*Journal of Fractional Calculus and Applications*
Vol. 14(1) Jan. 2023, pp. 88-102
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

# EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. In this paper, we investigate the existence of positive solutions for a class of nonlinear boundary value problem of fractional differential equations with integral boundary conditions. The procedure is based on the fixed point theorems due to the Avery-Peterson and the Guo-Krasnoselskii theorem for obtaining the main results to the fractional order integral boundary value problem.

## 1. Introduction

In this study, we consider the following fractional order integral boundary value problem (FIBVP)

$$\begin{cases} D^{\beta}(\varphi_p({}^cD^{\alpha}y(t))) + r(t)f(t,y(t),y'(t)) = 0, & t \in [0,1], \\ y(0) = y''(0) = 0, \quad y(1) = k\int_0^1 y(s)ds, \\ \varphi_p({}^cD^{\alpha}y(0)) = [\varphi_p({}^cD^{\alpha}y(0))]' = 0, \end{cases} \tag{1}$$

where $2 < \alpha \leq 3,\ 1 < \beta \leq 2,\ 0 < k < 2,\ {}^cD^{\alpha}$ and $D^{\beta}$ are the Caputo and Riemann-Liouville derivatives respectively and $\varphi_p(y) = |y|^{p-2}y$ such that $p > 1$, $\varphi_p^{-1} = \varphi_q$ with $1/p + 1/q = 1$.

Several studies have been aligned to fractional calculus and these aimed at a wide area of applications which include biological science, feedback amplifiers, electrical circuits and physics related areas, see [3],[5],[7],[8]. Numerous work on fractional calculus initially focused on solvability of linear initial value fractional differential equations with unique functions [12]. Recently, advances in the theory of fractional calculus has led to the inception and a notable appreciation of fractional differential equations with its widespread application in physics, chemistry, engineering, mechanics and so forth, see [9]-[11]. With time, much more work was reoriented to existence and multiplicity of positive solutions of nonlinear initial value fractional differential equations by means of fixed point theorems such as the Krasnosel'skii

2010 *Mathematics Subject Classification.* 30E25; 34A08; 34B18; 34G20; 34K06; 34K37.
*Key words and phrases.* Riemann-Liouville and Caputo fractional derivative, $p$-Laplacian operator, integral boundary value problem, Avery and Peterson fixed point theorem, Guo-Krasnoselski fixed point theorem.
Submitted Nov. 22, 2021. Revised March 4, 2022.

fixed point theorem among others [26]-[28]. Moreover, existence of positive solutions of boundary value problems with integer order differential equation extended to $p$-Laplacian boundary value problems were investigated by [12]-[16], of which many authors have concentrated on, in recent years. Some work incorporating boundary value problems with integral boundary conditions have been studied on the existence and nonexistence of positive solutions on such nonlinear fractional differential equations, see [12],[17]-[20].
A small number of papers have covered nonlinear fractional differential equations with a $p$-Laplacian operator some of which include [21],[22]. Furthermore, different types of fractional-order derivatives have been widely investigated exclusively from each other in various fractional differential equations. A mouthful of fractional differential equations with mixed fractional -order derivatives have not been sufficiently studied, some of the few include [23]-[25].
Despite the existence of the afore-mentioned literature and other prior work, to the best of the authors' knowledge, hardly any work involves the existence of multiple positive solutions of nonlinear fractional differential equations with integral boundary value conditions and mixed Caputo-Riemann Liouville fractional-order derivatives. In this study we address this lag, one aspect to note is that the nonlinear fractional differential equation we consider herein consists of two nonlinear terms, which are $p$-Laplacian operator and the $f$ term dependent on the first order derivative $y'$.

This paper is organized in such a manner, Section 2 presents some necessary background material, lemmas, definitions and Green's function with its properties. In Section 3, the main results are derived. This section deals with the existence of the single and the multiple positive solutions for the FIBVP (1) based on the fixed point theorems.

In the entire paper, we assume the following conditions hold:

$(H_1)$ $f:[0,1]\times[0,+\infty)\times(-\infty,+\infty)\to[0,\infty)$ is continuous;
$(H_2)$ $r\in C([0,1],[0,+\infty))$ and there exists $0<\omega<1$ such that $\int_\omega^1 G(1,s)\varphi_q(I^\beta r(s))ds>0$.

## 2. Basic Definitions and Preliminaries

In this section, we introduce some necessary definitions and lemmas.

**Definition 2.1.** [6] The integral

$$I^\beta g(t)=\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds, \tag{2}$$

where $\beta>0$, is the fractional integral of order $\beta$ for a function $g(t)$.

**Definition 2.2.** [6] For a function $g(t)$ the expression

$$D_{0+}^\beta g(t)=\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\beta-1}g(s)ds, \tag{3}$$

is called the Riemann-Liouville fractional derivative of order $\beta$, where $n=[\beta]+1$, and $[\beta]$ denotes the integer part of number $\beta$.

