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CONTROLLABILITY RESULTS FOR FRACTIONAL NEUTRAL DIFFERENTIAL SYSTEMS WITH NON-INSTANTANEOUS IMPULSES

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Abstract. The main concern of this manuscript is to investigate some sufficient conditions under which the ABC-fractional neutral dynamic system with non-instantaneous impulsive conditions is controllable. Many physical problems require to control the state of the system not only at the final time of the interval but also at each of the impulses points, i.e., the so-called total controllability. In this regard, this is the first attempt to establish the total controllability results for ABC-fractional system. We also establish these results for the considered system with integral term. The basic technique of our approach is to reduce the controllability problem into a solvability problem of an operator equation in some suitable function space and then we prove the solvability results for the operator equations which in turn imply the controllability of the system. Semi-group theory, functional analysis, measure of non-compactness and Mönch fixed point theorem have been used to establish these results. At last, an example is given to validate the obtained analytical outcomes.

1. Introduction

There are several real world problems in which the state of the system have some abrupt changes. These abrupt changes are known as the impulsive effect in the system and the corresponding differential equations are known as the impulsive differential equations. In the literature, there are mainly two types of impulsive systems: the first one is the instantaneous impulsive systems, in which the effect of impulses is active for a very short period of time as compare to the overall duration of the process, for example in shocks, harvesting and natural disasters [1, 2, 3]; the second one, we are considering is the non-instantaneous impulsive systems which were introduced by Hernández in [4], in which the impulsive effect remains active for a period of time, for instance, in the case of a hyperglycemic patient, an insulin can be injected. The introduction of the medicine in the circulation system causes an unexpected change in the systems, trailed by a consistent interaction until the drug is completely absorbed [4]. For the further details on non-instantaneous impulsive systems, we may refer e.g. to [5, 6, 7, 8] and the references cited therein.

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The dynamical systems that can be modelled by the fractional differential equations carried with a non-integer derivative are called the fractional order dynamic systems. There are many real-applications, for instance, fractal properties, power-law long-range dependence or power-law nonlocality, which evolves the integrals and derivatives of fractional orders. In the last few decades, the growth of science and engineering systems has considerably stimulated the employment of fractional calculus in various areas of the control theory, for example in controllability, fault estimation, observability, observer design, stability and stabilization. However, the applications of fractional calculus and their outcomes vary as much as the definitions of fractional integrals and derivatives, for example, Riesz-Caputo, Grunwald-Letnikov, Caputo, Riemann-Liouville, Hadamard, Weyl, Chen, and so on. For more details about fractional calculus see [9, 10] for instance.

Recently, in 2015, Caputo and Fabrizio [11] introduced a new fractional nonlocal derivative with non-singular kernel known as Caputo-Fabrizio fractional derivative given by

$$ {}^{CF}D^{q}_{a+}z(t) = \frac{M(q)}{1-q}\int_a^t \exp\left[-\frac{q}{1-q}(t-z)\right]z'(z)dz, $$

where $q \in (0,1)$ and $M(q)$ is a normalization function such that $M(0) = 1 = M(1)$. This definition made a significant challenge to its realization but soon it discovered its way into many applications of engineering and science, for example, mechanical engineering and thermal science. For more details, one can see [12, 13].

A year after, a new definition of nonlocal derivatives with non-singular kernel relied on the Mittag-Leffler function was introduced by Atangana and Baleanu [14]. This new definition upheld the Caputo-Fabrizios one relied on the exponential function. It expands the profundity of the connection between the Mittag-Leffler function and fractional calculus which leads to significant applications in science and engineering such as thermal physics, population dynamics, control problems and so on [15, 16, 17]. In the last five years, many authors considered the ABC-fractional differential equations and investigated many results such as existence, stability and data dependence of the solutions [18, 19, 20, 21, 22, 23].

Controllability is a significant idea in modern control theory. In general, controllability implies, we can transfer the state of a control system from an arbitrary initial state to an arbitrary state by using some control function. Over the most recent couple of years, many authors studied the controllability results for different types of systems such as functional differential equations, neutral functional differential equations and impulsive differential equations of integer as well as fractional order, see for instance [24, 25, 26, 27, 28, 29, 30, 31] and the references cited therein. Very recently, in [32], the authors considered the ABC-fractional semilinear differential equations with instantaneous impulsive conditions and investigated the controllability results by using the Darbo fixed point theorem, measures of noncompactness with semigroup theory. In [33], the authors extended the controllability results of [32] for the integro-differential equations with instantaneous impulses by using the Banach fixed point theorem.

However, the above results cannot be easily extended to the case of neutral system with non-instantaneous impulses. Since, in practicality, there is no impulse that occurs instantaneously rather it is non-instantaneous howsoever time of occurrence is very small. Therefore, it is beneficial to study a class of differential equations with non-instantaneous impulses and from the best of the authors knowledge there is no article which reported the controllability results for ABC-fractional neutral dynamic system with non-instantaneous impulses. Motivated from the above discussion, in this manuscript, we investigate the total controllability results for the following ABC-fractional neutral dynamic system with non-instantaneous impulses

$$\begin{aligned} {}^{ABC}D^q[z(t) - \mathfrak{F}(t, z_a(t))] &= \mathcal{A}[z(t) - \mathfrak{F}(t, z_a(t))] + \mathcal{B}u(t) + \mathfrak{G}(t, z_b(t)), \\ & \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, m, \\ z(t) &= \mathfrak{J}_i(t, z(t_i^-)),\ t \in (t_i, s_i],\ i = 1, \cdots, m, \\ z(0) &= z_0, \end{aligned} \tag{1}$$

where ${}^{ABC}D^q$ denote the ABC-fractional derivative of order $q \in (0, 1)$, z is the state variable, $z_a(t) = z(a(t)), z_b(t) = z(b(t)),\ a, b : I = [0, T] \to I$ with $a(t), b(t) \leq t$. The arbitrary points t_i and s_i satisfy the relations $0 = t_0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = T$, $z(t_i^-) = \lim_{h\to 0^+} z(t_i - h)$ denotes the left limit of $z(t)$ at $t = t_i$, $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is the infinitesimal generator of a q-resolvent family $(S_q(t))_{t\geq 0}$, $(T_q(t))_{t\geq 0}$ is solution operator defined on a complex Banach space $(X, \|\cdot\|)$ (for more details, please see [32, 34]) , $u(\cdot)$ in $L^2([0, T], U)$ is the control function where U is a Banach space. $\mathcal{B}$ is a linear and bounded operator from U into X. $\mathfrak{F}, \mathfrak{G}, \mathfrak{J}_i$ are the given functions which satisfies certain assumptions to be specified later on.

The primary contribution and advantage of this paper are as follows.

- In the paper, we study the controllability results for a class of ABC-fractional neutral differential system with non-instantaneous impulses.
- We define a new piecewise control function which control the system not only at the final time of the interval but also at each of the impulse points, i.e., we study the total controllability results.
- Also, we establish the total controllability results for the considered problem with the integral term.
- We use the concept of piecewise continuous mild solution to the proposed impulsive system for constructing a suitable operator and with the help of this operator, we derived the controllability results by using the Mönch fixed point technique in measure of noncompactness.
- At last, we provide an example to show the effectiveness of the obtained analytical results.

The remainder of the manuscript is structured as follows. In Section 2, we introduce some fundamental definitions, lemmas and theorems. Section 3 is devoted to the study of controllability results for the considered impulsive system (1). In Section 4, we study the system (1) with integral term. In the last Section 5, an example is given to validate the analytical outcomes of the manuscript.

2. Preliminaries

In this section, we introduce some fundamental definitions, lemmas and important results which are often used throughout the manuscript. $C(I, X)$ denotes the Banach space of all continuous functions from I into X. $\mathbb{B}(X)$ denotes the space of all bounded linear operators from X into X. The space of measurable functions which are square Bochner integrable with values space X is denoted by $L^2(I, X)$. For any subset $A \subset X$, $\overline{co}(A)$ denotes its closed convex hull.

Now, we introduce the space of all piecewise continuous functions by $PC(I,X) = \{z : I \to X : z \in C(\cup_{i=0}^{m}(t_i,t_{i+1}],X)$ and there exists $z(t_i^-)$ and $z(t_i^+)$, $i = 1,\cdots,m$ with $z(t_i^-) = z(t_i)\}$. Clearly, we can seen that $PC(I,X)$ forms a Banach space under the supremum norm, $\|z\|_{PC} = \sup_{t\in I}\|z(t)\|$.

Next, we define the ABC-fractional derivative and integral.

Definition 2.1. [14] *Let $F \in H^1(a,T), a < T$, be a function. Then, the ABC-fractional of F at a point $t \in (0,T)$ of order $q \in (0,1)$ is given by*

$$^{ABC}D_{a^+}^{q}F(t) = \frac{B(q)}{1-q}\int_a^t F'(z)E_q\left(-\epsilon(t-z)^q\right)dz,$$

where $\epsilon = \frac{q}{1-q}$, $E_q(\cdot)$ is the usual Mittag Leffler function and $B(q) = (1-q) + \frac{q}{\Gamma(q)}$ satisfying $B(0) = B(1) = 1$ is the normalization function.

