Journal of Fractional Calculus and Applications,
Vol. 4(2) July 2013, pp. 357 - 366.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

STOCHASTIC RESPONSE OF DUFFING OSCILLATOR WITH FRACTIONAL OR VARIABLE-ORDER DAMPING

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Abstract. This paper introduces a numerical technique for the estimation of stochastic response of the Duffing oscillator with fractional or variable order damping and driven by white noise excitation. The Wiener-Hermite expansion is integrated with the Grunwald-Letnikov approximation in case of fractional order damping and with Coimbra approximation in case of variable-order damping. The numerical solver was tested and validated with the analytic solution and with Monte-Carlo simulations. The developed technique was shown to be efficient in simulating the stochastic non-linear differential equations with fractional or variable order derivatives.

1. Introduction

In the last two decades, fractional derivative modeling of viscoelasticity has been applied in numerous studies and, for its capability of describing complex material behaviors at a macroscopic level, in form of equations involving a small number of parameters it is now a well-established approach to viscoelastic media [1]. The fractional order damping oscillator can be interpreted as an ensemble of non-identical harmonic oscillators that differ in phase or it can be interpreted due to the non-conservative nature of some forces (e.g. friction force), [2].

Variable order (VO) systems constitute a generalization of fractional order representations to fractional order. In VO systems the order of the derivative changes with respect to either the dependent or the independent variable (or both), or parametrically with respect to an external fractional behavior [3]. Variable order formulations have been utilized, among other applications, to describe the mechanics of an oscillating mass subjected to a variable viscoelasticity damper and a linear spring [4].

Recently, the research efforts are directed in estimating the response of fractional or variable-order damping oscillators with stochastic excitations. In [5], the stochastic stability of Duffing oscillator with fractional derivative damping under stochastic excitation has been studied using the stochastic averaging technique. In [6], the response of fractional oscillator to multiplicative trichotomous noise was estimated using Lapace transform of the fractional derivatives. In [1], the stochastic response

2000 *Mathematics Subject Classification.* 34A99.
Key words and phrases. Fractional differential equations, Wiener-Hermite expansion.
Submitted May 12, 2013.

of the fractional-damped Duffing oscillator was estimated numerically by transforming the fractional-order equation to equivalent ordinary system with additional degrees of freedom instead of using the Grunwald-Letnikov (GL) approximation. Other research efforts are done in [7] using GL approximation with the statistical linearization, in [8] using the reduced Fokker-Plank-Kolmogorov equation with the stochastic averaging technique and in [9] using Laplace transform with Monte-Carlo simulations.

The main objective of this work is to develop a new numerical technique that combines the Wiener-Hermite expansion (WHE) with the approximation schemes of the fractional or varible-order derivatives. The Grunwald-Letnikov approximation will be used in case of fractional order damping while the Coimbra approximation will be used instead in case of variable-order damping. The developed numerical schemes are used in estimating the response of the stochastic nonlinear systems with fractional or variable-order derivatives. In particular, the response of the Duffing oscillator with stochastic excitation and fractional or variable-order damping will be considered.
This paper is organized as follows; in Section 2, the problem formulation is described. In Section 3, the WHE is reviewed and the equivalent deterministic system is derived. The suggested solution strategy of the fractional order damping will be outlined in Section 4. Also, testing and validation of the developed solver against the analytical and Monte-Carlo solutions are shown. In Section 5, the variable order numerical scheme will be described and comparisons with the constant fractional order damping are given.