**Definition 2.3.** [6] The $\alpha$ order Caputo fractional derivatives for a function $f(t)$ is defined as follows:

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n-1 < \alpha < n. \tag{4}$$

**Definition 2.4.** [2] Let $P \subseteq K$ be a nonempty, convex closed set and $K$ a real Banach space. Then $P$ is called a cone in $K$ provided that

(1) $\lambda y \in P$, for all $y \in P$ and $\lambda \geq 0$;
(2) $y, -y \in P$ implies that $y = 0$.

**Definition 2.5.** [2] Let $P$ be a cone in real Banach space $K$. If the map $\Upsilon : P \to [0, \infty)$ is continuous and satisfies

$$\Upsilon(tx + (1-t)y) \geq t\Upsilon(x) + (1-t)\Upsilon(y), \quad x, y \in P, \ t \in [0,1],$$

then $\Upsilon$ is called a nonnegative continuous concave functional on $P$.
In a similar way, the map $\omega$ is a nonnegative continuous convex function on a cone $P$ of a real Banach space $K$ provided that $\omega \to [0, \infty)$ is continuous and

$$\omega(tx + (1-t)y) \leq t\omega(x) + (1-t)\omega(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

**Lemma 2.1.** [1] *Assume that $g \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\beta > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^\beta D^\beta g(t) = g(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_N t^{\beta-N}, \tag{5}$$

*for some $c_i \in \mathbb{R}, \ i = 1, 2, \cdots, N$, where $N$ is the smallest integer greater than or equal to $\beta$.*

**Lemma 2.2.** [2] *Assume that $\alpha > 0$ and $n = [\alpha] + 1$. If the function $y \in L[0,1] \cap C[0,1]$, then there exists $c_i \in \mathbb{R}, \ i = 1, 2, \ldots, n$, such that*

$$I^\alpha(^cD^\alpha f(t)) = f(t) - c_1 - c_2 t \cdots - c_n t^{n-1}. \tag{6}$$

**Lemma 2.3.** *The FIBVP (1) has a unique solution as follows:*

$$y(t) = \int_0^1 G(t,s)\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds, \quad t \in [0,1] \tag{7}$$

*where*

$$G(t,s) = \begin{cases} \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))-\alpha(2-k)(t-s)^{\alpha-1}}{(2-k)\Gamma(\alpha+1)}, & 0 \leq s \leq t \leq 1, \\ \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{8}$$

*Proof.* Let $u(t) = \varphi_p(^cD^\alpha y(t))$, we now show that the problem (1) can be expressed as the following IBVPs:

$$\begin{cases} D^\beta u(t) + r(t)f(t, y(t), y'(t)) = 0, \\ u(0) = u'(0) = 0 \end{cases} \tag{9}$$

and

$$\begin{cases} ^cD^\alpha y(t) = \varphi_q(u(t)), \\ y(0) = y''(0) = 0, \quad y(1) = k\int_0^1 y(s)ds. \end{cases} \tag{10}$$

Using Lemma 2.1 and (9), we get

$$u(t) = -I^{\beta}(r(t)f(t, y(t), y'(t))) + c_1 t^{\beta-1} + c_2 t^{\beta-2},$$

since $u(0) = u'(0) = 0$, then $c_1 = c_2 = 0$ and we have

$$\begin{aligned} u(t) =& -I^{\beta}(r(t)f(t, y(t), y'(t))) \\ =& \frac{-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} r(s) f(s, y(s), y'(s)) ds. \end{aligned} \tag{11}$$

Also, from (10) and Lemma 2.2

$$y(t) = -I^{\alpha}\varphi_q(I^{\beta}(r(t)f(t, y(t), y'(t)))) + c_0 + c_1 t + c_2 t^2,$$

since $y(0) = y''(0) = 0$, then $c_0 = c_2 = 0$,

$$y(t) = -I^{\alpha}\varphi_q(I^{\beta}(r(t)f(t, y(t), y'(t)))) + c_1 t. \tag{12}$$

From condition $y(1) = k\int_0^1 y(s)ds$ of (10), we get

$$y(1) = k\int_0^1 y(s)ds = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds + c_1,$$

then

$$c_1 = k\int_0^1 y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds$$

substituting for $c_1$ into (12) implies that

$$\begin{aligned} y(t) =& -I^{\alpha}\varphi_q(I^{\beta}(r(t)f(t, y(t), y'(t)))) + kt\int_0^1 y(s)ds \\ &+ t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds. \end{aligned} \tag{13}$$

Let $H = \int_0^1 y(t)dt$, then from (13), we have

$$\begin{aligned} H =& -\int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))dsdt + \int_0^1 ktHdt \\ &+ \int_0^1 t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))dsdt \\ =& -\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds + \frac{k}{2}H \\ &+ \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds \end{aligned}$$

and so

$$\begin{aligned} \frac{(2-k)}{2}H =& -\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds \\ &+ \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^{\beta}(r(s)f(s, y(s), y'(s))))ds \end{aligned}$$

thus

$$H = -\frac{2}{2-k}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds + \frac{1}{2-k}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds. \quad (14)$$

Substituting (14) into (13), we get

$$\begin{aligned} y(t) =& -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ &- \frac{2kt}{2-k}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ &+ \frac{kt}{2-k}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ &+ t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ =& \int_0^1 \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ &- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ =& \int_0^t \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s)) - \alpha(t-s)^{\alpha-1}(2-k)}{(2-k)\Gamma(\alpha+1)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ &+ \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds \\ =& \int_0^1 G(t,s)\varphi_q(I^{\beta}(r(s)f(s,y(s),y'(s))))ds. \end{aligned}$$

This completes the proof. □

Now, we will give some inequalities satisfied by the Green's function of the problem.