Definition 2.2. [14] *Let $F \in H^1(a,T), a < T$, be a function. Then, the Atangana-Baleanu fractional derivative at a point $t \in (0,T)$ of order $q \in (0,1)$ in Riemann-Liouville sense is given by*

$$^{ABR}D_{a^+}^{q}F(t) = \frac{B(q)}{1-q}\frac{d}{dt}\int_a^t F(z)E_q\left(-\epsilon(t-z)^q\right)dz.$$

Definition 2.3. [14] *The integral of fractional order associated to the AB derivative is given by*

$$^{AB}I_{a^+}^{q}F(t) = \frac{1-q}{B(q)}F(t) + \frac{q}{B(q)\Gamma(q)}\int_a^t (t-z)^{q-1}F(z)dz.$$

Next, we define some important properties about the measure of noncompactness.

Definition 2.4. [32] *Let D a bounded subset of X. The map $\beta : X \to [0,\infty)$ defined by $\beta(D) = \inf\{\varepsilon > 0 : B \subseteq \bigcup_{i=1}^{n} B_i$ and diameter$(B_i) \le \varepsilon\}$ for $B \in D$, is called Kuratowski measure of noncompactness.*

Throughout the manuscript, Kuratowski measure of noncompactness on the bounded set D of X, $C(I,X)$ and $PC(I,X)$ is denoted by $\beta(\cdot)$, $\beta_C(\cdot)$ and $\beta_{PC}(\cdot)$, respectively. Moreover, if $D \subset C(I,X)$ is bounded, then $D(t)$ is bounded in X and $\beta(D(t)) \le \beta_C(D)$.

Lemma 2.5. [32] *For any bounded subsets A and B of a real Banach space X, the Kuratowski measure of noncompactness satisfies the following properties*

(i) $\beta(B) = 0 \iff B$ *is relatively compact in X; (ii)* $\beta(B) = \beta(\overline{B})$.

(iii) $\beta(A+B) \le \beta(A) + \beta(B)$; *(iv)* $A \subset B \implies \beta(A) \le \beta(B)$.

(v) $\beta(convB) = \beta(B)$; *(vi)* $\beta(cB) = |c|\beta(B)$; $c \in \mathbb{R}$; *(vii)* $\beta(A \cup B) = \max\{\beta(A),\beta(B)\}$.

Definition 2.6. [32] *If $D \subset C([a,b],X)$ is equicontinuous and bounded over $[a,b]$, then for $t \in [a,b]$, $\beta(D(t))$ is continuous and*

$$\beta(D) = \sup_{t\in[a,b]}\beta(D(t)), \text{ where } D(t) = \{z(t) : t \in D\} \subset X.$$

Lemma 2.7. [32] *Let $D \subset X$ be bounded. Then, there exists a countable subset $D_0 \subset D$ for which the following inequality holds*

$$\beta(D) \le 2\beta(D_0).$$

Lemma 2.8. [32] *Let $D = \{u_n\}_{n=1}^{\infty} \subset C([a,b],X)$ be a bounded and countable set. Then $\beta(D(t))$ is Lebesgue integrable on $[a,b]$ and*

$$\beta\left(\left\{\int_a^b u_n(z)dz\right\}_{n=1}^{\infty}\right) \leq 2\int_a^b \beta(\{u_n(z)\}_{n=1}^{\infty})dz.$$

Next, we define some important semi-group properties which are generally used. We denote $D(\mathcal{A})$ and $\sigma(\mathcal{A})$ for the domain and spectrum of $\mathcal{A}$, respectively. The resolvent of an operator $\mathcal{A}$ at a point λ is defined by $R(\lambda,\mathcal{A}) = \{(\lambda I - \mathcal{A})^{-1},\ \lambda \in \rho(\mathcal{A})\}$, where $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} | (\lambda I - \mathcal{A}) : D(\mathcal{A}) \to X \text{ is invertible}\}$ called the resolvent set.

Definition 2.9. [23] *A linear and closed operator $\mathcal{A}$ is called the sectorial operator if there exist positive constants $c_1 > 0, a \in \mathbb{R}$ and $c \in [\pi/2, \pi]$, such that*

(1) $\sum_{(c,a)} = \{\lambda \in \mathbb{C} : \lambda \neq c, |arg(\lambda - 1)| < c\} \subset \rho(\mathcal{A})$;
(2) $\|R(\lambda,\mathcal{A})\| \leq \dfrac{c_1}{|\lambda - a|},\ \lambda \in \sum_{(c,a)}$.

Theorem 2.10. [32] *The mild solution of the Cauchy problem*

$$\begin{aligned} {}^{ABC}D^q z(t) &= \mathcal{A}z(t) + \mathfrak{F}(t,z(t)),\ q \in (0,1),\ t \in I, \\ z(0) &= z_0, \end{aligned}$$

is given by

$$\begin{aligned} z(t) = \mathfrak{G}T_q(t)z_0 &+ \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^t (t-z)^{q-1}\mathfrak{F}(z,z(z))dz \\ &+ \frac{q\mathfrak{G}^2}{B(q)}\int_0^t S_q(t-z)\mathfrak{F}(z,z(z))dz, \end{aligned}$$

where $\mathfrak{G} = \mu(\mu I - \mathcal{A})^{-1}$ and $\mathfrak{K} = -\epsilon\mathcal{A}(\mu I - \mathcal{A})^{-1}$ are linear operators with $\mu = \dfrac{B(q)}{1-q}$ and

$$\begin{aligned} T_q(t) &= E_q(-\mathfrak{K}t^q) = \frac{1}{2\pi i}\int_\Gamma e^{zt}z^{q-1}(z^q I - \mathfrak{K})^{-1}dz, \\ S_q(t) &= t^{q-1}E_{q,q}(-\mathfrak{K}t^q) = \frac{1}{2\pi i}\int_\Gamma e^{zt}(z^q I - \mathfrak{K})^{-1}dz, \end{aligned}$$

with $\mathfrak{F} \in C(I,X)$ and Γ is a specific path lying on $\sum_{(c,a)}$.

Now, using the above Theorem 2.10 and the Definition 2.14 of [23], we can define the solution of the system (1) as follows.

Definition 2.11. *A function $z \in PC(I,X)$ is called a mild solution of the system (1), if $z(t)$ satisfies the initial condition $z(0) = z_0$, the impulsive conditions $z(t) = \mathfrak{J}_i(t, z(t_i^-)),\ \forall\ t \in (t_i, s_i],\ i = 1, \cdots, m$ and the following integral equations*

$$\begin{aligned} z(t) &= \mathfrak{G}T_q(t)[z_0 - \mathfrak{F}(0,z_0)] + \mathfrak{F}(t,z_a(t)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^t (t-z)^{q-1}[\mathfrak{G}(z,z_b(z)) \\ &\quad + \mathcal{B}u(z)]dz + \frac{q\mathfrak{G}^2}{B(q)}\int_0^t S_q(t-z)[\mathfrak{G}(z,z_b(z)) + \mathcal{B}u(z)]dz,\ \forall\ t \in (0,t_1], \quad (2) \\ z(t) &= \mathfrak{G}T_q(t-s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))] + \mathfrak{F}(t, z_a(t)) \end{aligned}$$

$$+\frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^{t}(t-z)^{q-1}[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz$$
$$+\frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t}S_q(t-z)[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz,\ \forall\ t\in(s_i,t_{i+1}],i=1,\cdots,m. \tag{3}$$

The next theorem is the key theorem to prove our main results.

Theorem 2.12. *(Mönch fixed point theorem)* [35]. *Let D be convex and closed subset of X with $z\in D$. If a continuous operator $F:D\to D$ satisfies the following property : $A\subset D$ countable, $A\subset\overline{co}(z\cup F(A))$ implies A is relatively compact. Then F has a fixed point in D.*

To prove the main outcomes of this manuscript, we consider the following assumptions.

[A1] : The functions $\mathfrak{F},\mathfrak{G}:T_0\times X\to X, T_0=\cup_{i=0}^{m}[s_i,t_{i+1}]$ is continuous and there exist positive constants $M_{\mathfrak{F}},M_{\mathfrak{G}},L_{\mathfrak{F}}$ and $L_{\mathfrak{G}}$ such that

[A1a] : $\|\mathfrak{F}(t,z)\|\le M_{\mathfrak{F}}$ and $\|\mathfrak{G}(t,z)\|\le M_{\mathfrak{G}}$ for all $z\in X,\ t\in T_0$.

[A1b] : $\beta(\mathfrak{F}(t,D))\le L_{\mathfrak{F}}\beta(D)$ and $\beta(\mathfrak{G}(t,D))\le L_{\mathfrak{G}}\beta(D)$ for any bounded sets $D\subset X$ and $t\in T_0$.

[A2] : The functions $\mathfrak{J}_i:I_i\times X\to X, I_i=[t_i,s_i], i=1,\cdots,m$, are continuous and there exist positive constants $M_{\mathfrak{J}}$ and $L_{\mathfrak{J}}$ such that

[A2a] : $\|\mathfrak{J}_i(t,z)\|\le M_{\mathfrak{J}}$ for all $z\in X, t\in I_i$.

