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ESTIMATE OF FOURTH HANKEL DETERMINANT FOR A SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED SÃLÃGEAN OPERATOR

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ABSTRACT. In Geometric function theory, the estimation of upper bound of the Hankel determinants for various subclasses of analytic functions is an interesting topic of current research. Till now, extensive work has been done on the estimation of second and third Hankel determinants. The present investigation deal with the estimate of fourth Hankel determinant for a unified subclass of multivalent functions in the open unit disc $E = \{z : |z| < 1\}$. This work will set the stage in the direction of investigation of fourth Hankel determinant for several other classes.

1. INTRODUCTION

For $p \in \mathbb{N}$, let $\mathcal{A}_p$ denote the class of analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1}$$

in the unit disc $E = \{z : |z| < 1\}$ and further normalized by $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$, we denote the subclass of $\mathcal{A}_1 \equiv \mathcal{A}$ consisting of the functions of the form (1) and which are univalent in E.

Let $\mathcal{P}$ denote the class of analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

whose real parts are positive in E.

For $\delta \geq 0$ and $f \in \mathcal{A}_p$, Goyal et al. [8] introduced the following differential operator:

$$D_\delta^0 f(z) = f(z),$$
$$D_\delta^1 f(z) = (1-\delta) f(z) + \frac{\delta}{p} z f'(z) = D_\delta f(z),$$

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and in general,

$$D_\delta^n f(z) = D(D_\delta^{n-1} f(z)) = z^p + \sum_{k=p+1}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n a_k z^k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

with $D_\delta^o f(0) = 0$. For $p = 1$, the operator $D_\delta^n f(z)$ coincides with that introduced by Al-Oboudi [1] and for $p = 1, \delta = 1$, the operator $D_\delta^n f(z)$ reduces to well known Sãlãgean operator.

For understanding of main content, let us recall the following standard classes:

$\mathcal{R} = \{f : f \in \mathcal{A}, Re(f'(z)) > 0, z \in E\}$, the class of bounded turning functions introduced and studied by MacGregor [14].

$\mathcal{R}_1 = \{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0, z \in E\}$, the subclass of close-to-star functions studied by MacGregor [15].

$\mathcal{R}(\alpha) = \left\{f : f \in \mathcal{A}, Re\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > 0, 0 \le \alpha \le 1, z \in E\right\}$, the class studied by Murugusundramurthi and Magesh [17]. In particular, $\mathcal{R}(1) \equiv \mathcal{R}$ and $\mathcal{R}(0) \equiv \mathcal{R}_1$.

$\mathcal{R}'(\alpha) = \{f : f \in \mathcal{A}, Re\,(f'(z) + \alpha z f''(z)) > 0, \alpha \ge 0, z \in E\}$, the class studied by Sahoo [20]. Particularly, $\mathcal{R}'(0) \equiv \mathcal{R}$.

Motivated by the above defined classes, we introduce the following subclass of multivalent functions defined with operator D_δ^n:

$\mathcal{R}_p(\delta; n; \alpha) = \left\{f : f \in \mathcal{A}_p, Re\left\{(1-\alpha)\frac{D_\delta^n f(z)}{z^p} + \alpha\frac{(D_\delta^n f(z))'}{pz^{p-1}}\right\} > 0, 0 \le \alpha \le 1, z \in E\right\}.$

The following consequences can be easily observed:

(i) $\mathcal{R}_1(1; 0; \alpha) \equiv \mathcal{R}(\alpha)$.

(ii) $\mathcal{R}_1(1; 1; \alpha) \equiv \mathcal{R}'(\alpha)$.

(iii) $\mathcal{R}(1; 0; 1) \equiv \mathcal{R}$.

(iv) $\mathcal{R}(1; 0; 0) \equiv \mathcal{R}_1$.

In 1976, Noonan and Thomas [18] introduced the q^{th} Hankel determinant for $q \ge 1$ and $n \ge 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Particularly, for $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2 a_4 - a_3^2$.

The paper is concerned with the Hankel determinant in case of $q = 3, n = p$ as

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix},$$

which is known as the third Hankel determinant.