**Lemma 2.4.** [6] *The function $G(t,s)$ defined in (8) satisfies the following properties:*

1. $0 < G(t,s) \le \frac{2}{(2-k)\Gamma(\alpha)}$, *for $t,s \in (0,1)$ if and only if $0 < k < 2$.*
2. $tG(1,s) \le G(t,s) \le \frac{2\alpha}{k(\alpha-2)}G(1,s)$, *for all $t,s \in (0,1)$, $2 < \alpha < 3$ and $0 < k < 2$.*

The following fixed point theorems and the definition are fundamental and important to the proof of our main results.

**Theorem 2.5.** [31] *Let $K$ be a Banach space, $P \subset K$ a cone, and $\Omega_1, \Omega_2$ two bounded open subsets of $K$ centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Assume that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous operator such that either of the following holds:*

1. $\|Ty\| \le \|y\|, y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|, y \in P \cap \partial\Omega_2$,
2. $\|Ty\| \ge \|y\|, y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \le \|y\|, y \in P \cap \partial\Omega_2$.

*Then $T$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

**Definition 2.6.** [30] A completely continuous operator is continuous and maps bounded sets into pre-compact sets. If $\omega_1$ and $\omega_2$ be nonnegative continuous convex functionals on $P$, $\Upsilon$ be a nonnegative continuous concave functional on $P$ and $\vartheta$ be a nonnegative continuous functional on $P$. Therefore, for positive real numbers $a, b, c$ and $d$, we denote the following convex sets

$$\begin{aligned}
&P(\omega_1, d) = \{x \in P | \omega_1(x) < d\},\\
&P(\omega_1, \Upsilon, b, d) = \{x \in P | b \leq \Upsilon(x), \omega_1(x) \leq d\},\\
&P(\omega_1, \omega_2, \Upsilon, b, c, d) = \{x \in P | b \leq \Upsilon(x), \omega_2(x) \leq c, \omega_1(x) \leq d\}
\end{aligned}$$

and a closed set

$$R(\omega_1, \vartheta, a, d) = \{x \in P | a \leq \vartheta(x), \omega_1(x) \leq d\}.$$

**Theorem 2.6.** [30] *Let $P$ be a cone in a real Banach space $K$. Let $\omega_1$ and $\omega_2$ be nonnegative continuous convex functionals on $P$, $\Upsilon$ be a nonnegative continuous concave functional on $P$ and $\vartheta$ be a nonnegative continuous functional on $P$ satisfying $\vartheta(\lambda x) \leq \lambda \vartheta(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $X$ and $d$,*

$$\Upsilon(x) \leq \vartheta(x) \text{ and } \|x\| \leq X \omega_1(x) \tag{15}$$

*for all $x \in \overline{P(\omega_1, d)}$. Suppose $T : \overline{P(\omega_1, d)} \to \overline{P(\omega_1, d)}$ is completely continuous and there exist positive numbers $a, b$ and $c$ with $a < b$ such that*

- $(C_1)$ *$\{x \in P(\omega_1, \omega_2, \Upsilon, b, c, d) | \Upsilon(x) > b\} \neq \Phi$ and $\Upsilon(Tx) > b$ for $x \in P(\omega_1, \omega_2, \Upsilon, b, c, d)$;*
- $(C_2)$ *$\Upsilon(Tx) > b$ for $x \in P(\omega_1, \Upsilon, b, d)$ with $\omega_2(Tx) > c$;*
- $(C_3)$ *$0 \notin R(\omega_1, \vartheta, a, d)$ and $\vartheta(Tx) < a$ for $x \in R(\omega_1, \vartheta, a, d)$ with $\vartheta(x) = a$.*

*Then, $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\omega_1, d)}$ such that*

$$\begin{aligned}
&\omega_1(x_i) \leq d \quad \text{for} \quad i = 1, 2, 3,\\
&b < \Upsilon(x_1),\\
&a < \vartheta(x_2) \quad \text{with} \quad \Upsilon(x_2) < b,\\
&\vartheta(x_3) < a.
\end{aligned}$$

## 3. Main results

We consider the Banach space $K = (C^1[0,1], \|\cdot\|)$ endowed with maximum norm

$$\|y\| = \max \left\{ \max_{0 \leq t \leq 1} |y(t)|, \max_{0 \leq t \leq 1} |y'(t)| \right\}.$$

We denote $C^{1+}[0,1] = \{\eta \in C^1[0,1] | \eta(t) \geq 0, t \in [0,1]\}$. Let the cone $P \subset K$ be defined by

$$\begin{aligned}
P = \{y \in K | y(t) \geq 0, y(0) = y''(0) = \varphi_p({}^cD^\alpha y(0)) = [\varphi_p({}^cD^\alpha y(0))]' = 0,\\
y(1) = k \int_0^1 y(s) ds, y \text{ is a concave on } [0,1]\}.
\end{aligned}$$

We define an operator $T : P \to P$ as

$$Ty(t) := \int_0^1 G(t,s) \varphi_q(I^\beta(r(s) f(s, y(s), y'(s)))) ds, \tag{16}$$

where $G$ is defined in (8).

