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MULTIVARIABLE JACOBI POLYNOMIALS VIA FRACTIONAL CALCULUS

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Abstract. In recent years, many works on the subject of fractional calculus contain interesting accounts of the theory and applications of fractional calculus operators in a number of areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, summation of series, etc.). The main object of this paper is to construct multivariable extension of Jacobi polynomials by means of fractional derivative operator and to give various generating functions for these polynomials by making use of fractional calculus. Furthermore, we derive various families of multilinear and multilateral generating functions. Some special cases of the results presented in this study are also indicated.

1. Introduction

Fractional calculus is the theory of derivatives and integrals of non-integer order. The fractional calculus started from some speculations of G. W. Leibniz (1695, 1697) and L. Euler (1730) and it has developed progressively up to now. Famous mathematicians who have provided important contributions up to the middle of the 20th century are known: P. S. Laplace (1812), J. B. J. Fourier (1822), N. H. Abel (1823-1826), J. Liouville (1832-1873), B. Riemann (1847), H. Holmgren(1865-1867), A. K. Grünwald (1867-1872), A. V. Letnikov (1868-1872), O. Heaviside (1892-1912), H. Weyl (1917), G. H. Hardy and J. E. Littlewood (1917-1928), A. Erdélyi (1939-1965) and many others (see Machado et. al [7]).

A great deal of literature has appeared discussing the application of fractional calculus operators in a number of areas of mathematical analysis (cf., e.g., [2],[8],[10], [17],[18]). There are many examples of the use of fractional derivatives in the theory of hypergeometric functions, in solving ordinary and partial diffrential equations and integral equations (see, for instance, [10],[15],[17]). Furthermore, literature includes many works that refer to the applications of fractional calculus in several scientific areas including potential fields, control theory, chemical physics, stochastic processes, anomalous diffusion (see Machado et. al [7]).

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In recent years, many researchers have studied various special functions by using fractional calculus. Laguerre polynomials, generalized Legendre polynomials, Legendre polynomials and generalized ultraspherical or Gegenbauer functions of arbitrary (fractional) orders have been defined in [3, 5, 9, 13, 14] and fractional derivatives of various multivariable functions have been derived (see, for instance [1]).

One of the most frequently encountered tools in the theory of fractional calculus (that is, differentiation and integration of an arbitrary real or complex order) is given by the familiar differintegral operator ${}_cD_z^{\mu}$, defined by

$$
\begin{aligned}
&{}_cD_z^{\mu}\left\{f(z)\right\} \\
&:=\begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_c^z (z-\xi)^{-\mu-1} f(\xi)\,d\xi \quad , \quad (c\in\mathbb{R},\ \mathbb{R}(\mu)<0) \\ \dfrac{d^m}{dz^m}\ {}_cD_z^{\mu-m}\left\{f(z)\right\}\ ,\ (m-1\leq\mathbb{R}(\mu)<m\ ,\ m\in\mathbb{N}) \end{cases}
\end{aligned}
\tag{1}
$$

provided that the integral exists. Throught the paper, we consider the case of $c=0$ in (1). The operator D_z^{μ} given by

$$D_z^{\mu}\left\{f(z)\right\} :=\ {}_0D_z^{\mu}\left\{f(z)\right\}\ \ (\mu\in\mathbb{C}) \tag{2}$$

corresponds essentially to the classical Riemann-Liouville fractional derivative (or integral) of order μ (or -μ) (see [17]).

The main object of this paper is to obtain generating functions by making use of fractional calculus for multivariable Jacobi polynomials. The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n are defined by the Rodrigues formula

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} \\
&\quad\times\frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}
\end{aligned}
$$

or equivalently, by

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2}\right) \tag{3}$$

where ${}_2F_1$ denotes the familiar (Gauss) hypergeometric function which corresponds to the special case $r-1=s=1$ of the generalized hypergeometric function ${}_rF_s$ which is given by

$${}_rF_s\left(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{x^n}{n!}$$

where $(\lambda)_k{:=}\lambda(\lambda+1)...(\lambda+k-1)$ and $(\lambda)_0:=1$ denotes the Pochhammer symbol.