For $f \in \mathcal{A}_p, a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and by using the triangle inequality, we have

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+1}a_{p+2} - a_{p+3}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \quad (2)$$

For any $f \in \mathcal{A}_p$, we can represent the fourth Hankel determinant as

$$H_4(p) = a_{p+6}H_3(p) - a_{p+5}D_1 + a_{p+4}D_2 - a_{p+3}D_3, \quad (3)$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$D_1 = (a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) - a_{p+1}(a_{p+1}a_{p+5} - a_{p+2}a_{p+4}) + a_{p+3}(a_{p+1}a_{p+3} - a_{p+2}^2), \quad (4)$$

$$D_2 = (a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+1}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+2}(a_{p+2}a_{p+4} - a_{p+3}^2), \quad (5)$$

$$D_3 = a_{p+1}(a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+2}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+3}(a_{p+2}a_{p+4} - a_{p+3}^2). \quad (6)$$

Hankel determinant has been considered by several authors. For example, Noor [19] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions given by (1) with bounded boundary. Ehrenborg [7] studied the Hankel determinant of exponential polynomials and in [10], the Hankel transform of an integer sequence is defined and some of its properties have been discussed by Layman. Second Hankel determinant for various classes has been extensively studied by various authors including Singh [22], Mehrok and Singh [16], Janteng et al.[9] and many others. Third Hankel determinant has been studied by some of the researchers including Babalola [3, 4], Shanmugam et al.[21], Sudharsan et al.[24], Bucur et al. [5], Altinkaya and Yalcin [2] and Singh and Singh [23].

In this paper, we establish the upper bound for the functional $H_{4,p}(f)$ for the functions belonging to the class $\mathcal{R}_p(\delta; n; \alpha)$. This work will motivate the future researchers to work in this direction.

2. Main results

Lemma 2.1[6, 11] If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, then for $n, k \in \mathbb{N} = \{1, 2, 3, ...\}$, we have the following inequalities:

$$|c_{n+k} - \lambda c_n c_k| \leq 2, 0 \leq \lambda \leq 1$$

and

$$|c_n| \leq 2.$$

Lemma 2.2 If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, then for $n, k \in \mathbb{N} = \{1, 2, 3, ...\}$, we have the following inequalities:

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2, \lambda \geq 1.$$

Proof For $\lambda \geq 1$, we have

$$|c_{n+k} - \lambda c_n c_k| \leq |c_n c_k - c_{n+k}| + (\lambda - 1)|c_n c_k|.$$

Using Lemma 2.1, the above inequality yields

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2.$$

Lemma 2.3[12, 13] If $p \in \mathcal{P}$, then

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x, z satisfying $|x| \leq 1, |z| \leq 1$, and $c_1 \in [0, 2]$.

Lemma 2.4 [4] If $p \in \mathcal{P}$, then

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \leq \begin{cases} 2(1-\sigma) & \text{if } \sigma \leq 0, \\ 2 & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1) & \text{if } \sigma \geq 2. \end{cases}$$

Theorem 2.5 If $f \in \mathcal{R}_p(\delta; n; \alpha)$, then

$$|a_{p+k}| \leq \frac{2p^{k+1}}{[p+k\alpha][p+k\delta]^n}. \tag{7}$$

The bound is sharp.
Proof As $f \in \mathcal{R}_p(\delta; n; \alpha)$, therefore by definition, there exists a function $p \in \mathcal{P}$ such that

$$(1-\alpha)\frac{D_\delta^n f(z)}{z^p} + \alpha\frac{(D_\delta^n f(z))'}{pz^{p-1}} = p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

On expanding and equating the coefficients in the above equation, it yields

$$a_{p+k} = \frac{p^{n+1}c_k}{[p+k\alpha][p+k\delta]^n}. \tag{8}$$

Using Lemma 2.1 in (8), the result (7) is obvious.