Nonnegative continuous concave functional $\Upsilon$, nonnegative continuous convex functionals $\omega_2, \omega_1$ and the nonnegative continuous functional $\vartheta$ according to [29] are defined on the cone $P$ by

$$\omega_1(y) = \max_{0\le t\le 1} |y'(t)|, \vartheta(y) = \omega_2(y) = \max_{0\le t\le 1} |y(t)|, \Upsilon(y) = \min_{\frac{1}{4}\le t\le \frac{3}{4}} |y(t)|.$$

**Lemma 3.1.** [30] *If $y \in P$, then*

$$\max_{0\le t\le 1} |y(t)| \le \max_{0\le t\le 1} |y'(t)|.$$

**Lemma 3.2.** *$T : P \to P$ is completely continuous.*

*Proof.* By the continuity and the non-negativeness of $G$ and $f$ on their domains of definition, we see that if $y \in P$, then $Ty \in K$ and $Ty(t) \ge 0$ for all $t \in [0, 1]$.
We proceed to show that $T(P) \subset P$, we take $y \in P$, then

$$Ty(0) = \int_0^1 G(0, s)\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds = 0$$

and

$$\begin{aligned}
Ty(1) =& \int_0^1 G(1, s)\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds \\
=& \int_0^1 \frac{(1-s)^{\alpha-1}}{(2-k)\Gamma(\alpha+1)}[2(\alpha - k + ks) - (2-k)\alpha]\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds \\
=& k\int_0^1 \frac{(1-s)^{\alpha-1}}{(2-k)\Gamma(\alpha+1)}[2s - 2 + \alpha]\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds \\
=& k\int_0^1 Ty(s)ds,
\end{aligned}$$

also

$$\begin{aligned}
(Ty)''(t) =& \int_0^1 \frac{\partial^2 G(t, s)}{\partial t^2}\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds \\
=& \frac{\alpha(\alpha-1)(\alpha-2)}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-3}\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds \\
\le& 0.
\end{aligned}$$

We notice that $Ty(t)$ is concave and $(Ty)''(0) = 0$. By the continuity of $G$ and $f$, the operator $T : P \to P$ is continuous. Let $\Omega \subset P$ be bounded. Then, for all $t \in [0, 1]$ and $y \in \Omega$, there exists a positive constant $M$ such that $|f(t, y(t), y'(t))| \le M$. Thus,

$$\begin{aligned}
|(Ty)(t)| =& \left|\int_0^1 G(t, s)\varphi_q(I^\beta(r(s)f(s, y(s), y'(s))))ds\right| \\
\le& \int_0^1 |G(t, s)| \left(\int_0^s (s-\tau)^{\beta-1}d\tau\right)^{q-1} ds \left(\frac{\|r\|M}{\Gamma(\beta)}\right)^{q-1} \\
\le& \frac{(\|r\|M)^{q-1}}{(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{(q-1)\beta}|G(t, s)|ds \\
\le& \frac{2(\|r\|M)^{q-1}}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}.
\end{aligned}$$

This implies that $T(\Omega)$ is uniformly bounded. On the other hand, for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$,

$$|G(t_1,s) - G(t_2,s)| < \frac{(\|r\|M)^{q-1}}{(\Gamma(\beta+1))^{q-1}}\epsilon.$$

Hence, for all $y \in \Omega$,

$$\begin{aligned}
|(Ty)(t_2) - (Ty)(t_1)| = & \int_0^1 |G(t_2,s) - G(t_1,s)|\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \\
\leq & \int_0^1 |G(t_2,s) - G(t_1,s)| \left(\int_0^s (s-\tau)^{\beta-1}d\tau\right)^{q-1} ds \left(\frac{\|r\|M}{\Gamma(\beta)}\right)^{q-1} \\
\leq & \frac{(\|r\|M)^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{(q-1)\beta}|G(t_2,s) - G(t_1,s)|ds \\
\leq & \frac{(\|r\|M)^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 |G(t_2,s) - G(t_1,s)|ds \\
= & \epsilon,
\end{aligned}$$

which means that $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we see that $T : P \to P$ is completely continuous. The proof is complete. □

We now present the sufficient conditions of the operator $T$ applying the Avery-Peterson fixed theorem for the existence of at least three positive solutions to the FIBVP (1).
Let

$$\begin{aligned}
E = & \frac{\|r\|^{q-1}}{(\Gamma(\beta+1))^{q-1}} \max\left\{\left|\int_0^1 \frac{2s^{\beta(q-1)}(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\Gamma(\alpha+1)}ds\right|,\right. \\
& \left.\left|\int_0^1 s^{\beta(q-1)}\left(\frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\Gamma(\alpha+1)} - \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)ds\right|\right\}, \\
N = & \left[\frac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi_q(I^\beta r(s))ds\right]^{-1}, \\
Z = & \left[\frac{2\alpha}{k(\alpha-2)}\int_0^1 G(1,s)\varphi_q(I^\beta r(s))ds\right]^{-1},
\end{aligned}$$

where $\|r\| = \max_{0\leq t\leq 1}|r|$.