[A2b] : $\beta(\mathfrak{J}_i(t,D))\le L_{\mathfrak{J}}\beta(D)$ for any bounded sets $D\subset X$.

[A3] : The linear operators $\mathcal{W}_{s_i}^{t_{i+1}}:L^2(I,U)\to X$ given by

$$\mathcal{W}_{s_i}^{t_{i+1}}u=\frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\mathcal{B}u(z)dz$$
$$+\frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t_{i+1}}S_q(t_{i+1}-z)\mathcal{B}u(z)dz,\ i=0,1,\cdots,m,$$

have the bounded invertible operators $(\mathcal{W}_{s_i}^{t_{i+1}})^{-1},\ i=0,1,\cdots,m$, which take values in $L^2(I,U)\backslash\ker\mathcal{W}_{s_i}^{t_{i+1}}$ and there exist positive constants $M_{\mathcal{W}}^i, i=0,1,\cdots,m$, such that $\|(\mathcal{W}_{s_i}^{t_{i+1}})^{-1}\|\le M_{\mathcal{W}}^i$.

Also, $\mathcal{B}$ is continuous operator from U to X and there exists a positive constant $M_{\mathcal{B}}$ such that $\|\mathcal{B}\|\le M_{\mathcal{B}}$.

[A4] : $\mathfrak{G},\mathfrak{K}\in\mathbb{B}(X)$ and there exist positive constants $M_{\mathfrak{G}}$ and $M_{\mathfrak{K}}$ such that $\|\mathfrak{G}\|\le M_{\mathfrak{G}}$ and $\|\mathfrak{K}\|\le M_{\mathfrak{K}}$.

Now onwards, we set

$$\|T_q(t)\|\le M_T;\ \|S_q(t)\|\le t^{q-1}M_S;\ M=\left(\frac{M_{\mathfrak{K}}(1-q)}{\Gamma(q+1)}+M_{\mathfrak{G}}M_S\right);$$
$$\mathfrak{Q}_1^0=M_{\mathcal{W}}^0\left(L_{\mathfrak{F}}+\frac{2MM_{\mathfrak{G}}L_{\mathfrak{G}}t_1^q}{B(q)}\right),\ d_1^0=\left(L_{\mathfrak{F}}+\frac{2MM_{\mathfrak{G}}(L_{\mathfrak{G}}+M_{\mathcal{B}}\mathfrak{Q}_1^0)t_1^q}{B(q)}\right);$$
$$\mathfrak{Q}_1^i=M_{\mathcal{W}}^i\left(M_{\mathfrak{G}}M_T(L_{\mathfrak{J}}+L_{\mathfrak{F}})+L_{\mathfrak{F}}+\frac{2MM_{\mathfrak{G}}L_{\mathfrak{G}}T^q}{B(q)}\right),\ i=1,\cdots,m;$$
$$d_1^i=\left(M_{\mathfrak{G}}M_T(L_{\mathfrak{J}}+L_{\mathfrak{F}})+L_{\mathfrak{F}}+\frac{2MM_{\mathfrak{G}}(L_{\mathfrak{G}}+M_{\mathcal{B}}\mathfrak{Q}_1^i)T^q}{B(q)}\right),\ i=1,\cdots,m.$$

3. Main Results

Definition 3.1. *Control system (1) is said to be controllable over I, if for every $z_0, z_T \in X$, there exists a piece-wise continuous function $u \in L^2(I, X)$ such that the mild solution of (1) satisfies $z(0) = z_0$ and $z(T) = z_T$.*

Definition 3.2. *Control system (1) is said to be totally controllable on I, if it is controllable on $[0, t_1]$ and $[s_i, t_{i+1}]$, $i = 1, \cdots, m$, i.e., if for every $z_0, z_{t_{i+1}} \in X, l = 0, 1, \cdots, m$, there exists a piece-wise continuous function $u \in L^2(I, X)$ such that the mild solution of (1) satisfies $z(0) = z_0$ and $z(t_{i+1}) = z_{t_{i+1}}$ for $i = 0, 1, \cdots, m$.*

Remark 3.3. *From the above two definitions, one can see that if a system is totally controllable on an interval I, then it is also controllable on I, i.e., Definition 3.2 implies Definition 3.1.*

Lemma 3.4. *If the assumptions **(A1)-(A4)** hold, then the control function*

$$u(t) = (\mathcal{W}_0^{t_1})^{-1}\Bigg[z_{t_1} - \mathfrak{G}T_q(t_1)[z_0 - \mathfrak{F}(0, z_0)] - \mathfrak{F}(t_1, z_a(t_1)) - \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^{t_1}(t_1 - z)^{q-1}\mathfrak{G}(z, z_b(z))dz - \frac{q\mathfrak{G}^2}{B(q)}\int_0^{t_1} S_q(t_1 - z)\mathfrak{G}(z, z_b(z))dz\Bigg](t),\ t \in (0, t_1], \tag{4}$$

transfers the state $z(t)$ of the system (1) from z_0 to z_{t_1}. Further, the control function $u(t)$ has an estimate $\|u(t)\| \le M_u^0$ on $t \in (0, t_1]$, where

$$M_u^0 = M_{\mathcal{W}}^0\Bigg[\|z_{t_1}\| + M_{\mathfrak{G}}M_T(\|z_0\| + M_{\mathfrak{F}}) + M_{\mathfrak{F}} + \frac{MM_{\mathfrak{G}}t_1^qM_{\mathfrak{G}}}{B(q)}\Bigg].$$

Proof : By using the control function $u(t)$ given by the equation (4) in the mild solution $z(t)$ of the system (1) at $t = t_1$, we have

$$\begin{aligned}
z(t_1) = {} & \mathfrak{G}T_q(t_1)[z_0 - \mathfrak{F}(0, z_0)] + \mathfrak{F}(t_1, z_a(t_1)) + \frac{q\mathfrak{G}^2}{B(q)}\int_0^{t_1} S_q(t_1 - z)\mathfrak{G}(z, z_b(z))dz \\
& + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^{t_1}(t_1 - z)^{q-1}\mathfrak{G}(z, z_b(z))dz + (\mathcal{W}_0^{t_1})(\mathcal{W}_0^{t_1})^{-1}\Bigg[z_{t_1} \\
& - \mathfrak{F}(t_1, z_a(t_1)) - \mathfrak{G}T_q(t_1)[z_0 - \mathfrak{F}(0, z_0)] - \frac{q\mathfrak{G}^2}{B(q)}\int_0^{t_1} S_q(t_1 - z)\mathfrak{G}(z, z_b(z))dz \\
& - \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^{t_1}(t_1 - z)^{q-1}\mathfrak{G}(z, z_b(z))dz\Bigg] \\
= {} & z_{t_1}.
\end{aligned}$$

Also, the estimate for the control function $u(t)$ on $t \in (0, t_1]$ is calculated as

$$\begin{aligned}
\|u(t)\| \le {} & M_{\mathcal{W}}^0\Bigg[\|z_{t_1}\| + M_{\mathfrak{G}}M_T[\|z_0\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} \\
& + \frac{M_{\mathfrak{K}}M_{\mathfrak{G}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)}\int_0^{t_1}(t_1 - z)^{q-1}dz + \frac{qM_{\mathfrak{G}}^2M_{\mathfrak{G}}M_S}{B(q)}\int_0^{t_1}(t_1 - z)^{q-1}dz\Bigg] \\
\le {} & M_{\mathcal{W}}^0\Bigg[\|z_{t_1}\| + M_{\mathfrak{G}}M_T[\|z_0\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \Bigg(\frac{M_{\mathfrak{K}}M_{\mathfrak{G}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)}
\end{aligned}$$

$$+\frac{qM_{\mathfrak{G}}^2M_{\mathfrak{G}}M_S}{B(q)}\Big)\frac{t_1^q}{q}\Bigg]$$
$$= M_u^0.$$

Hence, the result follows.