Theorem 2.6 If $f \in \mathcal{R}_p(\delta; n; \alpha)$, then

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p^{n+1}}{(p+2\alpha)(p+2\delta)^n}. \tag{9}$$

The estimate is sharp.
Proof Using (8), we find that

$$|a_{p+2} - a_{p+1}^2| = \frac{p^{n+1}}{(p+2\alpha)(p+2\delta)^n}\left|c_2 - \frac{2p^{n+1}(p+2\alpha)(p+2\delta)^n}{(p+\alpha)^2(p+\delta)^{2n}}.\frac{c_1^2}{2}\right|.$$

Since $0 \leq \sigma = \dfrac{2p^{n+1}(p+2\alpha)(p+2\delta)^n}{(p+\alpha)^2(p+\delta)^{2n}} \leq 2$, so by Lemma 2.4, the result (9) is obvious.

Theorem 2.7 If $f \in \mathcal{R}_p(\delta; n; \alpha)$, then

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{4p^{2(n+1)}}{(p+2\alpha)^2(p+2\delta)^{2n}}. \tag{10}$$

Proof Using (8), we have

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \left|\frac{p^{2(n+1)}c_1c_3}{(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)} - \frac{p^{2(n+1)}c_2^2}{(p+2\delta)^{2n}(p+2\alpha)^2}\right|.$$

Using Lemma 2.3, rearranging the terms and applying the triangle inequality along with the inequality $|z| \leq 1$, it yields

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \tfrac{T}{4}\Big[[(p+2\delta)^{2n}(p+2\alpha)^2 - (p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)]c_1^4$$
$$+2[(p+2\delta)^{2n}(p+2\alpha)^2 - (p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)]c_1^2x(4-c_1^2)$$
$$+[\{(p+2\delta)^{2n}(p+2\alpha)^2 - (p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)\}c_1^2$$

$+4(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)](4-c_1^2)x^2$
$+2(p+2\delta)^{2n}(p+2\alpha)^2(4-c_1^2)c_1(1-|x|^2)\Bigg],$

where $T = \dfrac{p^{2(n+1)}}{(p+\delta)^n(p+2\delta)^{2n}(p+3\delta)^n(p+\alpha)(p+2\alpha)^2(p+3\alpha)}$.

For $c_1 = c \in [0,2]$ and $|x| = \mu$, we have

$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \dfrac{T}{4}\Bigg[[(p+2\delta)^{2n}(p+2\alpha)^2 - (p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)]c^4$
$+2(p+2\delta)^{2n}(p+2\alpha)^2(4-c^2)c + 2[(p+2\delta)^{2n}(p+2\alpha)^2$
$-(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)]c^2(4-c^2)\mu + \{(p+2\delta)^{2n}(p+2\alpha)^2$
$-(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)\}(4-c^2)(c-2)(c-\beta)\mu^2\Bigg]$=F(c,$\mu$),

where $\beta = \beta(\alpha) = \dfrac{2(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)}{(p+2\delta)^{2n}(p+2\alpha)^2 - (p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)}$.

Now $\frac{\partial F}{\partial \mu} = \frac{\{(p+2\delta)^{2n}(p+2\alpha)^2-(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)\}(4-c^2)[c^2+(c-2)(c-\beta)\mu]}{2(p+\delta)^n(p+2\delta)^{2n}(p+3\delta)(p+\alpha)(p+2\alpha)^2(p+3\alpha)} > 0.$

So, max.$F(c,\mu) = F(c,1) = G(c)$.

Therefore

$G'(c) = W(\alpha,\delta)\Bigg[\{(p+2\delta)^{2n}(p+2\alpha)^2 - (p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)\}c^3$
$+[4(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha) - 3(p+2\delta)^{2n}(p+2\alpha)^2]\Bigg] < 0,$

where

$W(\alpha,\delta) = -\dfrac{2p^{2(n+1)}}{(p+\delta)^n(p+2\delta)^{2n}(p+3\delta)(p+\alpha)(p+2\alpha)^2(p+3\alpha)}$.

So, max.$G(c) = G(0)$. Hence the result (10).