2. Problem Formulation

Consider the model equation

$$L(x(t)) = -\epsilon x^n + f(t) + g(t)N(t,w); \quad t \in (0,T] \tag{1}$$

with the proper set of initial conditions which will be assumed deterministic. The differential operator L is a general ordinary or non-ordinary (fractional or variable order) linear operator. The nonlinearity is introduced as losses of degree $n > 1$ strengthened by a deterministic small parameter (ϵ). The uncertainty is introduced through white noise $N(t;w)$ intensified by a deterministic envelope function $g(t)$. The function $f(t)$ is a deterministic excitation force.
The Duffing oscillator equation of motion in case of fractional damping and stochastic excitation is a special case of equation (1) with $n = 3$. Particularly, it takes the form:

$$m\ddot{x}(t) + C_\alpha(D_t^\alpha x)(t) + kx(t) + \epsilon x^3(t) = f(t) + g(t)N(t;w); \quad t \in (0,T] \tag{2}$$

where m is the mass, C_α is the fractional damping coefficient, k is the linear stiffness of the spring. The term $kx(t)+\epsilon x^3(t)$ represents the restoring force of the nonlinear spring while $C_\alpha(D_t^\alpha x)(t)$ is the attenuation force of the viscoelastic damper. When $\alpha = 0$, the damping is purely elastic and for $\alpha = 1$, the damping is purely viscous [1]. We shall assume the fractional derivative to be in Caputo's sense which is defined as:

$$(D_t^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x^{(j)}(s)}{(t-s)^\alpha} ds; \quad j-1 < \alpha < j; \quad j = \lceil \alpha \rceil \tag{3}$$

Equation (2) when divided by m will give;

$$\ddot{x}(t) + c_\alpha (D_t^\alpha x)(t) + \omega^2 x(t) + \epsilon_0 x^3(t) = f_0(t) + g_0(t) N(t;w) \tag{4}$$

where $\omega = \sqrt{\frac{k}{m}}$ is the natural frequency, $c_\alpha = \frac{C_\alpha}{m}$ and $\epsilon_0 = \frac{\epsilon}{m}$. We shall assume initially, a quiescent system (i.e., $x(0) = x_o^*(0) = 0$)

3. Description of the WHE Technique

The solution of stochastic PDEs (SPDEs) using WHE has the advantage of converting the problem into a system of deterministic equations that can be solved efficiently using the standard deterministic methods. The main statistics, such as the mean, covariance, and higher order statistical moments, can be calculated by simple formulae involving only the deterministic WHE coefficients [10]. In the WHE technique, the stochastic response function $x(t;w)$ is expanded as:

$$x(t;w) = \sum_{k=0}^{\infty} \int_{R^k} x^{(k)}(t;t_1,t_2,\cdots,t_k) H^{(k)}(t_1,t_2,\cdots,t_k) d\tau_k \tag{5}$$

where $x^{(k)}(t,t_1,t_2,\cdots,t_k)$ is the k^{th} deterministic kernel of $x(t;w)$, $d\tau_k = dt_1 dt_2 \cdots dt_k$ and $\int_{R^k}$ is a k-dimensional integral over the variables $t_1,t_2,\cdots,t_k$. The functional $H^{(k)}(t_1,t_2,\cdots,t_k)$ is k^{th} order Wiener-Hermite functional which is defined for 1D continuous problem as [11];

$$H^{(k)}(t_1,t_2,\cdots,t_k) = \delta^{n/2}(0) e^{\frac{1}{2}\sum_{i=1}^{k}\xi^2(x_i)} \prod_{j=1}^{k} \left(\frac{-\partial}{\partial \xi(x_j)}\right) e^{\frac{1}{2}\sum_{i=1}^{k}\xi^2(x_i)} \tag{6}$$

and they constitute a complete set. The set $\{\xi_k\}$ is a denumerable set of independent Gaussian random variables with zero mean and unit variance, and δ is the Dirac delta function.
The WHE technique for general nonlinear exponent (n) and general order (M) follow the steps [10]:

1. Truncate the expansion (5) to contain only $M+1$ kernals $x^{(j)}; 0 \le j \le M$, i.e., $x(t;w) = x^{(0)}(t) + \sum_{j=1}^{M} \int_{R^j} x^{(j)} H^{(j)} d\tau_j$, and then
2. Substitute into equation (1)
3. Use the multinomial theorem to expand the nonlinear term $x^n; n = 3$,
4. Multiply by $H^{(j)}; 0 \le j \le M$ and then apply the ensemble average.

