp^{[p-1]}((\overline{\sigma}_g^{(p,q)}(f) + \varepsilon)(\log^{[q-1]} r_1)^{\rho_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\overline{\sigma}_g^{(p,q)}(f) + \varepsilon)(\log^{[q-1]} r_2)^{\rho_g^{(p,q)}(f)})\big). \end{aligned} \tag{54}$$

Similarly from the definitions of $\sigma_h^{(m,p)}(g)$ and $\overline{\sigma}_h^{(m,p)}(g)$, it follows for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} M_g(r_1,r_2) \ \text{¡}\ & M_h\big(\exp^{[m-1]}((\sigma_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_1)^{\rho_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\sigma_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_2)^{\rho_h^{(m,p)}(g)})\big). \end{aligned} \tag{55}$$

$$\begin{aligned} M_g(r_1,r_2) \ \text{¿}\ & M_h\big(\exp^{[m-1]}((\overline{\sigma}_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_1)^{\rho_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\overline{\sigma}_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_2)^{\rho_h^{(m,p)}(g)})\big). \end{aligned} \tag{56}$$

Also for a sequence of values of r_1, r_2 tending to infinity, we obtain that

$$\begin{aligned} M_g(r_1,r_2) \ \text{¿}\ & M_h\big(\exp^{[m-1]}((\sigma_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_1)^{\rho_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\sigma_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_2)^{\rho_h^{(m,p)}(g)})\big), \end{aligned} \tag{57}$$

$$\begin{aligned} M_g(r_1,r_2) \ \text{¡}\ & M_h\big(\exp^{[m-1]}((\overline{\sigma}_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_1)^{\rho_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\overline{\sigma}_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_2)^{\rho_h^{(m,p)}(g)})\big). \end{aligned} \tag{58}$$

Further from the definitions of $\tau_g^{(p,q)}(f)$ and $\overline{\tau}_g^{(p,q)}(f)$,we have for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} M_f(r_1,r_2) \ \text{¡}\ & M_g\big(\exp^{[p-1]}((\tau_g^{(p,q)}(f)+\varepsilon)(\log^{[q-1]} r_1)^{\lambda_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\tau_g^{(p,q)}(f)+\varepsilon)(\log^{[q-1]} r_2)^{\lambda_g^{(p,q)}(f)})\big), \end{aligned} \tag{59}$$

$$\begin{aligned} M_f(r_1,r_2) \ \text{¿}\ & M_g\big(\exp^{[p-1]}((\overline{\tau}_g^{(p,q)}(f)-\varepsilon)(\log^{[q-1]} r_1)^{\lambda_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\overline{\tau}_g^{(p,q)}(f)-\varepsilon)(\log^{[q-1]} r_2)^{\lambda_g^{(p,q)}(f)})\big). \end{aligned} \tag{60}$$

and also for a sequence of values of r_1, r_2 tending to infinity, we get that

$$\begin{aligned} M_f(r_1,r_2) \ \text{¿}\ & M_g\big(\exp^{[p-1]}((\tau_g^{(p,q)}(f)-\varepsilon)(\log^{[q-1]} r_1)^{\lambda_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\tau_g^{(p,q)}(f)-\varepsilon)(\log^{[q-1]} r_2)^{\lambda_g^{(p,q)}(f)})\big), \end{aligned} \tag{61}$$

$$\begin{aligned} M_f(r_1,r_2) \ \text{¡}\ & M_g\big(\exp^{[p-1]}((\overline{\tau}_g^{(p,q)}(f)+\varepsilon)(\log^{[q-1]} r_1)^{\lambda_g^{(p,q)}(f)}), \\ & \exp^{[p-1]}((\overline{\tau}_g^{(p,q)}(f)+\varepsilon)(\log^{[q-1]} r_2)^{\lambda_g^{(p,q)}(f)})\big). \end{aligned} \tag{62}$$

Similarly from the definitions of $\tau_h^{(m,p)}(g)$ and $\overline{\tau}_h^{(m,p)}(g)$, it follows for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} M_g(r_1,r_2) \ \text{¡}\ & M_h\big(\exp^{[m-1]}((\tau_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_1)^{\lambda_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\tau_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_2)^{\lambda_h^{(m,p)}(g)})\big), \end{aligned} \tag{63}$$

$$\begin{aligned} M_g(r_1,r_2) \ \text{¿}\ & M_h\big(\exp^{[m-1]}((\overline{\tau}_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_1)^{\lambda_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\overline{\tau}_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_2)^{\lambda_h^{(m,p)}(g)})\big). \end{aligned} \tag{64}$$