Theorem 2.8 If $f \in \mathcal{R}_p(\delta;n;\alpha)$, then

$$|a_{p+1}a_{p+2} - a_{p+3}| \tag{11}$$

$\leq \frac{2p^{n+1}[3(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-2p^{n+1}(p+3\delta)^n(p+3\alpha)]^{\frac{3}{2}}}{3(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)\sqrt{3[(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-(p+3\delta)^n(p+3\alpha)]}}.$

Proof From (8), we have

$$|a_{p+1}a_{p+2} - a_{p+3}| = \left|\frac{p^{2(n+1)}c_1c_2}{(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)} - \frac{p^{n+1}c_3}{(p+3\delta)^n(p+3\alpha)}\right|.$$

Using Lemma 2.3 and rearranging the terms, it yields

$|a_{p+1}a_{p+2} - a_{p+3}| = \Bigg|\dfrac{\{2p^{2(n+1)}(p+3\delta)^n(p+3\alpha) - p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)\}c_1^3}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}$
$-\dfrac{2\{p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha) - p^{2(n+1)}(p+3\delta)^n(p+3\alpha)\}}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}c_1(4-c_1^2)x$
$+\dfrac{p^{n+1}c_1(4-c_1^2)x^2}{4(p+3\delta)^n(p+3\alpha)} - \dfrac{p^{n+1}(4-c_1^2)(1-|x|^2)z}{2(p+3\delta)^n(p+3\alpha)}\Bigg|.$

On applying the triangle inequality and using $c_1 = c \in [0,2]$ and $|x| = \rho, |z| \leq 1$, we have

$|a_{p+1}a_{p+2} - a_{p+3}| \leq \dfrac{\{2p^{2(n+1)}(p+3\delta)^n(p+3\alpha) - p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)\}c^3}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}$
$+\dfrac{p^{n+1}\{(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha) - p^{2(n+1)}(p+3\delta)^n(p+3\alpha)\}}{2(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}c(4-c^2)\rho$

$+\dfrac{p^{n+1}(4-c^2)}{2(p+3\delta)^n(p+3\alpha)}+\dfrac{p^{n+1}(c-2)(4-c^2)\rho^2}{4(p+3\delta)^n(p+3\alpha)}=F(c,\rho).$

$\frac{\partial F}{\partial \rho}=\frac{p^{n+1}\{(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-p^{2(n+1)}(p+3\delta)^n(p+3\alpha)\}c(4-c^2)}{2(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}+\frac{p^{n+1}(c-2)(4-c^2)\rho}{2(p+3\delta)^n(p+3\alpha)}>0.$

Now $F(\rho)\le F(1)$ and

$F(c,1)=\frac{\{4p^{2(n+1)}(p+3\delta)^n(p+3\alpha)-3p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)\}c^3}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}$

$+\frac{8\{p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-p^{2(n+1)}(p+3\delta)^n(p+3\alpha)\}c}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}+\frac{p^{n+1}(4c-c^3)}{4(p+3\delta)^n(p+3\alpha)}=G(c).$

$G'(c)=\dfrac{3\{4p^{2(n+1)}(p+3\delta)^n(p+3\alpha)-3p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)\}c^2}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}$

$+\dfrac{8\{p^{n+1}(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-p^{2(n+1)}(p+3\delta)^n(p+3\alpha)\}}{4(p+\delta)^n(p+2\delta)^n(p+3\delta)^n(p+\alpha)(p+2\alpha)(p+3\alpha)}+\dfrac{p^{n+1}(4-3c^2)}{4(p+3\delta)^n(p+3\alpha)}.$

$G''(c)=0$ gives,

$$c=\sqrt{\frac{3(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-2p^{n+1}(p+3\delta)^n(p+3\alpha)}{3\{(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-p^{n+1}(p+3\delta)^n(p+3\alpha)\}}}=c_0.$$

Since $G''(c_0)<0$, so max. $G(c)=G(c_0)$ and hence the result (11) is obvious.

Theorem 2.9 If $f\in\mathcal{R}_p(\delta;n;\alpha)$, then

$|H_3(p)|\le\frac{4p^{2(n+1)}}{(p+2\delta)^n(p+2\alpha)}\Bigg[\frac{2p^{n+1}}{(p+2\delta)^{2n}(p+2\alpha)^2}+\frac{1}{(p+4\delta)^n(p+4\alpha)}$

$+\frac{[3(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-2p^{n+1}(p+3\delta)^n(p+3\alpha)]^{\frac{3}{2}}}{3(p+\delta)^n(p+3\delta)^{2n}(p+\alpha)(p+3\alpha)^2\sqrt{3[(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-p^{n+1}(p+3\delta)^n(p+3\alpha)]}}\Bigg].$

Proof Using Lemma 2.2, Theorem 2.5, Theorem 2.6 and the result (9) in (2), the result is obvious.