This will lead to $(M+1)$ equations in the deterministic kernels $x^{(j)} : 0 \le j \le M$ as

$$(j!) L(x^{(j)}) + \epsilon_0 \sum_f c_f \int_{R^z} \left(\prod_{i=0}^{M} [x^{(i)}]^{k_j^i} \right) E_f^j d\tau_z = \delta_{j0}\ f_0(t) + \delta_{jl}\ g_0(t)\delta(t-t_1);\ 0 \le j \le M \tag{7}$$

The expectations E_f^j are computed as $E_f^j =< H^{(j)} \prod_{i=0}^{m} (H^{(i)})^{k_f^j} >$. It was explained in [10] how to get E_f^j in terms of the Dirac delta functions and then used to reduce the order of integration. The Kronecker delta function δ_{jl} is zero expect with $j = l$. The counter f, in the summation in the left hand side of (7) runs over all the $\binom{n+m}{n}$ combinations of the positive integers $k_f^0, k_f^l, \cdots, k_f^M$ such that $\sum_{i=0}^{M} k_f^i = n$. After solving for the kernels, the expectation and variance are obtained as;

$$E[x(t)] = x^{(0)}$$

$$Var[x(t)] = \sum_{k=l}^{M}(k!)\int_{R^k}[x^{(k)}]^2 d\tau_k \tag{8}$$

For first order (Gaussian) solution, the following deterministic system is obtained after applying the above described WHE technique with $M = 1$:

$$L(x^{(0)}) + \epsilon_0 x^{(0)}\left([x^{(0)}]^2 + 3\int_R [x^{(1)}(t_1)]^2 dt_1\right) = f_0(t)$$

$$L(x^{(1)}) + 3\epsilon_0 x^{(1)}(t_1)\left([x^{(0)}]^2 + \int_R [x^{(1)}(t_1)]^2 dt_1\right) = g_0(t)\delta(t-t_1) \tag{9}$$

This means that we have two simultaneous integro-differential equation in the deterministic kernels ($x^{(0)}$ and $x^{(1)}$). Higher-order solutions can be obtained similarly. Many solution methods can be used to solve equations (9). Two of them are Picard's successive approximation and the WHEP technique [10]. In the current work. Picard's successive approximation technique will be considered.

As in Theorem (1) in [10], when using the Picard's successive approximation technique, the solution of Equation (1), if exists, will be a power series in ϵ_0, i.e., $x(t) = \sum_{i=0}^{\infty}\epsilon_0^i x_i(t)$. The proof is introduced in [10] for ordinary linear operator L and extension to non-ordinary linear operator is straightforward. This theorem tells us that there is a condition on ϵ_0 to get a convergent solution using Picard's successive technique. Particularly, we should have:

$$\epsilon_0 \leq \left|\frac{x_i}{x_{i+1}}\right|.$$

This means that ϵ_0 should obey an upper bound condition after which divergent solution is obtained. To solve (9) for larger values of ϵ_0, a different technique should be considered (e.g. using Newton's approximation). Practically, ϵ_0 has small values. Solution of the Duffing oscillator with integer-order damping using technique similar to Picard's approximation was considered in [2]. The numerical Picard's successive approximation technique will take the form

$$L(x^{(0)})\big|_{k+1} = f_0(t) - \epsilon_0 x^{(0)}\left([x^{(0)}]^2 + 3\int_R [x^{(1)}(t_1)]^2 dt_1\right)\big|_k$$

$$L(x^{(1)})\big|_{k+1} = g_0(t)\delta(t-t_1) - 3\epsilon_0 x^{(1)}(t_1)\left([x^{0)}]^2 + \int_R [x^{(1)}(t_1)]^2 dt_1\right)\big|_k;\ k = 0, 1, 2, \cdots \tag{10}$$

