Journal of Fractional Calculus and Applications,
Vol. 4(2) July 2013, pp. 367- 372.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

MAJORIZATION FOR CERTAIN CLASS OF MULTIVALENT FUNCTIONS DEFINED BY DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, we obtain majorization results for certain class of multivalent functions defined by a differential operator .

1. INTRODUCTION

Let $A(p,j)$ be the class of functions which are analytic and p-valent in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{k=p+j}^{\infty} a_k z^k \ (p, j \in \mathbb{N} = \{1, 2, ...\}). \tag{1}$$

For $g(z) \in A(p,j)$, given by $g(z) = z^p + \sum\limits_{k=p+j}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+j}^{\infty} a_k b_k z^k = (g * f)(z). \tag{2}$$

For $f(z) \in A(p,j)$, we have (see [6]):

$$f^{(q)}(z) = \delta(p,q) z^{p-q} + \sum_{k=p+j}^{\infty} \delta(k,q) a_k z^{k-q} \ (q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p > q), \tag{3}$$

where

$$\delta(x,y) = \frac{x!}{(x-y)!} = \begin{cases} 1 & (y = 0) \\ x(x-1)...(x-y+1) & (y \neq 0) \end{cases}.$$

For $f(z) \in A(p,j)$, Aouf ([3] and [4]) defined the operator $D_p^m f^{(q)}(z)$ as follows:

$$\begin{aligned} D_p^0 f^{(q)}(z) &= f^{(q)}(z); \\ D_p^1 f^{(q)}(z) &= D_p f^{(q)}(z) = \frac{z}{(p-q)} (f^{(q)}(z))' = \frac{z}{(p-q)} f^{(1+q)}(z) \end{aligned}$$

1991 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Multivalent functions, majorization , subordination, differential operator.
Submitted may 1, 2013 Revised June 22, 2013.

and (in general):

$$\begin{aligned} D_p^m f^{(q)}(z) &= D_p(D_p^{(m-1)} f^{(q)}(z)) \\ &= \delta(p,q)z^{p-q} + \sum_{k=p+j}^{\infty} \delta(k,q) \left(\frac{k-q}{p-q}\right)^m a_k z^{k-q} \\ (p,j &\in \mathbb{N}; m,q \in \mathbb{N}_0; p>q).4 \end{aligned} \quad (1)$$

We note that, for $q = 0$, $D_p^m f^{(0)}(z) = D_p^m f(z)$, where the operator D_p^m was introduced and studied by Kamali and Orhan [9] and Aouf and Mostafa [5] which for $p = 1$ reduces to the Salagean operator D^m (see [15]).

From (4), one can easily verify that:

$$z\left(D_p^m f^{(q)}(z)\right)' = (p-q) D_p^{m+1} f^{(q)}(z). \quad (5)$$

For two analytic functions f, $g \in A(p,j)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

If $f(z)$ and $g(z)$ are analytic functions in U, then $f(z)$ is majorized by $g(z)$ in U and written

$$f(z) << g(z) \quad (z \in U), \quad (6)$$

if there exists a function $\phi(z)$, analytic in U, such that (see [10]):

$$|\phi(z)| \le 1 \text{ and } f(z) = \phi(z)g(z) \quad (z \in U). \quad (7)$$

It is noted that the notation of majorization is closely related to the concept of quasi-subordination between analytic functions.

Definition 1. *For $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, -1 \le B < A \le 1, p \in \mathbb{N}$, $m,q \in \mathbb{N}_0, p > q$ and $|\gamma(A-B)+B| \le p-q$, a function $f(z) \in A(p,j)$ is said to be in the class $S_{p,j,q}(m,A,B,\gamma)$ of p-valently functions in U, if and only if*

$$1 + \frac{1}{\gamma}\left(\frac{D_p^{m+1} f^{(q)}(z)}{D_p^m f^{(q)}(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \quad \text{(it8)}$$

where $D_p^m f^{(q)}(z)$ is given by (4).