For $p=1,\delta=1,n=0,\alpha=1$, Theorem 2.9 agrees with the following result due to Babalola [2]:

Corollary 2.9.1 If $f\in\mathcal{R}$, then

$$|H_3(1)|\le 0.7423.$$

Theorem 2.10 If $f\in\mathcal{R}_p(\delta;n;\alpha)$, then

$$|H_4(p)|\le\frac{8p^{3(n+1)}}{(p+2\delta)^n(p+6\delta)^n(p+2\alpha)(p+6\alpha)}\Bigg[\frac{2p^{n+1}}{(p+2\delta)^{2n}(p+2\alpha)^2}+\frac{1}{(p+4\delta)^{2n}(p+4\alpha)} \tag{12}$$

$+\frac{[3(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-2p^{n+1}(p+3\delta)^n(p+3\alpha)]^{\frac{3}{2}}}{3(p+\delta)^n(p+3\delta)^{2n}(p+\alpha)(p+3\alpha)^2\sqrt{3[(p+\delta)^n(p+2\delta)^n(p+\alpha)(p+2\alpha)-p^{n+1}(p+3\delta)^n(p+3\alpha)]}}\Bigg]$

$+\frac{2p^{n+1}}{(p+5\delta)^n(p+5\alpha)}u(p,\delta,\alpha)+\frac{2p^{n+1}}{(p+4\delta)^n(p+4\alpha)}v(p,\delta,\alpha)+\frac{2p^{n+1}}{(p+3\delta)^n(p+3\alpha)}w(p,\delta,\alpha),$

where

$$u(p,\delta,\alpha)=2p^{2(n+1)}(4p^{n+1}-2)\Bigg[\frac{1}{(p+\delta)^{2n}(p+5\delta)^n(p+\alpha)^2(p+5\alpha)}+\frac{1}{(p+2\delta)^{2n}(p+3\delta)^n(p+3\alpha)(p+2\alpha)^2} \tag{13}$$

$+\frac{1}{(p+\delta)^n(p+3\delta)^{2n}(p+\alpha)(p+3\alpha)^2}\Bigg]+\frac{172p^{2(n+1)}(4p^{n+1}-2)+4p^{2(n+1)}}{48(p+\delta)^n(p+2\delta)^n(p+4\delta)^n(p+\alpha)(p+2\alpha)(p+4\alpha)},$

$$v(p,\delta,\alpha)=\Bigg[\frac{63p^{2(n+1)}(4p^{n+1}-2)}{25(p+\delta)^n(p+2\delta)^n(p+5\delta)^n(p+\alpha)(p+2\alpha)(p+5\alpha)} \tag{14}$$

$$+\frac{18p^{2(n+1)}(4p^{n+1}-2)}{5(p+2\delta)^{2n}(p+4\delta)^n(p+4\alpha)(p+2\alpha)^2}+\frac{150p^{2(n+1)}(4p^{n+1}-2)+4p^{2(n+1)}}{75(p+2\delta)^n(p+3\delta)^{2n}(p+2\alpha)(p+3\alpha)^2}\Bigg]$$

and

$$w(p,\delta,\alpha)=2p^{2(n+1)}(4p^{n+1}-2)\Bigg[\frac{1}{(p+2\delta)^{2n}(p+5\delta)^n(p+2\alpha)^2(p+5\alpha)} \tag{15}$$

$$+\frac{1}{(p+\delta)^n(p+3\delta)^n(p+5\delta)^n(p+\alpha)(p+3\alpha)(p+5\alpha)}+\frac{2}{(p+3\delta)^{3n}(p+3\alpha)^3}+\frac{1}{(p+\delta)^n(p+4\delta)^{2n}(p+\alpha)(p+4\alpha)^2}$$

$$+\frac{17}{16(p+2\delta)^n(p+3\delta)^n(p+4\delta)^n(p+2\alpha)(p+3\alpha)(p+4\alpha)}\Bigg]$$