The computations will be repeated until the convergence criterion decreases below a certain value. The convergence criterion in the current work is taken as:

$$err = \sum_{j=0}^{M}\left\|x_{k+1}^{(j)} - x_k^{(j)}\right\|$$

4. Numerical Fractional-Order Scheme

We need to develop a numerical solver for a model equation in the form

$$L(x(t)) = h(t) \tag{11}$$

With the non-ordinary operator L be:

$$L = \frac{d^2}{dt^2} + a_1 D_t^{\alpha} + a_0$$

where a_1 and a_0 are assumed to be constants. This form of the model equation can be easily observed from Equation (10) for the different $(M+1)$ kernels. We can use any difference scheme to discretize (11), but as we have a white noise excitation, the Dirac delta function $\delta(t-\tau)$ appears in the equarions of some kernels. So, an integral numerical scheme such as FEM or FVM will be more suitable. Integration of the Dirac delta function is easier to be handled. In the current work, the FVM will be applied. The time axis where $t \in [0,T]$ will be divided into N_i equal intervals of size Δt. The interval extends from t_{i-1} to t_i is taken as the control volume. Integrating $D_t^\alpha x(t)$ over the control volume from t_{i-1} to t_1 and using the trapezoidal rule, to get

$$\int_{t_{i-1}}^{t_i} D_t^\alpha x(t)dt = \frac{\Delta t}{2}[D_t^\alpha x_i + D_t^\alpha x_{i-1}] \tag{12}$$

The fractional derivative terms, $D_t^\alpha x_i$ and $D_t^\alpha x_{i-1}$, can be discretized using the Grunwald-Letnikov approximation as [13]:

$$D_t^\alpha x_i \simeq \lim_{\Delta t\to 0}(\Delta t)^\alpha \sum_{j=0}^{i} w_j\, x(t_i - j\Delta t) \tag{13}$$

where the coefficients $w_j; 0 \le j \le i$ are calculated as:

$$w_0 = 1; \quad w_j = \left(1 - \frac{\alpha-1}{j}\right) w_{j-1}; j > 0$$

Integrating equation (11) along the control volume and using (13) to get

$$x_i = \frac{2F_{i-1}\Delta t + (4 - a_0(\Delta t)^2)x_{i-1} - 2x_{i-2} - a_1(\Delta t)^{2-\alpha}\sum_{j=1}^{i}(w_j + w_{j-1})x_{i-j}}{2 + a_1(\Delta t)^{2-\alpha} + a_0(\Delta t)^2} \tag{14}$$

where $F_{i-1} = \int_{t_{i-1}}^{t_1} h(t)dt = 0.5(h_{i-1} + h_i)\Delta t$. If the Dirac delta function appears in the right hand side, a special treatment for F_{i-1} is considered. In this case $F_{i-1} = \int_{t_{i-1}}^{t_i} q(t)\delta(t-t_1)dt = q(t_i)$ when $t_i = t_1$ and $F_{i-1} = 0$ when $t_i \ne t_1$. The FVM numerical scheme (14) can be validated with the analytical solution that can be obtained in some special cases (e.g. $h(t) = 1$; Heaviside unit step function). The analytical solution can be obtained using the fractional Green's functions for three term fractional D. E. with constant coefficients as [13]:

$$x(t) = \int_0^t G_3(t-\tau)h(\tau)dt \tag{15}$$

$$G_3(t) = \frac{1}{a_0}\sum_{j=0}^{\infty}(-a_1)^j \sum_{k=0}^{\infty}(-1)^k(a_0)^{k+1}\frac{(j+k)!t^{2j+2k+1-\alpha j}}{j!k!\Gamma(2j+2k+2-\alpha j)} \tag{16}$$

For $x_0 = X_0^* = 0, a_1 = a_0 = h(t) = 1$, the analytical solution will be

$$x(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{(j+k)!t^{2(j+k+1)-\alpha j}}{j!k!\Gamma(2j+2k+3-\alpha j)}. \tag{17}$$

With $T = 15$ and $\Delta t = 0.01$, the comparison in Figure (1, left) shows that the numerical solution has a good accuracy compared with the analytical solution.
The convergence order of the developed FVM fractional solver is tested numerically as shown also in Figure (1, right). The scheme converges with $O(\Delta t)$, i.e., it is a

first order accurate scheme. This is a known property of the GL approximation. Higher order schemes can also be derived [13].