Also for a sequence of values of r_1, r_2 tending to infinity, we obtain that

$$\begin{aligned} M_g(r_1,r_2) \ \text{¿}\ & M_h\big(\exp^{[m-1]}((\tau_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_1)^{\lambda_h^{(m,p)}(g)}), \\ & \exp^{[m-1]}((\tau_h^{(m,p)}(g)-\varepsilon)(\log^{[p-1]} r_2)^{\lambda_h^{(m,p)}(g)})\big), \end{aligned} \tag{65}$$

$$M_g(r_1,r_2) \quad \text{¡} \quad M_h\Big(\exp^{[m-1]}((\overline{\tau}_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_1)^{\lambda_h^{(m,p)}(g)}), \\ \exp^{[m-1]}((\overline{\tau}_h^{(m,p)}(g)+\varepsilon)(\log^{[p-1]} r_2)^{\lambda_h^{(m,p)}(g)})\Big). \tag{66}$$

Since in view of Theorem 13 $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \geq \rho_g^{(p,q)}(f)$, we get from (51) and (63) for all sufficiently large values of r_1, r_2 that

$$M_f(r_1,r_2) \\ < M_h\Big(\exp^{[m-1]}((\tau_h^{(m,p)}(g)+\varepsilon)(\sigma_g^{(p,q)}(f)+\varepsilon)^{\lambda_h^{(m,p)}(g)}(\log^{[q-1]} r_1)^{\rho_h^{(m,q)}(f)}), \\ \exp^{[m-1]}((\tau_h^{(m,p)}(g)+\varepsilon)(\sigma_g^{(p,q)}(f)+\varepsilon)^{\lambda_h^{(m,p)}(g)}(\log^{[q-1]} r_2)^{\rho_h^{(m,q)}(f)})\Big).$$

Since $\varepsilon(>0)$ is arbitrary, we obtain

$$\sigma_h^{(m,q)}(f) \leq \tau_h^{(m,p)}(g)(\sigma_g^{(p,q)}(f))^{\lambda_h^{(m,p)}(g)}$$

$$i.e.,\ \sigma_g^{(p,q)}(f) \geq \Big(\frac{\sigma_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{67}$$

Analogously from (51) and (66), we get that

$$\sigma_g^{(p,q)}(f) \geq \Big(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \tag{68}$$

as in view of Theorem 13 it follows that $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \geq \rho_g^{(p,q)}(f)$. Further in view of Theorem 13, since $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$, we obtain from (53) and (56) for a sequence of values of r_1, r_2 tending to infinity that

$$M_f(r_1,r_2) \\ > M_h\Big(\exp^{[m-1]}((\overline{\sigma}_h^{(m,p)}(g)-\varepsilon)(\sigma_g^{(p,q)}(f)-\varepsilon)^{\rho_h^{(m,p)}(g)}(\log^{[q-1]} r_1)^{\rho_h^{(m,q)}(f)}), \\ \exp^{[m-1]}((\overline{\sigma}_h^{(m,p)}(g)-\varepsilon)(\sigma_g^{(p,q)}(f)-\varepsilon)^{\rho_h^{(m,p)}(g)}(\log^{[q-1]} r_2)^{\rho_h^{(m,q)}(f)})\Big).$$

Since $\varepsilon(>0)$ is arbitrary, we obtain

$$\sigma_h^{(m,q)}(f) \geq \overline{\sigma}_h^{(m,p)}(g)(\sigma_g^{(p,q)}(f))^{\rho_h^{(m,p)}(g)}$$

$$i.e.,\ \sigma_g^{(p,q)}(f) \leq \Big(\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}}. \tag{69}$$

Thus the theorem follows from (67), (68) and (69). □

From Theorem 13, it follows that $\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$ and $\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$, therefore the conclusion of the following theorem can be carried out from (53) and (56); (53) and (64) respectively after applying the same technique of Theorem 22. So its proof is omitted.

Theorem 23. *Let f, g and h be any three entire functions of two variables such that $0 < \lambda_h^{(m,q)}(f) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$. Then*

$$\sigma_g^{(p,q)}(f) \leq \min\Big\{\Big(\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \Big(\frac{\tau_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}}\Big\}.$$

Similarly in the line of Theorem 22 and with the help of Theorem 13, one may easily carry out the following theorem from pairwise inequalities numbers (62) and (63); (56) and (60); (57) and (60); respectively and therefore its proofs is omitted:

Theorem 24. *Let f, g and h be any three entire functions of two variables such that $0<\lambda_h^{(m,q)}(f)\le\rho_h^{(m,q)}(f)<\infty$ and $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$. Then*

$$\begin{aligned}\Big(\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}} &\le \overline{\tau}_g^{(p,q)}(f)\\ &\le \min\Big\{\Big(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}},\Big(\frac{\tau_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}}\Big\}.\end{aligned}$$

Theorem 25. *Let f, g and h be any three entire functions of two variables such that $0<\rho_h^{(m,q)}(f)<\infty$ and $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$. Then*

$$\overline{\tau}_g^{(p,q)}(f)\ge\max\Big\{\Big(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}},\Big(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\Big\}.$$

With the help of Theorem 13, the conclusion of the above theorem can be carried out from (55), (62) and (62), (63) respectively after applying the same technique of Theorem 22 and therefore its proof is omitted.