$$+\frac{p^{2(n+1)}}{(p+\delta)^n(p+2\delta)^{2n}(p+3\delta)^n(p+4\delta)^{2n}(p+5\delta)^n(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Proof Using (8) in (4), (5) and (6), we get

$$D_1=\frac{p^{2(n+1)}c_2c_5}{(p+2\delta)^n(p+5\delta)^n(p+2\alpha)(p+5\alpha)}-\frac{p^{2(n+1)}c_3c_4}{(p+3\delta)^n(p+4\delta)^n(p+3\alpha)(p+4\alpha)} \tag{16}$$

$$-\frac{p^{3(n+1)}c_1^2c_5}{(p+\delta)^{2n}(p+5\delta)^n(p+\alpha)^2(p+5\alpha)}+\frac{p^{3(n+1)}c_1c_2c_4}{(p+\delta)^n(p+2\delta)^n(p+4\delta)^n(p+\alpha)(p+2\alpha)(p+4\alpha)}$$

$$+\frac{p^{3(n+1)}c_1c_3^2}{(p+\delta)^n(p+3\delta)^n(p+\alpha)(p+3\alpha)^2}-\frac{p^{3(n+1)}c_3c_2^2}{(p+2\delta)^{2n}(p+3\delta)^n(p+3\alpha)(p+2\alpha)^2},$$

$$D_2=\frac{p^{2(n+1)}c_3c_5}{(p+3\delta)^n(p+5\delta)^n(p+3\alpha)(p+5\alpha)}-\frac{p^{3(n+1)}c_4^2}{(p+4\delta)^{2n}(p+4\alpha)^2} \tag{17}$$

$$-\frac{p^{3(n+1)}c_1c_2c_5}{(p+\delta)^n(p+2\delta)^n(p+5\delta)^n(p+\alpha)(p+2\alpha)(p+5\alpha)}+\frac{p^{3(n+1)}c_1c_3c_4}{(p+\delta)^n(p+3\delta)^n(p+4\delta)^n(p+\alpha)(p+3\alpha)(p+4\alpha)}$$

$$+\frac{p^{3(n+1)}c_4c_2^2}{(p+2\delta)^{2n}(p+4\delta)^n(p+2\alpha)^2(p+4\alpha)}-\frac{p^{3(n+1)}c_2c_3^2}{(p+2\delta)^n(p+3\delta)^{2n}(p+2\alpha)(p+3\alpha)^2}$$

and

$$D_3=\frac{p^{3(n+1)}c_1c_3c_5}{(p+\delta)^n(p+3\delta)^n(p+5\delta)^n(p+\alpha)(p+3\alpha)(p+5\alpha)} \tag{18}$$

$$-\frac{p^{3(n+1)}c_1c_4^2}{(p+\delta)^n(p+4\delta)^{2n}(p+\alpha)(p+4\alpha)^2}-\frac{p^{3(n+1)}c_2^2c_5}{(p+2\delta)^{2n}(p+5\delta)^n(p+2\alpha)^2(p+5\alpha)}$$

$$+\frac{2p^{3(n+1)}c_2c_3c_4}{(p+2\delta)^n(p+3\delta)^n(p+4\delta)^n(p+2\alpha)(p+3\alpha)(p+4\alpha)}-\frac{p^{3(n+1)}c_3^3}{(p+3\delta)^{3n}(p+3\alpha)^3}.$$

On rearranging the terms in (16), (17) and (18), it yields

$$D_1=\frac{p^{2(n+1)}c_5(c_2-p^{n+1}c_1^2)}{(p+\delta)^{2n}(p+5\delta)^n(p+\alpha)^2(p+5\alpha)}+\frac{p^{2(n+1)}c_3(c_4-p^{n+1}c_2^2)}{(p+2\delta)^{2n}(p+3\delta)^n(p+3\alpha)(p+2\alpha)^2} \tag{19}$$

$$-\frac{p^{2(n+1)}c_3(c_4-p^{n+1}c_1c_3)}{(p+\delta)^n(p+3\delta)^{2n}(p+\alpha)(p+3\alpha)^2}-\frac{67p^{2(n+1)}c_4(c_3-p^{n+1}c_1c_2)}{48(p+\delta)^n(p+2\delta)^n(p+4\delta)^n(p+\alpha)(p+2\alpha)(p+4\alpha)}$$