These polynomials are orthogonal over the interval $(-1,1)$ with respect to the weight function

$$\omega(x)=(1-x)^{\alpha}(1+x)^{\beta}.$$

As a result of this, Jacobi polynomials

$$P_n^{*(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(2x-1)$$

are orthogonal with respect to the weight function $\omega(x) = x^{\beta}(1-x)^{\alpha}$ over the interval $(0,1)$. The multivariable extension of the Jacobi polynomials $P_n^{*(\alpha,\beta)}(x)$ is defined by

$$\begin{aligned} P_{n_1,\dots,n_r}^{*(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r) &= P_{n_1}^{*(\alpha_1,\beta_1)}(x_1)\dots P_{n_r}^{*(\alpha_r,\beta_r)}(x_r) \\ &= P_{\mathbf{n}}^{*(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(\mathbf{x}) \end{aligned} \tag{4}$$

where $\mathbf{x} = (x_1,\dots,x_r)$ and $|\mathbf{n}| = n_1 + \dots + n_r$; $n_1,\dots,n_r \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0\} \cup \{1,2,\dots\}$. The multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(\mathbf{x})$ are orthogonal with respect to the weight function

$$\omega(x_1,\dots,x_r) = \omega_1(x_1)\dots\omega_r(x_r) = (1-x_1)^{\alpha_1} x_1^{\beta_1} \dots (1-x_r)^{\alpha_r} x_r^{\beta_r}$$

over the domain

$$\Omega = \{(x_1,\dots,x_r) : \ 0 < x_i < 1; \ i = 1,2,\dots,r\}.$$

We organize the paper as follows:

In the next section, we recall some known applications of Riemann-Liouville fractional derivative. In section 3, the multivariable extension of Jacobi polynomials $P_n^{*(\alpha,\beta)}(x)$ is expressed in terms of fractional derivative operators and various mixed generating functions are obtained via fractional derivative operators. In section 4, multilinear and multilateral generating functions are derived for the multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(\mathbf{x})$. In section 5, some applications of the results obtained in section 4 are indicated.

2. Preliminaries

We recall here applications of Riemann-Liouville fractional derivative to some special functions. For proofs and more on the subject, see [6, 17].

The Riemann-Liouville fractional derivative of the power function holds that

$$D_z^{\mu}\{z^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} z^{\lambda-\mu}$$
$$(\ \mathbb{R}(\lambda) > -1 \ , \ \mathbb{R}(\lambda-\mu) > -1).$$

Applications of Riemann-Liouville fractional derivative to some special functions are as follows:

$$\begin{aligned} & D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\right\} \\ = \ & \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_D^{(3)}[\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz] \end{aligned} \tag{5}$$

$$(\mathbb{R}(\lambda) > 0 \ , \ \mathbb{R}(\mu) > 0 \ , \ |az| < 1 \ , \ |bz| < 1 \ , \ |cz| < 1)$$

and

$$\begin{aligned} & D_y^{\lambda-\mu}\left\{y^{\lambda-1}(1-y)^{-\alpha}\ {}_2F_1\left(\alpha,\beta;\gamma;\frac{x}{1-y}\right)\right\} \\ = \ & \frac{\Gamma(\lambda)}{\Gamma(\mu)} y^{\mu-1} F_2[\alpha,\beta,\lambda;\gamma,\mu;x,y] \end{aligned} \tag{6}$$

$$(\mathbb{R}(\lambda) > 0 \ , \ \mathbb{R}(\mu) > 0 \ , \ |x| + |y| < 1)$$

where $F_D^{(3)}[a, b_1, b_2, b_3; c; x_1, x_2, x_3]$ is Lauricella hypergeometric function, defined by

$$\begin{aligned} & F_D^{(3)}[a, b_1, b_2, b_3; c; x_1, x_2, x_3] \\ = & \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b_1)_m (b_2)_n (b_3)_p}{(c)_{m+n+p}} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!} \\ & (\max\{|x_1|, |x_2|, |x_3|\} < 1) \end{aligned} \tag{7}$$

and $F_2[\alpha, b_1, b_2; c_1, c_2; x, y]$ is the second kind of Appell's double hypergeometric function given by

$$F_2[a, b_1, b_2; c_1, c_2; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}$$
$$(|x| + |y| < 1).$$

The case of $a = 1,\ b = c = 0$ in (5) reduces to

$$\begin{aligned} D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha}\right\} &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\ {}_2F_1[\lambda, \alpha; \mu; z] \\ & (\mathbb{R}(\lambda) > 0\ ,\ \mathbb{R}(\mu) > 0\ ,\ |z| < 1). \end{aligned} \tag{8}$$