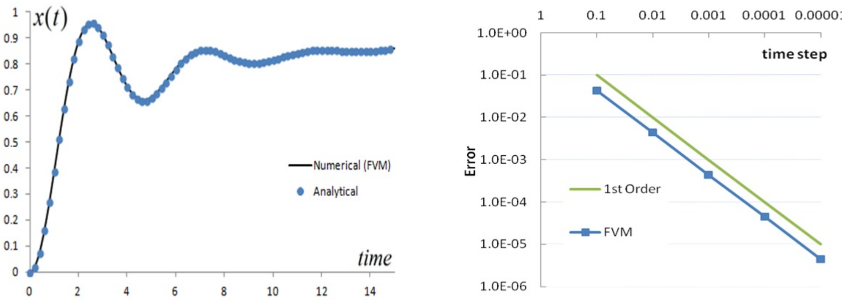

FIGURE 1. Comparison between the FVM solver and the analytical solution for $\alpha = 0.5$ (left) and Convergence order of the developed FVM (right)

The stochastic response of the developed FVM solver can be obtained using a numerical stochastic integration technique (e.g. [14]). The stochastic response for first order (Gaussian) solution, can be obtained as:

$$x(t;w) = x^{(0)}(t) + \int_R x^{(1)}(t;t_1)H^{(1)}(t_1)dt_1$$
$$= x^{(0)}(t) + \int_R x^{(1)}(t,t_1)dW = x^{(0)}(t) + \sum_{j=1}^{N_j} x^{(1)}(t;t_1)\Delta W_j \tag{18}$$

where ΔW_j is an independent Gaussian random variable in the form $\sqrt{\Delta t}N_G(0,1)$, with $N_G(0,1)$ is the standard Gaussian random variable of zero mean and unit variance. The stochastic response along the time axis for $\alpha = 0.5$ and $f_0(t) = 0$ is shown in Figure (2) at different values of the noise intensity $g_0(t)$ which will be assumed constant, i.e. $g_0(t) = S$. The noise intensity affects the convergence rate of the solution as shown in Figure (3). As the noise intensity increases, the solution converges in a slower rate and requires more interations to reach the required accuracy,

Comparison with the Monte-Carlo (MC) simulations was done to test the developed solver. An arbitrary sample of the white noise $N(t;w)$ is built on the well-known spectral representation [1]:

$$N(t,w) = \sum_{j=1}^{M_\omega} \sqrt{4S_0\Delta\omega}\cos(\omega_j t + \phi_j) \tag{19}$$

Where $\Delta\omega$ is a constant step on the frequency axis, ω_j are M_ω equally spaced frequencies and ϕ_j are M_ω random phases uniformly distributed in the interval $[0, 2\pi]$. In this work, we set $\Delta\omega = 0.05$ and $M_\omega = 500$. Figure (4) shows the stochastic response and a comparison between the FVM solver with the solution obtained with $10000MC$ simulations. The solution at each Monte-Carlo sample is computed as:

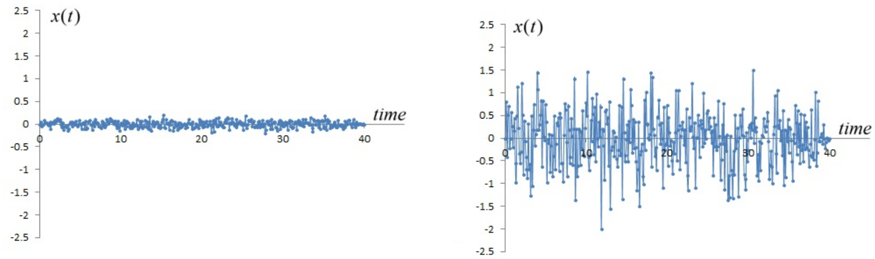

FIGURE 2. Response $x(t)$ for $\alpha = 0.5, \delta_0 = 0.25$, Noise intensity S = 0.1 (left) and S= 1.0 (right)