Lemma 3.5. *If the assumptions **(A1)-(A4)** hold, then for $i = 1, \cdots, m$, the control function*

$$u(t) = (\mathcal{W}_{s_i}^{t_{i+1}})^{-1}\Bigg[z_{t_{i+1}} - \mathfrak{G}T_q(t_{i+1}-s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))]$$
$$- \mathfrak{F}(t_{i+1}, z_a(t_{i+1})) - \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\mathfrak{G}(z, z_b(z))dz$$
$$- \frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t_{i+1}}S_q(t_{i+1}-z)\mathfrak{G}(z, z_b(z))dz\Bigg](t),\ t \in (s_i, t_{i+1}], \tag{5}$$

transfers the state $z(t)$ of the system (1) from z_0 to $z_{t_{i+1}}$. Further, the control function $u(t)$ has an estimate $\|u(t)\| \leq M_u^i$ on $t \in (s_i, t_{i+1}]$, where

$$M_u^i = M_{\mathcal{W}}^i\Bigg[\|z_{t_{i+1}}\| + M_{\mathfrak{G}}M_T(M_{\mathfrak{J}} + M_{\mathfrak{F}}) + M_{\mathfrak{F}} + \frac{MM_{\mathfrak{G}}t_{i+1}^qM_{\mathfrak{G}}}{B(q)}\Bigg].$$

Proof : By using the control function $u(t)$ given by the equation (5) in the mild solution $z(t)$ of the system (1) at $t = t_{i+1}, i = 1, \cdots, m$, we have

$$z(t_{i+1}) = \mathfrak{G}T_q(t_{i+1}-s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))] + \mathfrak{F}(t_{i+1}, z_a(t_{i+1}))$$
$$+ \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\mathfrak{G}(z, z_b(z))dz$$
$$+ \frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t_{i+1}}S_q(t_{i+1}-z)\mathfrak{G}(z, z_b(z))dz$$
$$+ (\mathcal{W}_{s_i}^{t_{i+1}})(\mathcal{W}_{s_i}^{t_{i+1}})^{-1}\Bigg[z_{t_{i+1}} - \mathfrak{G}T_q(t_{i+1}-s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))]$$
$$- \mathfrak{F}(t_{i+1}, z_a(t_{i+1})) - \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\mathfrak{G}(z, z_b(z))dz$$
$$- \frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t_{i+1}}S_q(t_{i+1}-z)\mathfrak{G}(z, z_b(z))dz\Bigg]$$
$$= z_{t_{i+1}}.$$

Also, the estimate for the control function $u(t)$ on $t \in (s_i, t_{i+1}],\ i = 1, \cdots, m$, is calculated as

$$\|u(t)\| \leq M_{\mathcal{W}}^i\Bigg[\|z_{t_1}\| + M_{\mathfrak{G}}M_T[\|M_{\mathfrak{J}}\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M_{\mathfrak{K}}M_{\mathfrak{G}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)}$$
$$\times\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}dz + \frac{qM_{\mathfrak{G}}^2M_{\mathfrak{G}}M_S}{B(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}dz\Bigg]$$
$$\leq M_{\mathcal{W}}^0\Bigg[\|z_{t_1}\| + M_{\mathfrak{G}}M_T[\|M_{\mathfrak{J}}\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \Bigg(\frac{M_{\mathfrak{K}}M_{\mathfrak{G}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)}$$
$$+ \frac{qM_{\mathfrak{G}}^2M_{\mathfrak{G}}M_S}{B(q)}\Bigg)\frac{t_{i+1}^q}{q}\Bigg]$$

$$= M_u^i.$$

Hence, the result follows.

Theorem 3.6. *If all the assumptions **(A1)-(A4)** are fulfilled and*

$$d = \max\{\max_{0\le i\le m} d_1^i, L_{\mathfrak{J}}\} < 1, \tag{6}$$

then the system (1) is totally controllable on I.

Proof : Define an operator $\Xi : PC(I,X) \to X$ by

$$\begin{aligned}
(\Xi z)(t) &= \mathfrak{G}T_q(t)[z_0 - \mathfrak{F}(0,z_0)] + \mathfrak{F}(t,z_a(t)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^t (t-z)^{q-1}[\mathfrak{G}(z,z_b(z)) \\
&\quad + \mathcal{B}u(z)]dz + \frac{q\mathfrak{G}^2}{B(q)}\int_0^t S_q(t-z)[\mathfrak{G}(z,z_b(z)) + \mathcal{B}u(z)]dz,\ \forall\, t\in(0,t_1], \\
(\Xi z)(t) &= \mathfrak{J}_i(t, z(t_i^-)), \quad \forall\, t\in(t_i,s_i],\ i=1,\cdots,m, \\
(\Xi z)(t) &= \mathfrak{G}T_q(t-s_i)[\mathfrak{J}_i(s_i,z(t_i^-)) - \mathfrak{F}(s_i,z_a(s_i))] + \mathfrak{F}(t,z_a(t)) \\
&\quad + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^t (t-z)^{q-1}[\mathfrak{G}(z,z_b(z)) + \mathcal{B}u(z)]dz \\
&\quad + \frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^t S_q(t-z)[\mathfrak{G}(z,z_b(z)) + \mathcal{B}u(z)]dz, \forall t\in(s_i,t_{i+1}], i=1,\cdots,m,
\end{aligned}$$

where $u(t)$ is defined by the equations (4) and (5) for $(0,t_1]$ and $(s_i,t_{i+1}], i = 1,\cdots,m$, respectively. From the Lemmas 3.4 and 3.5, we can see that z satisfies $z(t_1) = z_{t_1}$ and $z(t_{i+1}) = z_{t_{i+1}}, i = 1,\cdots,m$, respectively. Clearly, if z is a fixed point of Ξ, then the system (1) is controllable. Now, we shall show that Ξ has a fixed point. For the convenience, we divided the proof into the following steps :

Step 1 : Ξ is continuous. For this let $\{z^n\}_{n=1}^{\infty}$ be a sequence in $PC(I,X)$ such that $z^n \to z$ as $n\to\infty$ for some $z\in PC(I,X)$. Then, for any $t\in(0,t_1]$, we have

$$\begin{aligned}
\|(\Xi z^n)t - (\Xi z)t\| &\le \|\mathfrak{F}(t,z_a^n(t)) - \mathfrak{F}(t,z_a(t))\| + \frac{M_{\mathfrak{K}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \\
&\quad \times \int_0^t (t-z)^{q-1}\|\mathfrak{G}(z,z_b^n(z)) - \mathfrak{G}(z,z_b(z))\|dz \\
&\quad + \frac{M_{\mathfrak{K}}M_{\mathfrak{G}}M_{\mathcal{B}}(1-q)}{B(q)\Gamma(q)}\int_0^t (t-z)^{q-1}\|u_{z^n}(z) - u_z(z)\|dz \\
&\quad + \frac{qM_{\mathfrak{G}}^2}{B(q)}\int_0^t \|S_q(t-z)\|\|\mathfrak{G}(z,z_b^n(z)) - \mathfrak{G}(z,z_b(z))\|dz \\
&\quad + \frac{qM_{\mathfrak{G}}^2M_{\mathcal{B}}M_S}{B(q)}\int_0^t (t-z)^{q-1}\|u_{z^n}(z) - u_z(z)\|dz.
\end{aligned} \tag{7}$$

Furthermore,

$$\begin{aligned}
&\|u_{z^n}(z) - u_z(z)\| \\
&\le M_{\mathcal{W}}^0\Big[\frac{M_{\mathfrak{K}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)}\int_0^{t_1}(t_1-z)^{q-1}\|\mathfrak{G}(z,z_b^n(z)) - \mathfrak{G}(z,z_b(z))\|dz \\
&\quad + \frac{qM_{\mathfrak{G}}^2M_S}{B(q)}\int_0^{t_1}(t_1-z)^{q-1}\|\mathfrak{G}(z,z_b(z)) - \mathfrak{G}(z,z_b(z))\|dz\Big].
\end{aligned} \tag{8}$$

Now, using the equations (7), (8) and (A1) along with Lebesgue dominated convergence theorem, we can conclude that, for all $t \in (0, t_1]$:

$$\|(\Xi z^n)t - (\Xi z)t\| \to 0, \text{ as } n \to \infty. \tag{9}$$

Similarly, for any $t \in (s_i, t_{i+1}], i = 1, \cdots, m$, we have

$$\begin{aligned}\|(\Xi z^n)t - (\Xi z)t\| \leq & M_{\mathfrak{G}} M_T(\|\mathfrak{J}_i(t, z^n(t_i^-)) - \mathfrak{J}_i(t, z(t_i^-))\| + \|\mathfrak{F}(s_i, z_a^n(s_i)) \\ & - \mathfrak{F}(s_i, z_a(s_i))\|) + \|\mathfrak{F}(t, z_a^n(t)) - \mathfrak{F}(t, z_a(t))\| \\ & + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \int_{s_i}^t (t-z)^{q-1} \|\mathfrak{G}(z, z_b^n(z)) - \mathfrak{G}(z, z_b(z))\| dz \\ & + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}} M_{\mathcal{B}}(1-q)}{B(q)\Gamma(q)} \int_{s_i}^t (t-z)^{q-1} \|u_{z^n}(z) - u_z(z)\| dz \\ & + \frac{q M_{\mathfrak{G}}^2}{B(q)} \int_{s_i}^t \|S_q(t-z)\| \|\mathfrak{G}(z, z_b^n(z)) - \mathfrak{G}(z, z_b(z))\| dz \\ & + \frac{q M_{\mathfrak{G}}^2 M_S M_{\mathcal{B}}}{B(q)} \int_{s_i}^t (t-z)^{q-1} \|u_{z^n}(z) - u_z(z)\| dz \end{aligned} \tag{10}$$

and

$$\begin{aligned}\|u_{z^n}(z) - u_z(z)\| \leq M_{\mathcal{W}}^i \Big[& M_{\mathfrak{G}} M_T(\|\mathfrak{J}_i(t_{i+1}, z^n(t_i^-)) - \mathfrak{J}_i(t_{i+1}, z(t_i^-))\| \\ & + \|\mathfrak{F}(s_i, z_a^n(s_i)) - \mathfrak{F}(s_i, z_a(s_i))\|) \\ & + \|\mathfrak{F}(t_{i+1}, z_a^n(t)) - \mathfrak{F}(t_{i+1}, z_a(t))\| + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \\ & \times \int_{s_i}^{t_{i+1}} (t_{i+1}-z)^{q-1} \|\mathfrak{G}(z, z_b^n(z)) - \mathfrak{G}(z, z_b(z))\| dz \\ & + \frac{q M_{\mathfrak{G}}^2 M_S}{B(q)} \int_{s_i}^{t_{i+1}} (t-z)^{q-1} \|\mathfrak{G}(z, z_b^n(z)) - \mathfrak{G}(z, z_b(z))\| dz \Big]. \end{aligned} \tag{11}$$