The special case of $c = 0$ in (5) gives

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1[\lambda, \alpha, \beta; \mu; az, bz] \tag{9}$$
$$(\mathbb{R}(\lambda) > 0\ ,\ \mathbb{R}(\mu) > 0\ ,\ |az| < 1\ ,\ |bz| < 1\ )$$

where the first kind of Appell's double hypergeometric function F_1 is defined by

$$F_1\left[\alpha, \beta, \beta'; \gamma; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m \left(\beta'\right)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \max\{|x|, |y|\} < 1. \tag{10}$$

3. Multivariable Jacobi Polynomials via Fractional Derivative

In this section, we present the multivariable Jacobi polynomials in terms of fractional derivative operator and derive various generating functions for them.

Theorem 3.1. Multivariable Jacobi polynomials are expressed as

$$\begin{aligned} P_{\mathbf{n}}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(\mathbf{x}) &= P_{n_1,\ldots,n_r}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(x_1, \ldots, x_r) \\ &= \prod_{i=1}^{r} \frac{(-1)^{n_i}\Gamma(\beta_i + \rho_i + n_i + 1)\, x_i^{-(\beta_i+\rho_i)}}{n_i!\Gamma(\alpha_i + \beta_i + 1)} \\ & \quad \times D_{x_1}^{\alpha_1-\rho_1} \ldots D_{x_r}^{\alpha_r-\rho_r}\left\{\prod_{j=1}^{r} x_j^{\alpha_j+\beta_j}(1-x_j)^{n_j}\right\} \end{aligned} \tag{11}$$

where $D_{x_i}^{\alpha_i-\rho_i}\ (i = 1, 2, ..., r)\ ,\ (m_i - 1 \le \mathbb{R}(\alpha_i - \rho_i) < m_i\ ,\ m_i \in \mathbb{N},\ i = 1, 2, ..., r)$ is fractional derivative operator defined by (2) and

$$\mathbb{R}(\beta_i + \rho_i) > -1\ ,\ \mathbb{R}(\alpha_i + \beta_i) > -1\ ,\ i = 1, 2, ..., r,$$
$$(\alpha - \rho - \mathbf{n}; \beta + \rho) = (\alpha_1 - \rho_1 - n_1, ..., \alpha_r - \rho_r - n_r; \beta_1 + \rho_1, ..., \beta_r + \rho_r).$$

Theorem 3.2. The multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(\mathbf{x})$ defined by (11) are generated by

(i)

$$\sum_{n_1,...,n_r=0}^{\infty} \frac{(\lambda_1)_{n_1} ... (\lambda_r)_{n_r} (-1)^{n_1+...+n_r}}{(\beta_1+\rho_1+1)_{n_1} ... (\beta_r+\rho_r+1)_{n_r}} \tag{12}$$

$$\times P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{n_1,...,n_r} (x_1,...,x_r)\, t_1^{n_1} ... t_r^{n_r}$$

$$= \prod_{i=1}^{r} (1-t_i)^{-\lambda_i}\ {}_2F_1 \left[\alpha_i+\beta_i+1, \lambda_i; \beta_i+\rho_i+1; -\frac{x_i t_i}{1-t_i} \right]$$

$$\left(\left| \frac{x_i t_i}{1-t_i} \right| < 1\ ,\ i=1,2,...,r \right)$$

(ii) $\sum\limits_{n=0}^{\infty} y_n (x_1,...,x_r)\, t^n$

$$= \prod_{i=1}^{r} (1-t)^{-\lambda_i}\ {}_2F_1 \left[\alpha_i+\beta_i+1, \lambda_i; \beta_i+\rho_i+1; -\frac{x_i t}{1-t} \right] \tag{13}$$

$$\left(\left| \frac{x_i t}{1-t} \right| < 1\ ,\ i=1,2,...,r \right)$$

where

$$y_n (x_1,...,x_r)$$

$$= \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} ... \sum_{n_{r-1}=0}^{n-n_1-...-n_{r-2}} \delta_n (n_1,...,n_{r-1})$$

$$\times P^{*(\alpha_1-\rho_1-n+n_1+...+n_{r-1}, \alpha_2-\rho_2-n_1,...,\alpha_r-\rho_r-n_{r-1};\ \beta_1+\rho_1,...,\beta_r+\rho_r)}_{n-(n_1+...+n_{r-1}),n_1,...,n_{r-1}} (x_1,...,x_r)$$

and also

$$\delta_n (n_1,...,n_{r-1}) = \frac{(-1)^n (\lambda_1)_{n-(n_1+...+n_{r-1})} (\lambda_2)_{n_1} ... (\lambda_r)_{n_{r-1}}}{(\beta_1+\rho_1+1)_{n-(n_1+...+n_{r-1})} (\beta_2+\rho_2+1)_{n_1} ... (\beta_r+\rho_r+1)_{n_{r-1}}}.$$

