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SOLITARY SOLUTIONS OF THE FRACTIONAL KDV EQUATION USING MODIFIED REIMANN-LIOUVILLE DERIVATIVE

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Abstract. In this paper, reliable variational iteration method using Adomian's polynomials is employed to construct the solitary pattern solutions of nonlinear fractional KdV equations.The iteration procedure is based on a relatively new approach which Jumarie's derivative. Solution of two variants of fractional KdVs is presented to elucidate effectiveness of the proposed algorithm.

1. Introduction

Korteweg-de Vries (KdV) equation has been used to describe a wide range of physics phenomena as a model for the evolution and interaction of nonlinear waves. It was first derived as an evolution equation that governing a one dimensional, small amplitude, long surface gravity waves propagating in a shallow channel of water [1]. Subsequently the KdV equation has arisen in a number of other physical contexts as collision-free hydro-magnetic waves, stratified internal waves, ion-acoustic waves, plasma physics, and lattice dynamics [2]. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior, and mass transport. All of the physical phenomena may be considered as nonconservative and nonlinear, so they can be described using fractional differential equations. These nonlinear phenomena can be modeled to wave and dispersive equations.There is a strong interest in explicit soliton solutions. The solitons defined by Wadati [3] as a nonlinear wave of localized propagation. Several techniques including Adomian's decomposition, variational iteration, differential transform, variation of parameters, finite difference, finite volume, spline, sink glarkin, Tanh h, Sech and homotopy perturbation have been proposed to tackle verstaility of linear and nonlinear fractional differntial equatiosn of complex physical nature, see [1]-[25] and the refernces therein. Recently, Odibat [10],[11] used homotopy perturbation method

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to obtain solitary solutions of the variants of the KdV equations with fractional time derivatives. The bsic inspiration of this paper is the extension of variational iteration method using modified Riemann-Liouville derivative to find analytical approximate solutions to time fractional KdV equation. Solution procedure reflects the complete reliability of the proposedscheme and numerical results shows the fast convergence of suggested algorithm.

Definition 1. Assume $f : R \to R,\ x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition in the interval . Jumarie's derivative [15] is defined through the fractional difference

$$\Delta^{\alpha} = (FW-1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{a}{k} f(x+(\alpha-k)) \tag{1}$$

where, $FWf(x) = f(x+k)$. Then the fractional derivative is defined as the following limit.

$$f^{\alpha} = \lim_{k \to 0} \frac{\Delta^{\alpha}[f(x)-f(0)]}{h^{\alpha}}. \tag{2}$$

This definition is close to the standard definition of derivative, and as a direct result, the αth derivative of a constant $0 < \alpha < 1$; is zero.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ for a function $f \in C_{\mu}, \mu \geq -1$ in [3] defined as

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi,\ \alpha > 0, \tag{3}$$

Definition 3. The Jumarie's modified Riemann–Liouville derivatives [15]defined as

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-\xi)^{m-\alpha} f(\xi - f(0)) d\xi,\ \alpha > 0, \tag{4}$$

where, $x \in [0,1],\ m-1 \prec \alpha \prec m,\ m \geq 1$. The proposed modified Riemann–Liouville derivative as shown in equation (4) is strictly equivalent to equation (2).

Definition 4. Fractional derivative of compounded functions [16] is defined as

$$d^{\alpha} f(x) \cong \Gamma(1+\alpha) df \qquad 0 < \alpha < 1. \tag{5}$$

Definition 5. The integral with respect to $(dx)^{\alpha}$ is defined as the solution of fractional differential equation [16]given by equation.

$$dy \cong f(x)(dx)^{\alpha} \quad x \geq 0\ , \quad y(0) = 0\ , \quad 0 < \alpha < 1, \tag{6}$$

$$y \cong \int_0^x f(\xi)(d\xi)^{\alpha} = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi,\ \ 0 < \alpha \leq 1, \tag{7}$$

For example $f(x) = x^{\beta}$ in equation (7), one obtains

$$\int_0^x \xi^{\beta} (d\xi)^{\alpha} = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} x^{\alpha+\beta},\ \ 0 < \alpha \leq 1. \tag{8}$$

Definition 6. Assume that the continuous function $f : R \to R,\ x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any α; $0 < \alpha \leq 1$, then the following equality holds [3], which is

$$f(x+h)=\sum_{k=0}^{\infty}\frac{h^{\alpha k}}{\alpha k}\ f^{\alpha k}(x)\ ,\quad 0<\alpha\le 1, \tag{9}$$

On making the substitution $h\to x$ and $x\to 0$, we obtain the fractional Mc-Laurin series

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\alpha k}\ f^{\alpha k}(0)\ ,\quad 0<\alpha\le 1. \tag{10}$$

Definition 7. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha>0$ with is defined by the following series representation, valid in the whole complex plane [3]