Now, using the equations (7), (8) and (A1)-(A2) along with Lebesgue dominated convergence theorem, we can conclude that, for all $t \in (s_i, t_{i+1}], i = 1, \cdots, m$:

$$\|(\Xi z^n)t - (\Xi z)t\| \to 0, \text{ as } n \to \infty. \tag{12}$$

Similarly, for any $t \in (t_i, s_i], i = 1, \cdots, m$, we have

$$\|(\Xi z^n)t - (\Xi z)t\| \to 0, \text{ as } n \to \infty. \tag{13}$$

From the equations (9), (12) and (13), for any $t \in I$, we have

$$\|(\Xi z^n) - (\Xi z)\|_{PC} \to 0, \text{ as } n \to \infty.$$

Hence, Ξ is continuous.

Step 2 : We shall show that Ξ maps $B(0, r)$ into $B(0, r)$, where

$$r \geq \max\{\max_{0 \leq i \leq m} \mathfrak{N}_1^i, M_{\mathfrak{J}}\},$$

is a positive number and

$$\mathfrak{N}_1^0 = M_{\mathfrak{G}} M_T[\|z_0\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M M_{\mathfrak{G}}(M_{\mathfrak{G}} + M_{\mathcal{B}} M_u^0) t_1^q}{B(q)},$$

$$\mathfrak{N}_1^i = M_{\mathfrak{G}} M_T[M_J + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M M_{\mathfrak{G}}(M_{\mathfrak{G}} + M_{\mathcal{B}} M_u^i) T^q}{B(q)}, \ i = 1, \cdots, m.$$

For this, let for any $t \in (0, t_1]$, we have

$$\begin{aligned}
\|(\Xi z)t\| &\leq M_{\mathfrak{G}} M_T[\|z_0\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \int_0^t (t-z)^{q-1}[\|\mathfrak{G}(z, z_b(z)) \\
&\quad + \mathcal{B}u(z)\|]dz + \frac{q M_{\mathfrak{G}}^2}{B(q)} \int_0^t \|S_q(t-z)\|[\|\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z)\|]dz \\
&\leq M_{\mathfrak{G}} M_T[\|z_0\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)(M_{\mathfrak{G}} + M_{\mathcal{B}} M_u^0) t_1^q}{B(q)\Gamma(q+1)} \\
&\quad + \frac{M_{\mathfrak{G}}^2 M_S (M_{\mathfrak{G}} + M_{\mathcal{B}} M_u^0) t_1^q}{B(q)} \\
&= \mathfrak{N}_1^0.
\end{aligned} \tag{14}$$

Similarly, for any $t \in (s_i, t_{i+1}], i = 1, \cdots, m$, we have

$$\begin{aligned}
\|(\Xi z)t\| &\leq M_{\mathfrak{G}} M_T[M_{\mathfrak{J}} + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \int_{s_i}^t (t-z)^{q-1}[\|\mathfrak{G}(z, z_b(z)) \\
&\quad + \mathcal{B}u(z)\|]dz + \frac{q M_{\mathfrak{G}}^2}{B(q)} \int_{s_i}^t \|S_q(t-z)\|[\|\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z)\|]dz \\
&\leq M_{\mathfrak{G}} M_T[M_{\mathfrak{J}} + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)(M_{\mathfrak{G}} + M_{\mathcal{B}} M_u^i) t_{i+1}^q}{B(q)\Gamma(q)} \\
&\quad + \frac{M_{\mathfrak{G}}^2 M_S (M_{\mathfrak{G}} + M_{\mathcal{B}} M_u^i) t_{i+1}^q}{B(q)} \\
&\leq \mathfrak{N}_1^i.
\end{aligned} \tag{15}$$

Similarly, for any $t \in (t_i, s_i], i = 1, \cdots, m$, we have

$$\|(\Xi z)t\| \leq M_{\mathfrak{J}}. \tag{16}$$

From the above equations (14), (15) and (16), for any $t \in I$, we have

$$\|\Xi z\|_{PC} \leq r.$$

Hence, Ξ maps $B(0, r)$ into $B(0, r)$.
Step 3 : We shall show that if $D \subset B(0, r)$ is countable and

$$D \subset \overline{co}(\{a\} \cup \Xi(D)), \tag{17}$$

where $a \in B(0, r)$, then D is relatively compact. Without loss of generality, suppose that $D = \{z^n\}_{n=1}^{\infty}$. First we show $\{\Xi z^n\}_{n=1}^{\infty}$ is equicontinuous. If this is true then $\overline{co}(\{a\} \cup \Xi(D))$ is also equicontinuous. For this, let for any $z \in D$, $\tau_2, \tau_1 \in (0, t_1]$ such that $\tau_1 < \tau_2$, we have

$$\begin{aligned}
\|(\Xi z)\tau_2 - (\Xi z)\tau_1\| \leq\ & M_{\mathfrak{G}} \|T_q(\tau_2) - T_q(\tau_1)\|(\|z_0\| + M_{\mathfrak{F}})\| \\
& + \|\mathfrak{F}(\tau_2, z_a(\tau_2)) - \mathfrak{F}(\tau_1, z_a(\tau_1))\| \\
& + \frac{M_{\mathfrak{K}} M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \left\| \int_0^{\tau_2} (\tau_2 - z)^{q-1}[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z)]dz \right. \\
& \left. - \int_0^{\tau_1} (\tau_1 - z)^{q-1}[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z)]dz \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{qM_{\mathfrak{G}}^2}{B(q)}\Big\|\int_0^{\tau_2} S_q(\tau_2-z)[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz\\
&-\int_0^{\tau_1} S_q(\tau_1-z)[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz\Big\|\\
\le\, & M_{\mathfrak{G}}\|T_q(\tau_2)-T_q(\tau_1)\|(\|z_0\|+M_{\mathfrak{F}})\|+\|\mathfrak{F}(\tau_2,z_a(\tau_2))\\
&-\mathfrak{F}(\tau_1,z_a(\tau_1))\|+\frac{M_{\mathfrak{K}}M_{\mathfrak{G}}(1-q)(M_{\mathfrak{G}}+M_{\mathcal{B}}M_u^0)}{B(q)\Gamma(q)}\\
&\times\left(\int_0^{\tau_1}((\tau_2-z)^{q-1}-(\tau_1-z)^{q-1})dz+\int_{\tau_1}^{\tau_2}(\tau_2-z)^{q-1}dz\right)\\
&+\frac{qM_{\mathfrak{G}}^2(M_{\mathfrak{G}}+M_{\mathcal{B}}M_u^0)}{B(q)}\left(\int_0^{\tau_1}\|S_q(\tau_2-z)-S_q(\tau_1-z)\|dz\right.\\
&\left.+\int_{\tau_1}^{\tau_2}\|S_q(\tau_1-z)\|dz\right). \qquad (18)
\end{aligned}
$$

Similarly, for any $z\in D$, $\tau_2,\tau_1\in(s_i,t_{i+1}], i=1,\cdots,m$, such that $\tau_1<\tau_2$, we have

$$
\begin{aligned}
\|(\Xi z)\tau_2-(\Xi z)\tau_1\|\le\, & M_{\mathfrak{G}}\|T_q(\tau_2-s_i)-T_q(\tau_1-s_i)\|(\|M_{\mathfrak{J}}+M_{\mathfrak{F}})\|\\
&+\|\mathfrak{F}(\tau_2,z_a(\tau_2))-\mathfrak{F}(\tau_1,z_a(\tau_1))\|\\
&+\frac{M_{\mathfrak{K}}M_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)}\Big\|\int_{s_i}^{\tau_2}(\tau_2-z)^{q-1}[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz\\
&-\int_{s_i}^{\tau_1}(\tau_1-z)^{q-1}[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz\Big\|\\
&+\frac{qM_{\mathfrak{G}}^2}{B(q)}\Big\|\int_{s_i}^{\tau_2} S_q(\tau_2-z)[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz\\
&-\int_{s_i}^{\tau_1} S_q(\tau_1-z)[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)]dz\Big\|\\
\le\, & M_{\mathfrak{G}}\|T_q(\tau_2-s_i)-T_q(\tau_1-s_i)\|(M_{\mathfrak{J}}+M_{\mathfrak{F}})\|+\|\mathfrak{F}(\tau_2,z_a(\tau_2))\\
&-\mathfrak{F}(\tau_1,z_a(\tau_1))\|+\frac{M_{\mathfrak{K}}M_{\mathfrak{G}}(1-q)(M_{\mathfrak{G}}+M_{\mathcal{B}}M_u^i)}{B(q)\Gamma(q)}\\
&\times\left(\int_{s_i}^{\tau_1}((\tau_2-z)^{q-1}-(\tau_1-z)^{q-1})dz\right.\\
&\left.+\int_{\tau_1}^{\tau_2}(\tau_2-z)^{q-1}dz\right)+\frac{qM_{\mathfrak{G}}^2(M_{\mathfrak{G}}+M_{\mathcal{B}}M_u^i)}{B(q)}\\
&\times\left(\int_{s_i}^{\tau_1}\|S_q(\tau_2-z)-S_q(\tau_1-z)\|+\int_{\tau_1}^{\tau_2}\|S_q(\tau_1-z)\|dz\right). \qquad (19)
\end{aligned}
$$