Proof. (i) Consider the equality

$$\prod_{i=1}^{r} [1-(1-x_i)\, t_i]^{-\lambda_i} = \prod_{i=1}^{r} (1-t_i)^{-\lambda_i} \left(1+\frac{x_i t_i}{1-t_i}\right)^{-\lambda_i}. \tag{14}$$

If we rewrite the left side of (14), we obtain

$$\sum_{n_1,...,n_r=0}^{\infty} \frac{(\lambda_1)_{n_1} ... (\lambda_r)_{n_r}}{n_1!...n_r!} (1-x_1)^{n_1} t_1^{n_1} ... (1-x_r)^{n_r} t_r^{n_r} = \prod_{i=1}^{r} (1-t_i)^{-\lambda_i} \left(1+\frac{x_i t_i}{1-t_i}\right)^{-\lambda_i} \tag{15}$$

for $|(1-x_i)\, t_i| < 1\ ,\ i=1,2,...,r$. Multiplying both side of (15) by $x_1^{\alpha_1+\beta_1} ... x_r^{\alpha_r+\beta_r}$ and applying the fractional derivative operators $D_{x_1}^{\alpha_1-\rho_1} ... D_{x_r}^{\alpha_r-\rho_r}$, we find the first generating function from (8) and (11).

(ii) In (14), taking t instead of t_i $(i=1,...,r)$ and making similar calculations, we obtain the second relation.

Corollary 3.3. If we take $\lambda_i = -k_i$ $(i = 1, ..., r)$ in Theorem 3.2 (i), we have

$$\sum_{n_1,...,n_r=0}^{k_1,...,k_r} \frac{(-k_1)_{n_1} ... (-k_r)_{n_r} (-1)^{n_1+...+n_r}}{(\beta_1+\rho_1+1)_{n_1} ... (\beta_r+\rho_r+1)_{n_r}} P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{n_1,...,n_r} (x_1, ..., x_r)\, t_1^{n_1} ... t_r^{n_r}$$
$$= \prod_{i=1}^{r} (1-t_i)^{k_i}\ {}_2F_1 \left[\alpha_i+\beta_i+1, -k_i; \beta_i+\rho_i+1; -\frac{x_i t_i}{1-t_i} \right].$$

If we apply fractional derivative operators to Theorem 3.2 (i), we have the following mixed generating functions for the multivariable Jacobi polynomials $P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{\mathbf{n}} (\mathbf{x})$:

Theorem 3.4. Mixed generating function for the multivariable Jacobi polynomials $P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{\mathbf{n}} (\mathbf{x})$ and Appell hypergeometric function F_1 is as follows:

$$\sum_{n_1,...,n_r=0}^{\infty} (-1)^{n_1+...+n_r} P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{n_1,...,n_r} (\mathbf{x})\, F_1(\gamma_1, \rho_1, -n_1; \delta_1; w_1 y_1, y_1)$$
$$\times ... \times F_1(\gamma_r, \rho_r, -n_r; \delta_r; w_r y_r, y_r)\, t_1^{n_1} ... t_r^{n_r}$$
$$= \prod_{i=1}^{r} \Big\{ (1-t_i)^{\alpha_i-\rho_i} (1-t_i+x_i t_i)^{-(\alpha_i+\beta_i+1)}$$
$$\times F_D^{(3)} \left(\gamma_i, \rho_i, -\alpha_i+\rho_i, \alpha_i+\beta_i+1; \delta_i; w_i y_i, \frac{t_i y_i}{t_i-1}, \frac{(x_i-1)\, t_i y_i}{1-t_i+x_i t_i} \right) \Big\}$$
$$\left(\max \left\{ |y_i|, |w_i y_i|, \left| \frac{t_i y_i}{t_i-1} \right|, \left| \frac{(x_i-1)\, t_i y_i}{1-t_i+x_i t_i} \right| \right\} < 1\ ,\ i = 1, 2, ..., r \right)$$

where F_1 and $F_D^{(3)}$ are defined by (10) and (7), respectively.