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}. \tag{11}$$

2. Variational Iteration Method (VIM)

In order to elucidate the solution procedure of the VIM, we consider the following fractional differential equation of the form:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)=K[x]\ u(x,t)+q(x,t),\ t>0,\ x\in R, \tag{12}$$

$$u(x,0)=f(x)$$

Where $K[x]$ is the differential operator in x , $f(x)$and $q(x,t)$ are continuous functions. According to VIM introduced by He [20], we can construct a correction functional for equation (12) as follows

$$u_{n+1}(x,t)=u_n(x,t)+I^\alpha[\lambda(\frac{\partial^\alpha}{\partial t^\alpha}u_n(x,\xi)-k[x]u(x,t)-q(x,\xi))],$$

$$u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\lambda(\xi)(\frac{\partial^\alpha}{\partial t^\alpha}u_n(x,\xi)-k[x]u(x,t)-q(x,\xi))](d\xi)^\alpha, \tag{13}$$

Combining equations (7) and (13), we obtained a proposed correction functional

$$u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(\alpha+1)}\int_0^t\lambda(\xi)(\frac{\partial^\alpha}{\partial t^\alpha}u_n(x,\xi)-k[x]u(x,t)-q(x,\xi))](d\xi)^\alpha, \tag{14}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. It is obvious that the successive approximation $u_j,\ j\ge 0$ can be established by determining λ. The function $\widetilde{u}_n$ is a restricted variation which means $\delta\widetilde{u}_n=0$. Therefore, we first determine Lagrange's multiplier that will be identified optimally via integration by parts. The successive approximation of the $u_{n+1}(x,t),\ n\ge 0$ solution $u(x,t)$will be readily obtained upon using the Lagrange's multiplier and by using any selective function u_0. The initial values are usually used for selecting the zeroth approximation . With λ determined, several approximations follows immediately. Consequently, the exact solution may be obtained by using

$$u(x,t)=\lim_{n\to\infty}u_n(x,t). \tag{15}$$

3. Numerical Examples

In this section, some initial value problem are considered to show the efficiency of the method.

Example 1 Consider the following fractional KdV equation [10],[11]:

$$uD_t^\alpha u - a(u^2)_x + (u(u_{xx}))_x = 0, \qquad a > 0,\ t > 0,\ 0 < \alpha \leq 1, \tag{16}$$

with the initial condition

$$u(x,0) = \frac{2c}{a}\sinh^2(\frac{\sqrt{a}}{2}x).$$

where c is an arbitrary constants.The correction functional is read as

$$u_{n+1}(x,t) = u_n + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda(\xi)(\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - a(u_n^2)_x) + [u_n(u_{nxx})]_x]\ (d\xi)^\alpha,\ n \geq 0,$$

where $\lambda\,(\xi) = -1$ can be determined optimally via variational theory. We have the following iterative formula

$$u_{n+1} = u_n - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left\{\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - a(u_n^2)_x + [u_n(u_{nxx})_x]\right\}(d\xi)^\alpha.$$

The initial approximation is

$$u_0(x,t) = \frac{c}{a}(\cosh\sqrt{a}x - 1),$$

Consequently, we find the following approximations:

$$\begin{aligned} u_1 &= u_0 - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left\{\frac{\partial^\alpha u_0}{\partial \xi^\alpha} - \alpha(u_0^2)_x + [u_0(u_{0xx})]_x\right\}(d\xi)^\alpha \\ &= -\frac{c^2}{\sqrt{a}}\sinh(\sqrt{a}x)\frac{t^\alpha}{\Gamma(1+\alpha)}, \end{aligned}$$

$$\begin{aligned} u_2 &= u_1 - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left\{\frac{\partial^\alpha u_0}{\partial \xi^\alpha} - a(2u_0u_1)_x + [u_0u_{1xx} + u_1u_{0xx}]_x\right\}(d\xi)^\alpha \\ &= c^3\cosh(\sqrt{a}x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \end{aligned}$$

$$\vdots$$

The solitary patterns in a series form is given by

$$
\begin{aligned}
u(x,t) &= \frac{c}{a}\left[\cosh(\sqrt{a}x)(1+ac^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+a^2c^2\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}+...)-1\right] \\
&\quad -\frac{c}{a}\sinh(\sqrt{a}x)\left[\sqrt{a}c\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\sqrt{a^3}c^3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\sqrt{a^5}c^5\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}+...\right],
\end{aligned}
$$

The closed form solitary solution is

$$
u(x,t)=\frac{c}{a}\left[\cosh(\sqrt{a}x)\cosh(\sqrt{a}ct^{\alpha},\alpha)-\sinh(\sqrt{a}x)\sinh(\sqrt{a}ct^{\alpha},\alpha)-1\right]. \tag{17}
$$

where the functions $\cosh(z,\alpha)$ and $\sinh(z,\alpha)$ are defined as

$$
\cosh(z,\alpha)=\sum_{k=0}^{\infty}\frac{z^{2n}}{\Gamma(2n\alpha+1)},
$$

$$
\sinh(z,\alpha)=\sum_{k=0}^{\infty}\frac{z^{2n+1}}{\Gamma((2n+1)\alpha+1)},
$$