Also, for any $z\in D$, $\tau_2,\tau_1\in(t_i,s_i], i=1,\cdots,m$, such that $\tau_1<\tau_2$, we have

$$\|(\Xi z)\tau_2-(\Xi z)\tau_1\|\le\|\mathfrak{J}_i(\tau_2,z(t_i^-))-\mathfrak{J}_i(\tau_1,z(t_i^-))\|. \qquad (20)$$

From the compactness of $T_q(t)$ and $S_q(t)$ for $t>0$ along with (A1)-(A3) and the absolute continuity of the Lebesgue integral, we can see that the right-hand side of the above inequalities (18), (19) and (20), tends to zero as $\tau_2\to\tau_1$. Therefore, $\Xi(D)$ is equicontinuous.

Next, by using the Lemma 2.8 and assumption (A1b), we can notice that for any $t \in (0, t_1]$,

$$\begin{aligned}\beta(\{u_{z^n}(t)\}_{n=1}^{\infty}) &\le M_{\mathcal{W}}^0 \beta\Bigg(\Bigg\{\mathfrak{F}(t_1, z_a^n(t_1)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^{t_1}(t_1-z)^{q-1}\mathfrak{G}(z, z_b^n(z))dz \\ &\quad + \frac{q\mathfrak{G}^2}{B(q)}\int_0^{t_1} S_q(t_1-z)\mathfrak{G}(z, z_b^n(z))dz\Bigg](t)\Bigg\}_{n=1}^{\infty}\Bigg) \\ &\le M_{\mathcal{W}}^0\Bigg(L_{\mathfrak{F}} + \frac{2M_{\mathfrak{K}}M_{\mathfrak{G}}L_{\mathfrak{G}}(1-q)t_1^q}{B(q)\Gamma(q+1)} + \frac{2M_{\mathfrak{G}}^2 M_S L_{\mathfrak{G}} t_1^q}{B(q)}\Bigg)\beta_{PC}(\{z^n\}_{n=1}^{\infty}) \\ &= \mathfrak{Q}_1^0 \beta_{PC}(\{z^n\}_{n=1}^{\infty}).\end{aligned}$$

Henceforth, for any $t \in (0, t_1]$,

$$\begin{aligned}\beta(\{\Xi z^n(t)\}_{n=1}^{\infty}) &= \beta\Bigg(\Bigg\{\mathfrak{G}T_q(t)[z_0 - \mathfrak{F}(0, z_0)] + \mathfrak{F}(t, z_a^n(t)) \\ &\quad + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_0^t (t-z)^{q-1}[\mathfrak{G}(z, z_b^n(z)) + \mathcal{B}u_{z^n}(z)]dz \\ &\quad + \frac{q\mathfrak{G}^2}{B(q)}\int_0^t S_q(t-z)[\mathfrak{G}(z, z_b^n(z)) + \mathcal{B}u_{z^n}(z)]dz\Bigg\}_{n=1}^{\infty}\Bigg) \\ &\le \Bigg(L_{\mathfrak{F}} + \frac{2M_{\mathfrak{K}}M_{\mathfrak{G}}(L_{\mathfrak{G}} + M_{\mathcal{B}}\mathfrak{Q}_1^0)(1-q)t_1^q}{B(q)\Gamma(q+1)} \\ &\quad + \frac{2M_{\mathfrak{G}}^2 M_S(L_{\mathfrak{G}} + M_{\mathcal{B}}\mathfrak{Q}_1^0)t_1^q}{B(q)}\Bigg)\beta_{PC}(\{z^n\}_{n=1}^{\infty}) \\ &= d_1^0 \beta_{PC}(\{z^n\}_{n=1}^{\infty}). \end{aligned} \tag{21}$$

Similarly, for any $t \in (s_i, t_{i+1}], i = 1, \cdots, m$,

$$\begin{aligned}\beta(\{u_{z^n}(t)\}_{n=1}^{\infty}) &= \beta\Bigg(\Bigg\{(\mathcal{W}_{s_i}^{t_{i+1}})^{-1}\Bigg[z_{t_{i+1}} - \mathfrak{G}T_q(t_{i+1}-s_i)[\mathfrak{J}_i(s_i, z^n(t_i^-)) \\ &\quad - \mathfrak{F}(s_i, z_a^n(s_i))] - \mathfrak{F}(t_{i+1}, z_a^n(t_{i+1})) \\ &\quad - \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\mathfrak{G}(z, z_b^n(z))dz \\ &\quad - \frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t_{i+1}} S_q(t_{i+1}-z)\mathfrak{G}(z, z_b^n(z))dz\Bigg](t)\Bigg\}_{n=1}^{\infty}\Bigg) \\ &\le M_{\mathcal{W}}^i\Bigg(M_{\mathfrak{G}}M_T(L_{\mathfrak{J}}\beta(\{z^n(t_i^-)\}_{n=1}^{\infty}) + L_{\mathfrak{F}}\beta(\{z_a^n(s_i)\}_{n=1}^{\infty})) \\ &\quad + L_{\mathfrak{F}}\beta(\{z_a^n(t_{i+1})\}_{n=1}^{\infty}) + \frac{2M_{\mathfrak{K}}M_{\mathfrak{G}}L_{\mathfrak{G}}(1-q)}{B(q)\Gamma(q)} \\ &\quad \times \int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\beta(\{z_b^n(z)\}_{n=1}^{\infty})dz \\ &\quad + \frac{2qM_{\mathfrak{G}}^2 M_S L_{\mathfrak{G}}}{B(q)}\int_{s_i}^{t_{i+1}}(t_{i+1}-z)^{q-1}\beta(\{z_b^n(z)\}_{n=1}^{\infty})dz\Bigg) \\ &\le M_{\mathcal{W}}^i\Bigg(M_{\mathfrak{G}}M_T(L_{\mathfrak{J}} + L_{\mathfrak{F}}) + L_{\mathfrak{F}} + \frac{2M_{\mathfrak{K}}M_{\mathfrak{G}}L_{\mathfrak{G}}(1-q)t_{i+1}^q}{B(q)\Gamma(q+1)}\end{aligned}$$

$$+ \frac{2M_{\mathfrak{G}}^2 M_S L_{\mathfrak{G}} t_{i+1}^q}{B(q)}\Big)\beta_{PC}(\{z^n\}_{n=1}^{\infty})$$
$$\leq \mathfrak{Q}_1^i \beta_{PC}(\{z^n\}_{n=1}^{\infty})$$

and

$$\begin{aligned}\beta(\{\Xi z^n(t)\}_{n=1}^{\infty}) &= \beta\Bigg(\Bigg\{\mathfrak{G}T_q(t-s_i)[\mathfrak{J}_i(s_i, z^n(t_i^-)) - \mathfrak{F}(s_i, z_a^n(s_i))] + \mathfrak{F}(t, z_a^n(t)) \\ &\quad + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}\int_{s_i}^t (t-z)^{q-1}[\mathfrak{G}(z, z_b^n(z)) + \mathcal{B}u_{z^n}(z)]dz \\ &\quad + \frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^t S_q(t-z)[\mathfrak{G}(z, z_b^n(z)) + \mathcal{B}u_{z^n}(z)]dz\Bigg\}_{n=1}^{\infty}\Bigg) \\ &\leq \Bigg(M_{\mathfrak{G}}M_T(L_{\mathfrak{J}} + L_{\mathfrak{F}}) + L_{\mathfrak{F}} + \frac{2M_{\mathfrak{K}}M_{\mathfrak{G}}(L_{\mathfrak{G}} + M_{\mathcal{B}}\mathfrak{Q}_1^i)(1-q)t_{i+1}^q}{B(q)\Gamma(q+1)} \\ &\quad + \frac{2M_{\mathfrak{G}}^2 M_S(L_{\mathfrak{G}} + M_{\mathcal{B}}\mathfrak{Q}_1^i)t_{i+1}^q}{B(q)}\Bigg)\beta_{PC}(\{z^n\}_{n=1}^{\infty}) \\ &= d_1^i \beta_{PC}(\{z^n\}_{n=1}^{\infty}). \end{aligned} \tag{22}$$

Also, for $(t_i, s_i], i = 1, \cdots, m$,

$$\beta(\{\Xi z^n(t)\}_{n=1}^{\infty}) \leq L_{\mathfrak{J}}\beta_{PC}(\{z^n\}_{n=1}^{\infty}). \tag{23}$$

Hence, from the above equations (21), (22) and (23), for any $t \in I$, we have

$$\beta_{PC}(\{\Xi z^n(t)\}_{n=1}^{\infty}) \leq d\beta_{PC}(\{z^n\}_{n=1}^{\infty}). \tag{24}$$

Now, from the equations (6), (17), and (24), we get

$$\beta_{PC}(\{z^n\}_{n=1}^{\infty}) \leq \beta_{PC}(\{\Xi z^n(t)\}_{n=1}^{\infty}) \leq d\beta_{PC}(\{z^n\}_{n=1}^{\infty}),$$

which immediately shows that D is relatively compact.