$$+\frac{19p^{2(n+1)}c_2(c_5-p^{n+1}c_1c_4)}{48(p+\delta)^{2n}(p+2\delta)^n(p+4\delta)^n(p+\alpha)(p+2\alpha)(p+4\alpha)}$$

$$+\frac{p^{2(n+1)}c_2c_5}{48(p+\delta)^n(p+2\delta)^n(p+4\delta)^n(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$D_2=\frac{p^{2(n+1)}c_5(c_3-p^{n+1}c_1c_2)}{(p+\delta)^n(p+2\delta)^n(p+5\delta)^n(p+\alpha)(p+2\alpha)(p+5\alpha)} \tag{20}$$

$$-\frac{p^{2(n+1)}c_4(c_4-p^{n+1}c_2^2)}{(p+2\delta)^n(p+4\delta)^n(p+4\alpha)(p+2\alpha)^2}$$
$$+\frac{p^{2(n+1)}c_3(c_5-p^{n+1}c_2c_3)}{(p+2\delta)^n(p+3\delta)^{2n}(p+2\alpha)(p+3\alpha)^2}-\frac{4p^{2(n+1)}c_4(c_4-p^{n+1}c_1c_3)}{5(p+2\delta)^{2n}(p+4\delta)^n(p+4\alpha)(p+2\alpha)^2}$$
$$-\frac{13p^{2(n+1)}c_3(c_5-p^{n+1}c_1c_4)}{50(p+\delta)^n(p+2\delta)^n(p+5\delta)^n(p+\alpha)(p+2\alpha)(p+5\alpha)}+\frac{p^{2(n+1)}c_3c_5}{75(p+2\delta)^n(p+3\delta)^{2n}(p+2\alpha)(p+3\alpha)^2}$$

and

$$D_3=\frac{p^{2(n+1)}c_5(c_4-p^{n+1}c_2^2)}{(p+2\delta)^{2n}(p+5\delta)^n(p+2\alpha)^2(p+5\alpha)} \tag{21}$$
$$-\frac{p^{2(n+1)}c_5(c_4-p^{n+1}c_1c_3)}{(p+\delta)^n(p+3\delta)^n(p+5\delta)^n(p+\alpha)(p+3\alpha)(p+5\alpha)}$$
$$+\frac{p^{2(n+1)}c_3(c_6-p^{n+1}c_3^2)}{(p+3\delta)^{3n}(p+3\alpha)^3}-\frac{p^{2(n+1)}c_3(c_6-p^{n+1}c_2c_4)}{(p+3\delta)^{3n}(p+3\alpha)^3}+\frac{p^{2(n+1)}c_4(c_5-p^{n+1}c_1c_4)}{(p+\delta)^n(p+4\delta)^{2n}(p+\alpha)(p+4\alpha)^2}$$
$$-\frac{17p^{2(n+1)}c_4(c_5-p^{n+1}c_2c_3)}{16(p+2\delta)^n(p+3\delta)^n(p+4\delta)^{3n}(p+2\alpha)(p+3\alpha)(p+4\alpha)}$$
$$+\frac{p^{2(n+1)}c_4c_5}{4(p+\delta)^n(p+2\delta)^{2n}(p+3\delta)^n(p+4\delta)^n(p+5\delta)^n(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Using Lemma 2.1 and applying triangle inequality in (19), (20) and (21), we obtain

$$|D_1|\le u(p,\delta,\alpha), \tag{22}$$
$$|D_2|\le v(p,\delta,\alpha), \tag{23}$$

and

$$|D_3|\le w(p,\delta,\alpha), \tag{24}$$

where $u(p,\delta,\alpha)$, $v(p,\delta,\alpha)$ and $w(p,\delta,\alpha)$ are defined in (13), (14) and (15) respectively.
Hence, using Theorem 2.9, (9), (22), (23) and (24) and applying triangle inequality in (3), the result (12) is obvious.

For $p=1,\delta=1,n=0,\alpha=1$, Theorem 2.10 coincides with the following result:

Corollary 2.10.1 If $f\in\mathcal{R}$, then

$$|H_4(1)|\le 0.7973.$$

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