If we select the initial approximation $u(x,0)=-\frac{2c}{a}\cosh^2(\frac{\sqrt{a}}{2}x)$ then we have the following solitary patterns solution

$$
u(x,t)=-\frac{c}{a}\left[\cosh(\sqrt{a}x)\cosh(\sqrt{a}ct^{\alpha},\alpha)-\sinh(\sqrt{a}x)\sinh(\sqrt{a}ct^{\alpha},\alpha)-1\right]. \tag{18}
$$

by Setting $\alpha=1$,in equation (17) and (18), the solitary patterns solutions are

$$
u(x,t)=\frac{2c}{a}\sinh^2\left(\frac{\sqrt{a}}{2}(x-ct)\right)
$$

and

$$
u(x,t)=-\frac{2c}{a}\sinh^2\left(\frac{\sqrt{a}}{2}(x-ct)\right)
$$

Example 2 Consider the following fractional KdV equation [10],[11]:

$$
D_t^{\alpha}u-au(u)_x+(u(u_{xx}))_x=0, \qquad a>0,\ t>0,\ 0<\alpha\leq 1, \tag{19}
$$

with the initial condition

$$
u(x,0)=\frac{4c}{a}\sinh^2(\frac{1}{2}\frac{\sqrt{a}}{2}x).
$$

where c is an arbitrary constants.The correction functional is read as

$$
u_{n+1}(x,t)=u_n+\frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda(\xi)(\frac{\partial^{\alpha}u_n}{\partial\xi^{\alpha}}-au_n(u_n)_x)+[u_n(u_{nxx})]_x]\,(d\xi)^{\alpha},\ n\geq 0,
$$

where $\lambda(\xi)=-1$ can be determined optimally via variational theory.

The iterative formula is

$$u_{n+1} = u_n - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ \frac{\partial^\alpha u_n}{\partial \xi^\alpha} - a u_n (u_n)_x) + [u_n (u_{nxx})_x] \right\} (d\xi)^\alpha.$$

The initial approximation is

$$u_0(x,t) = \frac{2c}{a} \left(\cosh\left(\frac{\sqrt{a}}{2} x \right) - 1 \right),$$

Consequently, we have the following approximations:

$$u_1(x,t) = -\sqrt{\frac{2}{a}} c \sinh\left(\frac{\sqrt{a}}{2} x \right) \frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$u_2(x,t) = c^3 \cosh\left(\frac{\sqrt{a}}{2} x \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$u_3(x,t) = -\sqrt{\frac{2}{a}} c^4 \sinh\left(\frac{\sqrt{a}}{2} x \right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)},$$

.
.
.

The solitary patterns in a series form is given by

$$\begin{aligned} u(x,t) &= \frac{2c}{a} \left[\cosh(\sqrt{\frac{a}{2}} x)(1 + \frac{ac^2}{2} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^2 c^4}{4} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + ...) - 1 \right] \\ &\quad - \frac{2c}{a} \sinh(\sqrt{\frac{a}{2}} x) \left[\frac{\sqrt{a}}{\sqrt{2}} c \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{\sqrt{a^3}}{\sqrt{8}} c^3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ... \right], \end{aligned}$$

The closed form solitary solution is

$$u(x,t) = \frac{2c}{a} \left[\cosh(\sqrt{\frac{a}{2}} x) \cosh(\sqrt{\frac{a}{2}} ct^\alpha, \alpha) - \sinh(\sqrt{\frac{a}{2}} x) \sinh(\sqrt{\frac{a}{2}} ct^\alpha, \alpha) - 1 \right]. \tag{20}$$

If we select the initial approximation $u(x,0) = -\frac{4c}{a} \cosh^2(\frac{1}{2}\sqrt{\frac{a}{2}} x)$, we have the following solitary patterns solution

$$u(x,t) = -\frac{2c}{a} \left[\cosh(\sqrt{\frac{a}{2}} x) \cosh(\sqrt{\frac{a}{2}} ct^\alpha, \alpha) - \sinh(\sqrt{\frac{a}{2}} x) \sinh(\sqrt{\frac{a}{2}} ct^\alpha, \alpha) - 1 \right]. \tag{21}$$

by Setting $\alpha = 1$ equation (20) and (21), the solitary patterns solutions are

$$u(x,t) = \frac{4c}{a} \sinh^2 \left(\frac{1}{2} \sqrt{\frac{a}{2}} (x - ct) \right),$$

and

$$u(x,t) = -\frac{4c}{a}\cosh^2\left(\frac{1}{2}\sqrt{\frac{a}{2}}(x-ct)\right).$$

4. Conclusion

The variational iteration method (VIM) using Adomian's polynomials have been employed successfully to obtain the solitary pattern solutions of two variants of the fractional KdV equation with time derivative of fractional order. The obtained results are exactly the same with those obtained by homotopy perturbation method [11]. The method has been used in a direct way without linearization, perturbation or restrictive assumption and it can be concluded that VIM is powerful and efficient in finding the analytical approximate solutions as well as numerical solutions of differential equations of fractional order.

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