**Theorem 3.3.** *Suppose there exist constants $0 < a < b < c < d$ where $c = \frac{8\alpha}{k(\alpha-2)}b$, and assume that $f$ satisfies the following conditions:*

$(N_1)$ $f(t,u,v) \leq \varphi_p\left(\frac{d}{E}\right)$ *for* $(t,u,v) \in [0,1] \times [0,d] \times [-d,d]$;
$(N_2)$ $f(t,u,v) > \varphi_p(Nb)$ *for* $(t,u,v) \in [\frac{1}{4}, \frac{3}{4}] \times [b,c] \times [-d,d]$;
$(N_3)$ $f(t,u,v) < \varphi_p(Za)$ *for* $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$.

*Then, the FIBVP (1) has at least three positive solutions* $y_1, y_2$ *and* $y_3$ *satisfying*

$$\begin{aligned}
&\max_{0\leq t\leq 1}|y_i'(t)|\leq d, \text{ for } i=1,2,3;\\
&b<\min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|y_1(t)|;\\
&a<\max_{0\leq t\leq 1}|y_2(t)|, \text{ with } \min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|y_2(t)|<b;\\
&\max_{0\leq t\leq 1}|y_3(t)|<a.
\end{aligned}$$

*Proof.* Problem (1) has a solution $y=y(t)$ if and only if $y$ is a solution to the operator equation $y=Ty=\int_0^1 G(t,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds$. We now show that the operator $T$ satisfies the Avery-Peterson fixed point theorem which proves the existence of three fixed points of $T$. If $y\in\overline{P(\omega_1,d)}$, then $\omega_1(y)=\max_{0\leq t\leq 1}|y'(t)|\leq d$. From Lemma 3.1 we get

$$\max_{0\leq t\leq 1}|y(t)|\leq\max_{0\leq t\leq 1}|y'(t)|\leq d,$$

then, assumption $(N_1)$ implies that $f(t,y(t))\leq\varphi_q(\frac{d}{E})$, for $y$ nonnegative on $J$. Adversely, for $y\in P$, there exists $Ty\in P$, then $Ty$ is concave on $[0,1]$ and $\max_{t\in[0,1]}|(Ty)'(t)|=\max\{|(Ty)'(0)|,|(Ty)'(1)|\}$, thus

$$\begin{aligned}
\omega_1(Ty(t))=&\max_{0\leq t\leq 1}|(Ty)'(t)|\\
=&\max\{|(Ty)'(0)|,|(Ty)'(1)|\}\\
=&\max\left\{\left|\int_0^1\frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\Gamma(\alpha+1)}\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds\right|,\right.\\
&\left.\left|\int_0^1\left(\frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\Gamma(\alpha+1)}-\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds\right|\right\}\\
\leq&\frac{d\|r\|^{q-1}}{E(\Gamma(\beta+1))^{q-1}}\max\left\{\left|\int_0^1\frac{2s^{\beta(q-1)}(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\Gamma(\alpha+1)}ds\right|,\right.\\
&\left.\left|\int_0^1 s^{\beta(q-1)}\left(\frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\Gamma(\alpha+1)}-\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)ds\right|\right\}\\
=&d.
\end{aligned}$$

Therefore, $T:\overline{P(\omega_1,d)}\to\overline{P(\omega_1,d)}$.
To verify condition $(C_1)$ of Theorem 2.6, we assign $y(t)=\frac{1}{2}(b+c)$, for $0<t<1$. We can easily see that $y(t)=\frac{1}{2}(b+c)\in P(\omega_1,\omega_2,\Upsilon,b,c,d)$ and $\Upsilon(y)=\Upsilon(\frac{b+c}{2})=\frac{1}{2}\left(\frac{8\alpha}{k(\alpha-2)}b+b\right)=\frac{8\alpha+k(\alpha-2)}{2k(\alpha-2)}b>b$, thus $\{y\in P(\omega_1,\omega_2,\Upsilon,b,c,d)|\Upsilon(y)>b\}\neq\Phi$. If $y\in P(\omega_1,\omega_2,\Upsilon,b,c,d)$,

then $b < y(t) \leq c, |y'(t)| \leq d$ for $\frac{1}{4} \leq t \leq \frac{3}{4}$. From assumption $(N_2)$, we get

$$\begin{aligned}
\Upsilon(Ty(t)) &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} |Ty(t)| \\
&= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \left| \int_0^1 G(t,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\geq \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} t \left| \int_0^1 G(1,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\geq \frac{1}{4} \left| \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\geq \frac{1}{4} Nb \left| \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi_q(I^\beta r(s))ds \right| \\
&= b.
\end{aligned}$$