Specializing the parameters m, n, p, q, A, B and γ, we have the following classes:

i) $S_{p,j,0}(m,A,B,\gamma) = S_{p,j}(m,A,B,\gamma) = \left\{ f \in A(p,j) : 1 + \frac{1}{\gamma}\left(\frac{D_p^{m+1} f(z)}{D_p^m f(z)} - 1\right) \prec \frac{1+Az}{1+Bz} \right\}$;

ii) $S_{p,j,q}(m,1,-1,\gamma) = S_{p,j,q}(m,\gamma) = \left\{ f \in A(p,j) : Re\left[1 + \frac{1}{\gamma}\left(\frac{D_p^{m+1} f^{(q)}(z)}{D_p^m f^{(q)}(z)} - 1\right)\right] > 0 \right\}$;

iii) $S_{p,j,0}(m,1,-1,(1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda}) = S_{p,j}^{\lambda}(m,\alpha)$
$= \left\{ f \in A(p,j) : Re\left(e^{i\lambda}\frac{D_p^{m+1} f(z)}{D_p^m f(z)}\right) > \frac{\alpha}{p}\cos\lambda \right\}$ $(|\lambda| < \frac{\pi}{2}; 0 \le \alpha < p)$;

iv) $S_{p,j,0}(0,1,-1,(1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda}) = S_{p,j}^{\lambda}(\alpha)$
$= \left\{ f \in A(p,j) : Re\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right) > \frac{\alpha}{p}\cos\lambda \right\}$ $(|\lambda| < \frac{\pi}{2}; 0 \le \alpha < p)$ (see Srivastava et al. [16] with $j = 1$);

v) $S_{p,j,0}(1,1,-1,(1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda}) = C_{p,j}^{\lambda}(\alpha)$

$= \left\{ f \in A(p,j) : Re \left\{ e^{i\lambda}(1+\frac{zf''(z)}{f'(z)}) \right\} > \frac{\alpha}{p}\cos\lambda \ (|\lambda| < \frac{\pi}{2}; 0 \le \alpha < p) \right\}$ (see Srivastava et al. [16] with $j = 1$);

vi) $S_{p,0}(m, 1, -1, \gamma) = S_m(p, \gamma)$ (see Akbulut et al. [2]);

vii) $S_{1,1,0}(0, 1, -1, \gamma) = S(\gamma)$ (see Nasr and Aouf [12]);

viii) $S_{1,1,0}(1, 1, -1, \gamma) = S(\gamma)$ (see Nasr and Aouf [12]) and Wiatrowski [17];

ix) $S_{1,1,0}(0, 1, -1, 1-\alpha) = S^*(\alpha)$ $(0 \le \alpha < 1)$ (see Robertson [14]).

Majorization problems for the class $S^* = S^*(0)$ had been investigated by MacGregor [10], recently Altintas et al. [1] investigated a majorization problem for the class $S(\gamma)$. Very recently Goyal and Goswami [8] generalized these results for the fractional operator (see also Goswami and Aouf [7]). In this peper we investigated a majorization problem for the class $S_{p,j,q}(m, A, B, \gamma)$ and its special subclasses.

2. Main Results

Unless otherwcise mentioned, we assume that $\gamma \in C^*, -1 \le B < A \le 1, p \in N$, $m, q \in N_0$ and $p > q$.