Now, collecting the above three steps, we can conclude that all the required conditions of Theorem 2.12 are satisfied. Therefore, Ξ has a fixed point in $B(0, r)$ and hence, the system (1) is totally controllable on I.

4. Controllability of Integro-Differential Systems

In this section, we establish the total controllability results for the control system (1) with integral term of the form:

$$\begin{aligned} {}^{ABC}D^q[z(t) - \mathfrak{F}(t, z_a(t))] &= \mathcal{A}[z(t) - \mathfrak{F}(t, z_a(t))] + \mathcal{B}u(t) + \int_{s_i}^t \kappa(t, z)\mathfrak{H}(z, z(z))dz \\ &\quad + \mathfrak{G}(t, z_b(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, m, \\ z(t) &= \mathfrak{J}_i(t, z(t_i^-)),\ t \in (t_i, s_i],\ i = 1, \cdots, m, \\ z(0) &= z_0. \end{aligned} \tag{25}$$

Definition 4.1. *A function $z \in PC(I, X)$ is called a mild solution of the system (25), if $z(t)$ satisfies the initial condition $z(0) = z_0$, the impulsive conditions $z(t) = \mathfrak{J}_i(t, z(t_i^-)),\ \forall\ t \in (t_i, s_i],\ i = 1, \cdots, m$ and the following integral equations*

$$z(t) = \mathfrak{G}T_q(t)[z_0 - \mathfrak{F}(0, z_0)] + \mathfrak{F}(t, z_a(t)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}$$

$$\times \int_0^t (t-z)^{q-1} \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_0^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz$$
$$+ \frac{q\mathfrak{G}^2}{B(q)} \int_0^t S_q(t-z) \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_0^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz,$$
$$\forall\ t \in (0, t_1],$$

$$z(t) = \mathfrak{G}T_q(t-s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))] + \mathfrak{F}(t, z_a(t)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)}$$
$$\times \int_{s_i}^t (t-z)^{q-1} \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_{s_i}^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz$$
$$+ \frac{q\mathfrak{G}^2}{B(q)} \int_{s_i}^t S_q(t-z) \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_{s_i}^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz,$$
$$\forall\ t \in (s_i, t_{i+1}], i = 1, \cdots, m.$$

We need some more assumptions to establish the controllability results for the integro control system (25).

[A5] : Function $\kappa : T_0 \times T_0 \to \mathbb{R}$ is continuous and there exists a positive constant M_κ such that $\int_{s_i}^t |\kappa(t,s)|ds \leq M_\kappa$ for $i = 0, 1, ..., m$.

[A6] : Function $\mathfrak{H} : T_0 \times X \to X$ is continuous and there exist positive constants $M_{\mathfrak{H}}$ and $L_{\mathfrak{H}}$ such that

[A6a] : $\|\mathfrak{H}(t,z)\| \leq M_{\mathfrak{H}}$ for all $z \in X,\ t \in T_0$.

[A6b] : $\beta(\mathfrak{H}(t,D)) \leq L_{\mathfrak{H}}\beta(D)$ for any bounded sets $D \subset X$ and $t \in T_0$.

Lemma 4.2. *If the assumptions **(A1)-(A6)** hold, then the control function*

$$u(t) = (\mathcal{W}_0^{t_1})^{-1}\Bigg[z_{t_1} - \mathfrak{G}T_q(t_1)[z_0 - \mathfrak{F}(0, z_0)] - \mathfrak{F}(t_1, z_a(t_1))$$
$$- \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)} \int_0^{t_1} (t_1 - z)^{q-1}[\mathfrak{G}(z, z_b(z)) + \int_0^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau]dz$$
$$- \frac{q\mathfrak{G}^2}{B(q)} \int_0^{t_1} [S_q(t_1 - z)\mathfrak{G}(z, z_b(z)) + \int_0^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau]dz\Bigg](t), \quad (26)$$

for all $t \in (0, t_1]$, transfers the state $z(t)$ of the system (25) from z_0 to z_{t_1}. Further, the control function $u(t)$ has an estimate $\|u(t)\| \leq \hat{M}_u^0$ on $t \in (0, t_1]$, where

$$\hat{M}_u^0 = M_{\mathcal{W}}^0\left[\|z_{t_1}\| + M_{\mathfrak{G}}M_T(\|z_0\| + M_{\mathfrak{F}}) + M_{\mathfrak{F}} + \frac{MM_{\mathfrak{G}}t_1^q(M_{\mathfrak{G}} + M_\kappa M_{\mathfrak{H}})}{B(q)}\right].$$

Proof : This can be proved by using the same technique of Lemma 3.4. Hence, we omitted the proof.

Lemma 4.3. *If the assumptions **(A1)-(A6)** hold, then for $t \in (s_i, t_{i+1}], i = 1, \cdots, m$, the control function*

$$u(t) = (\mathcal{W}_{s_i}^{t_{i+1}})^{-1}\Bigg[z_{t_{i+1}} - \mathfrak{G}T_q(t_{i+1} - s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))]$$
$$- \mathfrak{F}(t_{i+1}, z_a(t_{i+1})) - \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)} \int_{s_i}^{t_{i+1}} (t_{i+1} - z)^{q-1}\mathfrak{G}(z, z_b(z))dz$$
$$- \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)} \int_{s_i}^{t_{i+1}} (t_{i+1} - z)^{q-1} \int_{s_i}^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau dz$$

$$- \frac{q\mathfrak{G}^2}{B(q)} \int_{s_i}^{t_{i+1}} S_q(t_{i+1}-z)[\mathfrak{G}(z, z_b(z)) + \int_{s_i}^{z} \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau]dz\Bigg](t), \tag{27}$$

transfers the state $z(t)$ *of the system (25) from* z_0 *to* $z_{t_{i+1}}$. *Further, the control function* $u(t)$ *has an estimate* $\|u(t)\| \leq \hat{M}_u^i$ *on* $t \in (s_i, t_{i+1}]$, *where*

$$\hat{M}_u^i = M_{\mathcal{W}}^i \left[\|z_{t_{i+1}}\| + M_{\mathfrak{G}} M_T (M_{\mathfrak{J}} + M_{\mathfrak{F}}) + M_{\mathfrak{F}} + \frac{MM_{\mathfrak{G}} + t_{i+1}^q (M_{\mathfrak{G}} + M_\kappa M_{\mathfrak{H}})}{B(q)}\right].$$

Proof : This can be proved by using the same technique of Lemma 3.5. Hence, we omitted the proof.

We set

$$\hat{\mathfrak{Q}}_1^0 = M_{\mathcal{W}}^0 \left(L_{\mathfrak{F}} + \frac{2MM_{\mathfrak{G}} t_1^q (L_{\mathfrak{G}} + M_\kappa L_{\mathfrak{H}})}{B(q)}\right),$$

$$\hat{d}_1^0 = \left(L_{\mathfrak{F}} + \frac{2MM_{\mathfrak{G}}(L_{\mathfrak{G}} + M_\kappa L_{\mathfrak{H}} + M_{\mathcal{B}} \hat{\mathfrak{Q}}_1^0) t_1^q}{B(q)}\right);$$

$$\hat{\mathfrak{Q}}_1^i = M_{\mathcal{W}}^i \left(M_{\mathfrak{G}} M_T (L_{\mathfrak{J}} + L_{\mathfrak{F}}) + L_{\mathfrak{F}} + \frac{2MM_{\mathfrak{G}} T^q (L_{\mathfrak{G}} + M_\kappa L_{\mathfrak{H}})}{B(q)}\right), \ i = 1, \cdots, m;$$

$$\hat{d}_1^i = \left(M_{\mathfrak{G}} M_T (L_{\mathfrak{J}} + L_{\mathfrak{F}}) + L_{\mathfrak{F}} + \frac{2MM_{\mathfrak{G}}(L_{\mathfrak{G}} + M_\kappa L_{\mathfrak{H}} + M_{\mathcal{B}} \hat{\mathfrak{Q}}_1^i) T^q}{B(q)}\right), \ i = 1, \cdots, m;$$

$$\hat{\mathfrak{N}}_1^0 = M_{\mathfrak{G}} M_T [\|z_0\| + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{MM_{\mathfrak{G}}(M_{\mathfrak{G}} + M_\kappa M_{\mathfrak{H}} + M_{\mathcal{B}} \hat{M}_u^0) t_1^q}{B(q)};$$

$$\hat{\mathfrak{N}}_1^i = M_{\mathfrak{G}} M_T [M_J + M_{\mathfrak{F}}] + M_{\mathfrak{F}} + \frac{MM_{\mathfrak{G}}(M_{\mathfrak{G}} + M_\kappa M_{\mathfrak{H}} + M_{\mathcal{B}} \hat{M}_u^i) T^q}{B(q)}, \ i = 1, \cdots, m.$$

Theorem 4.4. *If all the assumptions* ***(A1)-(A6)*** *are fulfilled and*

$$\hat{d} = \max\{\max_{0 \leq i \leq m} \hat{d}_1^i, L_{\mathfrak{J}}\} < 1, \tag{28}$$

then the system (25) is totally controllable on I.