This shows that condition $(C_1)$ of Theorem 2.6 is satisfied. We proceed by verifying condition $(C_2)$ of Theorem 2.6. For all $y \in P(\omega_1, \Upsilon, b, d)$ and $\omega_2(Ty) > c$, we have

$$\begin{aligned}
\Upsilon(Ty(t)) &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} |Ty(t)| \\
&\geq \frac{1}{4} \left| \int_0^1 G(1,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\geq \frac{1}{4} \frac{k(\alpha-2)}{2\alpha} \left| \int_0^1 \max_{0\leq t\leq 1}\{G(t,s)\}\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\geq \frac{k(\alpha-2)}{8\alpha} \max_{0\leq t\leq 1} \left| \int_0^1 G(t,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\geq \frac{k(\alpha-2)}{8\alpha} \omega_2(Ty(t)) \\
&> \frac{k(\alpha-2)}{8\alpha} c = b.
\end{aligned}$$

Therefore, condition $(C_2)$ of Theorem 2.6 is satisfied. We then show that $(C_3)$ of Theorem 2.6 also holds. Evidently, when $\vartheta(0) = 0 < a, 0 \notin R(\omega_1, \vartheta, a, d)$ holds. If $y \in R(\omega_1, \vartheta, a, d)$ with $\vartheta(y) = a$. Then, by assumption $(N_3)$, we have

$$\begin{aligned}
\vartheta(Ty(t)) &= \max_{0\leq t\leq 1} |Ty(t)| \\
&\leq \max_{0\leq t\leq 1} \left| \int_0^1 G(t,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\leq \left| \int_0^1 \max_{0\leq t\leq 1}\{G(t,s)\}\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&\leq \frac{2\alpha}{k(\alpha-2)} \left| \int_0^1 G(t,s)\varphi_q(I^\beta(r(s)f(s,y(s),y'(s))))ds \right| \\
&< \frac{2\alpha}{k(\alpha-2)} Za \left| \int_0^1 G(1,s)\varphi_q(I^\beta r(s))ds \right| = a.
\end{aligned}$$

Therefore, condition $(C_3)$ of Theorem 2.6 is satisfied. Thus, applying Theorem 2.6 implies that the integral boundary value problem (1) has at least three positive solutions $y_1, y_2$ and $y_3$. □

**Example 3.1.** Consider the fractional differential equation:

$$\begin{cases} D^{\frac{3}{2}}(\varphi_2({}^cD^{\frac{5}{2}}y(t))) + r(t)f(t,y(t),y'(t)) = 0, & t \in [0,1], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{1}{2}\int_0^1 y(s)ds, \\ \varphi_2({}^cD^{\frac{5}{2}}y(0)) = [\varphi_2({}^cD^{\frac{5}{2}}y(0))]' = 0. \end{cases} \tag{17}$$

where $r(t) = 7(1-t)$ and

$$f(t,y,y') = \begin{cases} \frac{t}{100} + 20y^3 + \left(\frac{|y'|}{1000}\right)^3, & y \leq 1, \\ \frac{t}{100} + 19 + y + \left(\frac{|y'|}{1000}\right)^3, & y \geq 1. \end{cases}$$

We set $a = 0.1, b = 1$ and $d = 100$. By computations, $c = 80, E = 0.81667, N = 19.169$ and $Z = 0.35880$. As a result, $f(t,y,v)$ satisfies

$$f(t,y,y') < \varphi_2\left(\frac{d}{E}\right) \approx 122.4485, \text{ for } (t,y,y') \in [0,1] \times [0,100] \times [-10^2, 10^2],$$

$$f(t,y,y') > \varphi_2(Nb) \approx 19.169, \text{ for } (t,y,y') \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [1,80] \times [-10^2, 10^2],$$

$$f(t,y,y') < \varphi_2(Za) \approx 0.035880, \text{ for } (t,y,y') \in [0,1] \times [0,0.1] \times [-10^2, 10^2].$$

Since all conditions of Theorem 3.3 hold. Therefore, the problem (17) has at least three positive solutions.

In this part, we impose some conditions on $f$ which allow us to apply Theorem 2.5 to establish the existence of at least a single positive solution for the FIBVP (1).

**Theorem 3.4.** *Assume* $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ *is continuous and there exist positive constants* $C, k_i$ *and* $\sigma_i \in (0,1)$ *where* $i = 1,2$ *such that*

$(A_1)$ $f(t,u,v) \leq \varphi_p(C + k_1|u|^{\sigma_1} + k_2|v|^{\sigma_2})$ *and* $f(t,u,v) \neq 0, (t,u,v) \in [0,1] \times [0,+\infty) \times (-\infty,+\infty)$.