Theorem 1. *Let the funtion* $f(z) \in A(p,j)$ and $g(z) \in S_{p,j,q}(m, A, B, \gamma)$. If $D_p^m f^{(q)}(z)$ is majorized by $D_p^m g^{(q)}(z)$ in U, then

$$\left| D_p^{m+1} f^{(q)}(z) \right| \le \left| D_p^{m+1} g^{(q)}(z) \right| \quad (|z| \le r_0), \tag{it9}$$

where $r_0 = r_0(p, q, \gamma, A, B)$ is the smallest root of the equation:

$$|\gamma(A-B)+B|\,(p-q)r^3 - [p-q+2\,|B|]r^2 - [2+(p-q)\,|\gamma(A-B)+B|]r + p - q = 0. \tag{it10}$$

Proof. Since $g(z) \in S_{p,j,q}(m, A, B, \gamma)$, then it follows from (8) that:

$$1 + \frac{1}{\gamma}\left(\frac{D_p^{m+1} g^{(q)}(z)}{D_p^m g^{(q)}(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{11}$$

where $w(z) = c_1 z + c_2 z^2 + ... \in P$, P denotes the well known class of bounded analytic functions in U which satisfy $w(0) = 0$ and $|w(z)| \le 1$.

From (11) we have:

$$\frac{D_p^{m+1} g^{(q)}(z)}{D_p^m g^{(q)}(z)} = \frac{1 + [\gamma(A-B)+B]w(z)}{(1 + Bw(z))}. \tag{12}$$

Hence

$$\left| D_p^m g^{(q)}(z) \right| \le \frac{(1 + |B|\,|z|)}{1 - |\gamma(A-B)+B|\,|z|} \left| D_p^{m+1} g^{(q)}(z) \right|. \tag{13}$$

Since, $D_p^m f^{(q)}(z)$ is majorized by $D_p^m g^{(q)}(z)$ in U, then we have:

$$D_p^m f^{(q)}(z) = \phi(z) D_p^m g^{(q)}(z). \tag{14}$$

Differentiating (14) with respect to z and then multiplying z, we get:

$$z\left(D_p^m f^{(q)}(z) \right)' = z\phi'(z) D_p^m g^{(q)}(z) + \phi(z) z \left(D_p^m g^{(q)}(z) \right)'. \tag{15}$$

Noting that the Schwarz function $\phi(z)$ satisfies (see [13]):

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2}, \tag{16}$$

and using (5) , (13) and (16) in (15), we have:

$$\left|D_p^{m+1} f^{(q)}(z)\right| \leq \left\{|\phi(z)| + \frac{|z|(1-|\phi(z)|^2)}{(p-q)(1-|z|^2)} \frac{(1+|B||z|)}{[1-|\gamma(A-B)+B||z|]}\right\} \left|D_p^{m+1} g^{(q)}(z)\right|. \tag{17}$$

Setting $|z| = r$ and $|\phi(z)| = \rho$ $(0 \leq \rho \leq 1)$, (17) reduces to

$$\left|D_p^{m+1} f^{(q)}(z)\right| \leq \frac{\Psi(\rho)}{(p-q)(1-r^2)\left[p-q-|\gamma(A-B)+B|\,r\right]} \left|D_p^{m+1} g^{(q)}(z)\right|, \tag{18}$$

where

$$\Psi(\rho) = \rho(p-q)(1-r^2)\left[1-|\gamma(A-B)+B|\,r\right] + r(1-\rho^2)(1+|B|\,r)$$

takes its maximum value at $\rho = 1$ with $r = r_0(p, q, \gamma, A, B)$ given by (10). Furthermore, if $0 \leq \sigma \leq r_0(p, q, \gamma, A, B)$, the the function $\Phi(\rho)$ defined by

$$\Phi(\rho) = \rho(p-q)(1-\sigma^2)\left[1-|\gamma(A-B)+B|\,\sigma\right] + \sigma(1-\rho^2)(1+|B|\,\sigma)$$

is an increasing function on $0 \leq \rho \leq 1$, so that

$$\begin{aligned} \Phi(\rho) &\leq \Phi(1) = (p-q)(1-\sigma^2)\left[1-|\gamma(A-B)+B|\,\sigma\right], \\ 0 &\leq \rho \leq 1; 0 \leq \sigma \leq r_0(p, q, \gamma, A, B). \end{aligned}$$

Then, setting $\rho = 1$ in (18), we conclude that (9) holds true for $|z| \leq r_0(p, q, \gamma, A, B)$. This completes the proof of Theorem 1.