Similarly in view of Theorem 13, the conclusion of the following theorem can be carried out from pairwise inequalities numbered (54) and (63); (52) and (57); (52) and (56) respectively after applying the same technique of Theorem 22 and therefore its proof is omitted.

Theorem 26. *Let f, g and h be any three entire functions of two variables such that $0<\rho_h^{(m,q)}(f)<\infty$ and $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$. Then*

$$\begin{aligned}\Big(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}} &\le \overline{\sigma}_g^{(p,q)}(f)\\ &\le \min\Big\{\Big(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}},\Big(\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}}\Big\}.\end{aligned}$$

Theorem 27. *Let f, g and h be any three entire functions of two variables such that $0<\lambda_h^{(m,q)}(f)<\infty$ and $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$. Then*

$$\begin{aligned}\overline{\sigma}_g^{(p,q)}(f) \le \min\Big\{&\Big(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}},\Big(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\Big)^{\frac{1}{\lambda_h^{(m,p)}(g)}},\\ &\Big(\frac{\tau_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}},\Big(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\Big)^{\frac{1}{\rho_h^{(m,p)}(g)}}\Big\}.\end{aligned}$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (52) and (64); (52) and (65); (52) and (57); (52) and (56) respectively after applying the same technique of Theorem 22 and with the help of Theorem 13. Therefore its proof is omitted.

Similarly in the line of Theorem 22 and with the help of Theorem 13, one may easily carry out the following theorem from pairwise inequalities numbered

(59) and (63); (59) and (66); (56) and (61) respectively and therefore its proof is omitted:

Theorem 28. *Let f, g and h be any three entire functions of two variables such that $0 < \lambda_h^{(m,q)}(f) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$. Then*

$$\max\left\{\left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq \tau_g^{(p,q)}(f) \leq \left(\frac{\tau_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Theorem 29. *Let f, g and h be any three entire functions of two variables such that $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$. Then*

$$\tau_g^{(p,q)}(f) \geq \max\left\{\left(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (58) and (59); (55) and (59); (59) and (63); (59) and (66) respectively after applying the same technique of Theorem 22 and with the help of Theorem 13. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because those can be derived easily using the same technique or with some easy reasoning with the help of Remark 4 and therefore left to the readers.

Theorem 30. *Let f, g and h be any three entire functions of two variables such that $0 < \rho_h^{(m,q)}(f) < \infty$ and $0 < \rho_h^{(m,p)}(g)$ $(= \lambda_h^{(m,p)}(g)) < \infty$. Then*

$$\begin{aligned}
&\left(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \leq \overline{\sigma}_g^{(p,q)}(f) \\
&\leq \min\left\{\left(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \\
&\leq \max\left\{\left(\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \\
&\leq \sigma_g^{(p,q)}(f) \leq \left(\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.
\end{aligned}$$

Remark 5. *In Theorem 30, if we will replace the conditions "$0 < \rho_h^{(m,q)}(f) < \infty$ and $0 < \rho_h^{(m,p)}(g)$ $(= \lambda_h^{(m,p)}(g)) < \infty$" by "$0 < \rho_h^{(m,q)}(f)$ $(= \lambda_h^{(m,q)}(f)) < \infty$ and $0 < \rho_h^{(m,p)}(g) < \infty$" respectively, then Theorem 30 remains valid with $\overline{\tau}_g^{(p,q)}(f)$ and $\tau_g^{(p,q)}(f)$ replaced by $\overline{\sigma}_g^{(p,q)}(f)$ and $\sigma_g^{(p,q)}(f)$ respectively.*

Theorem 31. *Let f, g and h be any three entire functions of two variables such that $0 < \rho_h^{(m,q)}(f)$ $(= \lambda_h^{(m,q)}(f)) < \infty$ and $0 < \lambda_h^{(m,p)}(g) < \infty$. Then*

$$\left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \overline{\sigma}_g^{(p,q)}(f)$$

$$\leq \min\left\{\left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$

$$\leq \max\left\{\left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$

$$\leq \sigma_g^{(p,q)}(f) \leq \left(\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

Remark 6. *In Theorem 31, if we will replace the conditions "$0 < \rho_h^{(m,q)}(f)$ $(= \lambda_h^{(m,q)}(f)) < \infty$ and $0 < \lambda_h^{(m,p)}(g) < \infty$" by "$0 < \lambda_h^{(m,q)}(f) < \infty$ and $0 < \rho_h^{(m,p)}(g)$ $(= \lambda_h^{(m,p)}(g)) < \infty$" respectively, then Theorem 31 remains valid with $\overline{\tau}_g^{(p,q)}(f)$ and $\tau_g^{(p,q)}(f)$ replaced by $\overline{\sigma}_g^{(p,q)}(f)$ and $\sigma_g^{(p,q)}(f)$ respectively.*

4. Conclusion

Throughout this article, we have generalized some previous results introducing the concepts of (p, q)-th order and (p,q)-th type of entire functions of two complex variables. Further it is interesting to extend the results of this paper for more than two complex variables which can easily be carried out by any interested reader or the involved author.

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