*Then the FIBVP* (1) *has at least one positive solution.*

*Proof.* Let $\Omega \subset P$ be bounded, that is, there exists a constant $\mathcal{M} > 0$ such that $\|y\| \leq \mathcal{M}$ for $y \in \Omega$. By definition of $\|y\|$, $0 \leq y(t), |y'(t)| \leq \mathcal{M}, \ y \in \Omega$, we let $C = \max_{0 \leq y, |y'| \leq \mathcal{M}} f(t,y,y')$. Let $\overline{P}_l = \{y \in P, \|y\| \leq l\}$, where $l \geq \max\{(3k_1L)^{\frac{1}{1-\sigma_1}}, (3k_2L)^{\frac{1}{1-\sigma_2}}, 3LC\}$ and

$$L = \frac{\|r\|^{q-1}}{(\Gamma(\beta+1))^{q-1}} \left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)}\right].$$

We now show that $T : \overline{P}_l \to \overline{P}_l$. Actually, if $y \in \overline{P}_l$, then $0 \leq y(t), |y'(t)| \leq l, \ t \in [0,1]$. By condition $(A_1)$

$$f(y,v) \leq \varphi_p(C + k_1 l^{\sigma_1} + k_2 l^{\sigma_2}) \tag{18}$$

By (18) and from Section 3, we can ascertain that

$$
\begin{aligned}
|(Ty)'(t)| =& \left| -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q(I^\beta r(s)f(s,y,y'))ds + \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\alpha\Gamma(\alpha)}\varphi_q(I^\beta r(s)f(s,y,y'))ds \right| \\
\le& \frac{\|r\|^{q-1}}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|f(\tau,y(\tau),y'(\tau))|d\tau\right)ds \\
&+ \frac{2\|r\|^{q-1}}{(2-k)\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha-1}(\alpha-k+ks)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|f(\tau,y(\tau),y'(\tau))|d\tau\right)ds \\
\le& \frac{\|r\|^{q-1}(C+k_1l^{\sigma_1}+k_2l^{\sigma_2})}{\Gamma(\alpha-1)(\Gamma(\beta+1))^{q-1}}\int_0^t s^{\beta(q-1)}(t-s)^{\alpha-2}ds \\
&+ \frac{2\alpha\|r\|^{q-1}(C+k_1l^{\sigma_1}+k_2l^{\sigma_2})}{(2-k)\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{\beta(q-1)}(1-s)^{\alpha-1}ds \\
\le& \frac{\|r\|^{q-1}(C+k_1l^{\sigma_1}+k_2l^{\sigma_2})}{\Gamma(\alpha-1)(\Gamma(\beta+1))^{q-1}}\int_0^t (t-s)^{\alpha-2}ds \\
&+ \frac{2\alpha\|r\|^{q-1}(C+k_1l^{\sigma_1}+k_2l^{\sigma_2})}{(2-k)\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}}\int_0^1 (1-s)^{\alpha-1}ds \\
\le& \frac{\|r\|^{q-1}(C+k_1l^{\sigma_1}+k_2l^{\sigma_2})}{(\Gamma(\beta+1))^{q-1}}\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\right] \qquad (19) \\
=& L(C+k_1l^{\sigma_1}+k_2l^{\sigma_2}), \qquad t\in[0,1];
\end{aligned}
$$

this means $\|Ty\| \le L(C+k_1l^{\sigma_1}+k_2l^{\sigma_2}) \le l$. By applying Schauder's fixed point theorem, condition $f(t,y,y') \ne 0$ implies that $T$ has at least one nontrivial fixed point in $\overline{P}_l$, which is a positive solution of the FIBVP (1). This completes the proof. □

**Example 3.2.** Consider the FIBVP:

$$
\begin{cases}
D^{\frac{3}{2}}(\varphi_2({}^cD^{\frac{5}{2}}y(t))) + r(t)f(t,y(t),y'(t)) = 0, \qquad t\in[0,1], \\
y(0) = y''(0) = 0, \quad y(1) = \frac{1}{2}\int_0^1 y(s)ds, \\
\varphi_2({}^cD^{\frac{5}{2}}y(0)) = [\varphi_2({}^cD^{\frac{5}{2}}y(0))]' = 0.
\end{cases} \tag{20}
$$

where $r(t) = 1-t$ and

$$
f(t,y,y') = 5 + \frac{|y|^{\frac{2}{3}}}{3} + \frac{|y'|^{\frac{2}{3}}}{4}, \ (t,y,y') \in [0,1]\times[0,+\infty)\times\mathbb{R}.
$$

It is evident that $f$ satisfies all the conditions of Theorem 3.4. Therefore, the FIBVP (20) has at least one positive solution.

**Theorem 3.5.** *Assume $f:[0,1]\times[0,+\infty)\times(-\infty,+\infty)\to[0,+\infty)$ is continuous and there exist two constants $\mathcal{R} > l > 0$ such that*

- $(A_2)$ $f(t,u,v) \ge \varphi_p(\mathcal{L}_1 l)$, $(t,u,v) \in [0,1]\times[0,l]\times[-l,l]$;
- $(A_3)$ $f(t,u,v) \le \varphi_p(\mathcal{L}\mathcal{R})$, $(t,u,v) \in [0,1]\times[0,\mathcal{R}]\times[-\mathcal{R},\mathcal{R}]$.