Proof : Define an operator $\Xi_1 : PC(I, X) \to X$ by

$$\begin{aligned}
(\Xi_1 z)(t) &= \mathfrak{G} T_q(t)[z_0 - \mathfrak{F}(0, z_0)] + \mathfrak{F}(t, z_a(t)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)} \\
&\quad \times \int_0^t (t-z)^{q-1} \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_0^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz \\
&\quad + \frac{q\mathfrak{G}^2}{B(q)} \int_0^t S_q(t-z) \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_0^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz, \\
&\quad \forall\ t \in (0, t_1], \\
(\Xi_1 z)(t) &= \mathfrak{J}_i(t, z(t_i^-)), \quad \forall\ t \in (t_i, s_i],\ i = 1, \cdots, m, \\
(\Xi_1 z)(t) &= \mathfrak{G} T_q(t-s_i)[\mathfrak{J}_i(s_i, z(t_i^-)) - \mathfrak{F}(s_i, z_a(s_i))] + \mathfrak{F}(t, z_a(t)) + \frac{\mathfrak{K}\mathfrak{G}(1-q)}{B(q)\Gamma(q)} \\
&\quad \times \int_{s_i}^t (t-z)^{q-1} \left[\mathfrak{G}(z, z_b(z)) + \mathcal{B}u(z) + \int_{s_i}^z \kappa(z,\tau)\mathfrak{H}(\tau, z(\tau))d\tau\right] dz
\end{aligned}$$

$$+\frac{q\mathfrak{G}^2}{B(q)}\int_{s_i}^{t} S_q(t-z)\left[\mathfrak{G}(z,z_b(z))+\mathcal{B}u(z)+\int_{s_i}^{z}\kappa(z,\tau)\mathfrak{H}(\tau,z(\tau))d\tau\right]dz,$$
$$\forall\ t\in(s_i,t_{i+1}],\ i=1,\cdots,m,$$

where $u(t)$ is defined by the equations (26) and (27) for $(0,t_1]$ and $(s_i,t_{i+1}], i=1,\cdots,m$, respectively. From the Lemmas 4.2 and 4.3, we can see that z satisfies $z(t_1)=z_{t_1}$ and $z(t_{i+1})=z_{t_{i+1}}, i=1,\cdots,m$, respectively. Clearly, if z is a fixed point of Ξ_1, then the system (25) is controllable. Since, using the same technique of Theorem 3.6, one can prove that Ξ_1 has a fixed point. Therefore, we are giving the brief proof of this theorem. By applying the same method as step 1 of Theorem 3.6, we can easily show that Ξ_1 is a continuous mapping while by applying the same method as step 2 of Theorem 3.6, we can show Ξ_1 maps $B(0,\hat{r})$ into $B(0,\hat{r})$, where

$$\hat{r}\geq\max\{\max_{0\leq i\leq m}\hat{\mathfrak{N}}_1^i, M_{\mathfrak{Z}}\},$$

is a positive number. Next, we show that if $\hat{D}\subset B(0,\hat{r})$ is countable and

$$\hat{D}\subset\overline{co}(\{a\}\cup\Xi(\hat{D})), \tag{29}$$

where $a\in B(0,\hat{r})$, then $\hat{D}$ is relatively compact. Without loss of generality, suppose that $\hat{D}=\{z^n\}_{n=1}^{\infty}$. Now, for any $z\in\hat{D}$, $\tau_2,\tau_1\in I$ such that $\tau_1<\tau_2$, one can easily show that $\|(\Xi_1 z)\tau_2-(\Xi_1 z)\tau_1\|\to 0$ as $\tau_2\to\tau_1$. Therefore, $\Xi_1(\hat{D})$ is equicontinuous. Also, by using the Lemma 2.8 and assumptions (A1b), (A2b) and (A6b), for any $t\in I$, we have

$$\beta_{PC}(\{\Xi_1 z^n(t)\}_{n=1}^{\infty})\leq\hat{d}\beta_{PC}(\{z^n\}_{n=1}^{\infty}). \tag{30}$$

Therefore, from the equations (28), (29) and (30), we get

$$\beta_{PC}(\{z^n\}_{n=1}^{\infty})\leq\beta_{PC}(\{\Xi_1 z^n(t)\}_{n=1}^{\infty})\leq\hat{d}\beta_{PC}(\{z^n\}_{n=1}^{\infty}),$$

which immediately shows that $\hat{D}$ is relatively compact. Thus, we can conclude that all the required conditions of Theorem 2.12 are satisfied. Therefore, Ξ_1 has a fixed point in $B(0,\hat{r})$ and hence, the system (25) is totally controllable on I.

5. An Illustrative Example

Consider the following fractional partial differential equation in $X=L^2(0,\pi)$

$$\begin{aligned}
{}^{ABC}D^q\left[Z(t,\eta)-\frac{t+\sin(Z(a(t),\eta))}{20e^{t+1}}\right]&=\frac{\partial^2}{\partial\eta^2}\left[Z(t,\eta)-\frac{t+\sin(Z(a(t),\eta))}{20e^{t+1}}\right]\\
&\quad+d(\eta)S(t,\eta)+\frac{t\cos(Z(b(t),\eta))}{10(t+2)^2},\\
&\quad t\in(0,0.5]\cup(0.6,1],\eta\in[0,\pi],q\in(0,1),\\
Z(t,0)=Z(t,\pi)&=0,\quad t\in I=[0,1],\\
Z(t,\eta)&=\frac{1}{5e^{t+1}}\frac{|Z(t,\eta)|}{3+|Z(t,\eta)|},\quad t\in I,\eta\in[0,\pi],\\
Z(0,\eta)&=z_0,\quad\eta\in[0,\pi].
\end{aligned}\tag{31}$$

Define an operator $\mathcal{A}$ by $\mathcal{A}z=z''$, $\forall\ z\in D(\mathcal{A})$, where

$$D(\mathcal{A})=\{z\in X: z \text{ and } z'' \text{ are absolutely continuous and } z(0)=z(\pi)=0\}.$$

Then, operator $\mathcal{A}$ has the following representation

$$\mathcal{A}z = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \quad z \in D(\mathcal{A}),$$

where $z_n(s) = \sqrt{2/\pi}\sin(ns), n = 1, 2, 3, ...$ is the orthogonal set of eigenvectors of $\mathcal{A}$.

Further, it is known that $\mathcal{A}$ is a generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ in X which is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} < z, z_n > z_n, \quad z \in D(\mathcal{A}),\ t > 0.$$

Clearly, $\{T(t)\}_{t\geq 0}$ is uniformly bounded compact semigroup and hence, the operator $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ is compact for each $\lambda \in \rho(\mathcal{A})$. Also, from the subordination principle of solution operator, we have $\|T_q(t)\| \leq M_T, \forall t \ \in I$.

Now, for $(t, \eta) \in I \times [0, \pi],\ \mathcal{B} \in \mathbb{B}(U, X)$ we set

$$\begin{aligned}
&z(t) = Z(t, \cdot),\ \text{i.e.,}\ \ z(t)(\eta) = Z(t, \eta),\ \mathfrak{F}(t, z_a(t))(\eta) = \frac{t + \sin(Z(a(t), \eta))}{20e^{t+1}},\\
&\mathfrak{G}(t, z_b(t))(\eta) = \frac{t\cos(Z(b(t), \eta))}{10(t+2)^2},\ \mathfrak{I}_i(t, z(t))(\eta) = \frac{1}{5e^{t+1}}\frac{|Z(t, \eta)|}{3 + |Z(t, \eta)|},\\
&\mathcal{B}u(t)(\eta) = d(\eta)S(t, \eta).
\end{aligned}$$

With this formulation, the equation (31) can be rewritten in the abstract form (1). Clearly, we can see that all the assumptions of Theorem 3.6 are hold. Therefore, based on Theorem 3.6, we can conclude that the system (31) is totally controllable on I.

Conclusion

We have successfully investigated some sufficient conditions for the controllability results of an ABC-fractional neutral differential equations with non-instantaneous impulses. Also, we studied the considered problem with integral term. To established these results, mainly we used the measure of noncompactness, Mönch fixed point theorem, functional analysis along with the semi-group to establish these results. Also, we have presented an example to validate the outcomes.

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