Putting $q = 0$ in Theorem 1, we have the following corollary:

Corollary 1. *Let the function* $f(z) \in A(p, j)$ and $g(z) \in S_{p,j}(m, A, B, \gamma)$. If $D_p^m f(z)$ is majorized by $D_p^m g(z)$ in U, then

$$\left|D_p^{m+1} f(z)\right| \leq \left|D_p^{m+1} g(z)\right| \quad (|z| \leq r_1),$$

where $r_1 = r_1(p, \gamma, A, B)$ *is the smallest root of the equation:*

$$|\gamma(A-B)+B|\,pr^3 - (p+2\,|B|)r^2 - [2+p\,|\gamma(A-B)+B|]r + p = 0.$$

Putting $A = 1$ and $B = -1$, in Theorem 1, (10) becomes

$$|2\gamma-1|\,(p-q)r^3 - (2+p-q)r^2 - [2+|2\gamma-1|\,(p-q)]r + p - q = 0, \tag{19}$$

which has $r = -1$ one of its roots and the other two roots are given by

$$|2\gamma-1|\,(p-q)r^2 - [|2\gamma-1|\,(p-q)+2+p-q]r + p - q = 0.$$

We may find the smallest postive root of (19).Hence, we have the following corollary:

Corollary 2. *Let the function* $f(z) \in A(p, j)$ and $g(z) \in S_{p,j,q}(m, \gamma)$. If $D_p^m f^{(q)}(z)$ is majorized by $D_p^m g^{(q)}(z)$ in U, then

$$\left|D_p^{m+1} f^{(q)}(z)\right| \leq \left|D_p^{m+1} g^{(q)}(z)\right| \quad (|z| \leq r_2),$$

where $r_2 = r_2(p, q, \gamma)$ is given by

$$r_2 = \frac{\eta - \{\eta^2 - 4(p-q)^2\,|2\gamma-1|\}^{\frac{1}{2}}}{2(p-q)\,|2\gamma-1|},$$

where $\eta = (p-q)\,|2\gamma-1| + 2 + p - q$.

Putting $\gamma = (1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda}(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < p)$ and $q = 0$ in Corollary 2, we have the following corollary:

Corollary 3. *Let the function* $f(z) \in A(p,j)$ and $g(z) \in S_{p,j}^{\lambda}(m,\alpha)$ $(|\lambda| < \frac{\pi}{2})$. If $D_p^m f(z)$ is majorized by $D_p^m g(z)$ in U, then

$$\left|D_p^{m+1} f(z)\right| \leq \left|D_p^{m+1} g(z)\right| \quad (|z| \leq r_3),$$

where $r_3 = r_3(p,\lambda,\alpha)$ is given by

$$r_3 = \frac{\delta - \left\{\delta^2 - 4p^2 \left|2(1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda} - 1\right|\right\}^{\frac{1}{2}}}{2p\left|2(1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda} - 1\right|}, \qquad \text{(it20)}$$

where $\delta = p\left|2(1-\frac{\alpha}{p})\cos\lambda e^{-i\lambda} - 1\right| + 2 + p$.

Putting $m = 0$ in Corollary 3, we have the following corollary:

Corollary 4. *Let the function* $f(z) \in A(p,j)$ and $g(z) \in S_{p,j}^{\lambda}(\alpha)$ $(|\lambda| < \frac{\pi}{2})$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r_3),$$

where $r_3 = r_3(p,\lambda,\alpha)$ is given by (20).

Remark. Specializing the parameters m, q, A, B and γ in Theorem 1, we obtain the majorization results for the corresponding classes defined in the introduction.

Acknowledgements. The author would like to thank the referees of the paper for their helpful suggestions

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