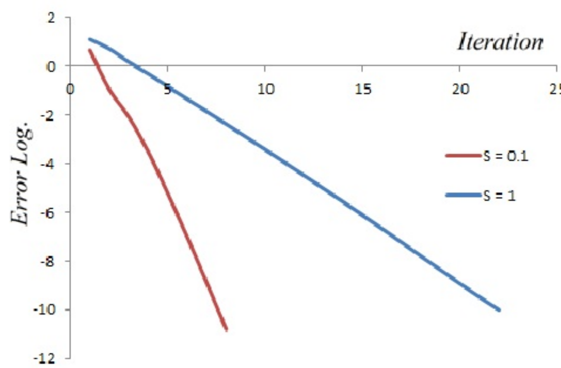

FIGURE 3. The convergence of FVM solver for different noise intensities; S=0.1 and S=1

$$L(x(t))\big|_{k+1} = -\epsilon x^3\big|_k + f(t) + g_0(t)N_i(t) \tag{20}$$

where $N_i(t)$ is the i^{th} Monte-Carlo sample of the white noise.

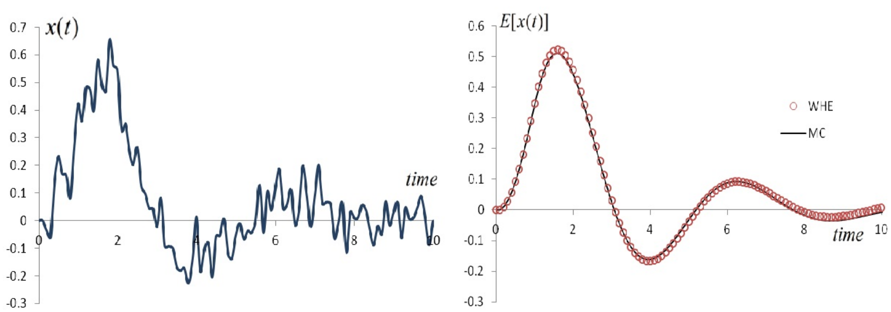

FIGURE 4. The stochastic response (left), mean response (right) for $\alpha = 0.5, \epsilon_0 = 0.25$, noise intensity $S = 0.1, f(t) = 1; 0 < t < 1.0$, 10,000 MC simulations.

The effect for different values of the fractional order α was tested at $\epsilon_0 = 0.25$ as shown in Figure (5). We can notice that as the fractional order α increases the stcady state variance decreases

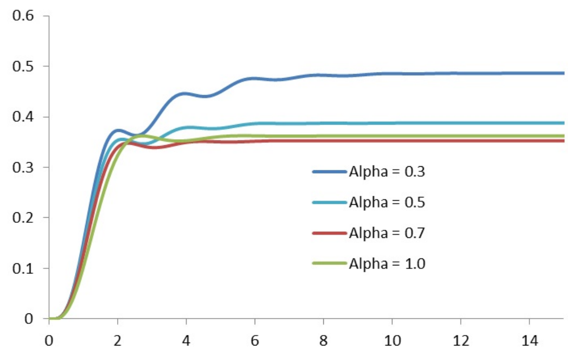

FIGURE 5. Variance obtained by FVM at different values of α

5. THE VARIABLE-ORDER NUMERICAL SCHEME

The VO derivative is defined, due to Coimbra [4], as:

$$D_i^{\alpha(t)}x(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0^+}^{t}(t-s)^{-\alpha(t)}\frac{dx(s)}{ds}ds + \frac{(x(0^+)-x(0^-))t^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \quad (21)$$

For zero initial conditions, we can write it in discrete form as [15]:

$$D_i^{\alpha_i}x_i = \frac{(\Delta t)^{2-\alpha_i}}{\Gamma(2-\alpha_i)}\sum_{j=0}^{i-1}(b_{j,i} - x_{i-j-2}) \quad (22)$$