*Then the FIBVP (1) has at least one positive solution where*

$$
\mathcal{L} = \left[\frac{\|r\|^{q-1}}{(\Gamma(\beta+1))^{q-1}}\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha+1)}\right]\right]^{-1} \quad \text{and} \quad \mathcal{L}_1 = \left[\frac{2\|r\|^{q-1}}{\alpha\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}}\right]^{-1}.
$$

*Proof.* Take $\Omega_1 = \{y\in P : \|y\| < l\}$, then, for $y \in \partial\Omega_1$ we get $0 \le y(t), |y'(t)| \le l,\ t\in[0,1]$. By condition $(A_2)$, $f(t,y,y') \ge \varphi_p(\mathcal{L}_1 l),\ t\in[0,1]$. By (19), we have

$$
\begin{aligned}
\|Ty\| \geq & |(Ty)'(0)| \\
= & \left| \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\alpha\Gamma(\alpha)} \varphi_q(I^\beta r(s) f(s,y,y'))ds \right| \\
= & \frac{2\|r\|^{q-1}}{(2-k)\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1}(\alpha-k+ks)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|f(\tau,y(\tau),y'(\tau))|d\tau\right) ds \\
\geq & \frac{2(\alpha-k)\|r\|^{q-1}\mathcal{L}_1 l}{(2-k)\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \int_0^1 (1-s)^{\alpha-1}ds \\
\geq & \frac{2\|r\|^{q-1}\mathcal{L}_1 l}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \int_0^1 (1-s)^{\alpha-1}ds \\
= & \frac{2\|r\|^{q-1}\mathcal{L}_1 l}{\alpha\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \\
= & l.
\end{aligned}
\tag{21}
$$

Take $\Omega_2 = \{y \in P : \|y\| < \mathcal{R}\}$, then, for $y \in \partial\Omega_2$ we get $0 \leq y(t), |y'(t)| \leq \mathcal{R},\ t \in [0,1]$. By condition $(A_3)$,

$$
\begin{aligned}
|(Ty)'(t)| = & \left| -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_q(I^\beta r(s) f(y,v))ds + \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\alpha\Gamma(\alpha)} \varphi_q(I^\beta r(s) f(y,v))ds \right| \\
\leq & \frac{\|r\|^{q-1}}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|f(y(\tau),v(\tau))|d\tau\right) ds \\
& + \frac{2\|r\|^{q-1}}{(2-k)\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1}(\alpha-k+ks)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|f(y(\tau),v(\tau))|d\tau\right) ds \\
\leq & \frac{\|r\|^{q-1}\mathcal{L}\mathcal{R}}{\Gamma(\alpha-1)(\Gamma(\beta+1))^{q-1}} \int_0^t s^{\beta(q-1)}(t-s)^{\alpha-2}ds \\
& + \frac{2\alpha\|r\|^{q-1}\mathcal{L}\mathcal{R}}{(2-k)\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)}(1-s)^{\alpha-1}ds \\
\leq & \frac{\|r\|^{q-1}\mathcal{L}\mathcal{R}}{\Gamma(\alpha-1)(\Gamma(\beta+1))^{q-1}} \int_0^t (t-s)^{\alpha-2}ds \\
& + \frac{2\alpha\|r\|^{q-1}\mathcal{L}\mathcal{R}}{(2-k)\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \int_0^1 (1-s)^{\alpha-1}ds \\
\leq & \frac{\|r\|^{q-1}\mathcal{L}\mathcal{R}}{(\Gamma(\beta+1))^{q-1}} \left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)}\right] \\
= & \mathcal{R}, \qquad t \in [0,1].
\end{aligned}
$$

As a result,

$$
\max_{t\in[0,1]} |(Ty)'(t)| \leq \mathcal{R}. \tag{22}
$$

It follows from (22) that $\|Ty\| \leq \mathcal{R}$ for $y \in \partial\Omega_2$. Thus, by Theorem 2.5, $T$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is the positive solution of the FIBVP (1). This completes the proof. □

**Example 3.3.** Consider the fractional differential equation:

$$\begin{cases} D^{\frac{3}{2}}(\varphi_2({}^cD^{\frac{5}{2}}y(t))) + r(t)f(t,y(t),y'(t)) = 0, \qquad t\in[0,1], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{1}{2}\int_0^1 y(s)ds, \\ \varphi_2({}^cD^{\frac{5}{2}}y(0)) = [\varphi_2({}^cD^{\frac{5}{2}}y(0))]' = 0. \end{cases} \tag{23}$$

where $r(t) = 7(1-t)$ and

$$f(t,y,y') = 2 + \frac{y}{45} + \frac{|y'|^{\frac{3}{2}}}{100}.$$

We set $\mathcal{R} = 35$ and $l = 2$. By computations, $\mathcal{L} = 0.78892$ and $\mathcal{L}_1 = 0.14025$. Thus, $f(t,y,y')$ satisfies

$$f(t,y,y') \approx 2 \geq \varphi_2(\mathcal{L}_1 l) \approx 1.5778, \text{ for } (t,y,y')\in[0,1]\times[0,2]\times[-2,2],$$
$$f(t,y,y') \approx 4.8484 \leq \varphi_2(\mathcal{L}\mathcal{R}) \approx 4.9088, \text{ for } (t,y,y')\in[0,1]\times[0,35]\times[-35,35].$$

Since all conditions of Theorem 3.5 hold, the FIBVP (23) has at least one positive solution.

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