Where $x_i = x(t_i), \alpha_i = \alpha(t_i)$ and $b_{j,i} = (j+1)^{1-\alpha_i} - (j)^{1-\alpha_i}$. This discretization is also first order i.e., $O(\Delta t)$, [15]. The above developed FVM solver (14) is adopted using the variable order discretization (22) to get:

$$x_i = \frac{2F_{i-1}\Delta t + (4 - a_0(\Delta t)^2)x_{i-1} + a_{1i}\sum_{j=1}^{i-2} b_{j,i}(x_{i-j} - x_{i-j-2})}{2 + a_{1i} + a_0(\Delta t)^2} \quad (23)$$

where $a_{1i} = \frac{a_1(\Delta t)^{2-\alpha_i}}{\Gamma(2-\alpha_i)}$. Two different linear variable-order variations are used to test the new VO solver. The two different variations are taken as

$$VO-1: \qquad \alpha_1(t) = \begin{cases} \alpha_0+0.8\,\frac{t}{T} & ; \ 0\le t\le 35 \\ 1 & ; \ t>35 \end{cases}$$

$$VO-2: \qquad \alpha_2(t) = \begin{cases} \alpha_0+1.2\,\frac{t}{T} & ; \ 0\le t\le 17.5 \\ 1 & ; \ t>17.5 \end{cases}$$

The deterministic excitation $f_0(t)$ is taken to be zero except in the interval $t \in [0,1]$, it will be unity. The expectation and variance are shown in Figure (6) for $\epsilon_0 = 0.25$ and $\alpha_0 = 0.3$.

Comparison between the constant fractional order damping and variable-order damping (VO-1) is shown in Figure (7). We can notice that the variable-order responcse has a different behavior than the constant fractional order damping.

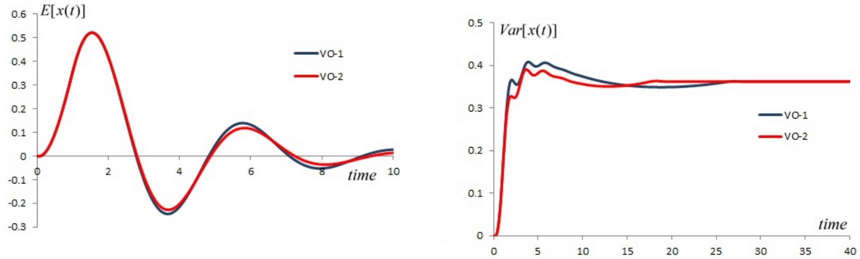

FIGURE 6. The expectation (left) and variance (right) for the two variable-order variations

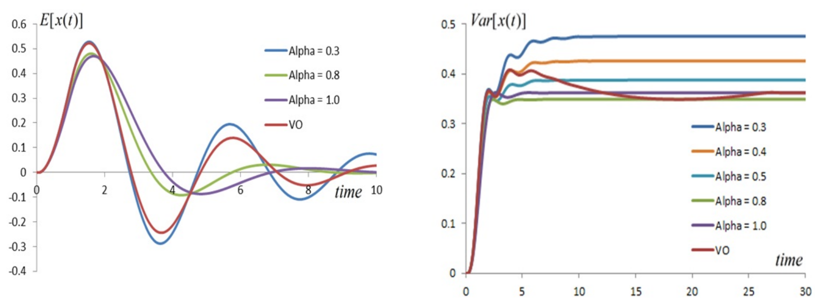

FIGURE 7. The expectation (left) and variance (right) of the variable order damping compared with the constant fractional order damping

6. Summary and Conclusions

In the current work, the response of the Duffing oscillator with fractional and variable-order damping under stochastic excitation is estimated. The GL approximation was combined with the WHE in the framework of a FVM to estimate the response of the Duffing oscillator. In case of variable-order damping the Coimbra approximation is combined with the WHE. The solver convergence was shown numerically and the expection and variance of the response are given. The developed solvers are first order accurate but can be extended easily to higher orders. The developed solvers are shown to be efficient in estimating the stochastic response of differential equations with fractional order or variable-order derivatives.

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