Proof. For the multivariable Jacobi polynomials $P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{n_1,...,n_r} (\mathbf{x})$, we have the following equality from (3):

$$\prod_{i=1}^{r} {}_2F_1(-n_i, \alpha_i+\beta_i+1; \beta_i+\rho_i+1; x_i)$$
$$= \prod_{i=1}^{r} \frac{n_i!}{(\beta_i+\rho_i+1)_{n_i}} P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{n_1,...,n_r} (\mathbf{x}) (-1)^{n_1+...+n_r} \tag{16}$$

and also the hypergeometric function ${}_2F_1$ is generated by (see [17])

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!}\ {}_2F_1(-n, \alpha; \beta; x)\, t^n = (1-t)^{\alpha-\beta} (1-t+xt)^{-\alpha}.$$

From these equalities, we obtain

$$\sum_{n_1,...,n_r=0}^{\infty} P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{n_1,...,n_r} (\mathbf{x}) (-1)^{n_1+...+n_r}\, t_1^{n_1} ... t_r^{n_r}$$
$$= \prod_{i=1}^{r} (1-t_i)^{\alpha_i-\rho_i} (1-t_i+x_i t_i)^{-(\alpha_i+\beta_i+1)}. \tag{17}$$

Replacing t_i by $(1-y_i)t_i$ $(i = 1, 2, ..., r)$ in (17), multiplying both side of (17) by $y_i^{\gamma_i-1}(1-w_i y_i)^{-\rho_i}$ $(i = 1, 2, ..., r)$ and applying the operator $D_{y_1}^{\gamma_1-\delta_1} ... D_{y_r}^{\gamma_r-\delta_r}$, we obtain mixed generating function for the multivariable Jacobi polynomials and Appell hypergeometric function F_1.

Corollary 3.5. Taking $\rho_i = 0$ $(i = 1, 2, ..., r)$ in Theorem 3.4, we have

$$\begin{aligned} &\sum_{n_1,...,n_r=0}^{\infty} (-1)^{n_1+...+n_r} P^{*(\alpha-\mathbf{n};\ \beta)}_{n_1,...,n_r}(\mathbf{x})\ {}_2F_1(\gamma_1, -n_1; \delta_1; y_1) \\ &\times ... \times\ {}_2F_1(\gamma_r, -n_r; \delta_r; y_r)\, t_1^{n_1} ... t_r^{n_r} \\ = &\prod_{i=1}^{r} \Big\{ (1-t_i)^{\alpha_i} (1-t_i+x_i t_i)^{-(\alpha_i+\beta_i+1)} \\ &\times F_1\left(\gamma_i, -\alpha_i, \alpha_i+\beta_i+1; \delta_i; \frac{t_i y_i}{t_i-1}, \frac{(x_i-1) t_i y_i}{1-t_i+x_i t_i}\right)\Big\} \end{aligned}$$

where

$$(\alpha - \mathbf{n}; \beta) = (\alpha_1 - n_1, ..., \alpha_r - n_r; \beta_1, ..., \beta_r).$$

Theorem 3.6. For the multivariable Jacobi polynomials and the hypergeometric function ${}_2F_1$,

$$\begin{aligned} &\sum_{n_1,...,n_r=0}^{\infty} H(n_1, ...n_r)\, P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{\mathbf{n}}(\mathbf{x})\ {}_2F_1(\gamma_1, \lambda_1+n_1; \delta_1; y_1) \\ &\times ... \times\ {}_2F_1(\gamma_r, \lambda_r+n_r; \delta_r; y_r)\, t_1^{n_1} ... t_r^{n_r} \\ = &\prod_{i=1}^{r} (1-t_i)^{-\lambda_i} F_2\left(\lambda_i, \alpha_i+\beta_i+1, \gamma_i; \beta_i+\rho_i+1, \delta_i; \frac{-x_i t_i}{1-t_i}, \frac{y_i}{1-t_i}\right) \\ &\left(|y_i| < 1\ ,\ \left|\frac{x_i t_i}{1-t_i}\right| + \left|\frac{y_i}{1-t_i}\right| < 1\ ,\ i = 1, 2, ..., r\right) \end{aligned}$$

gives another mixed generating function, where

$$H(n_1, ...n_r) = \prod_{i=1}^{r} \frac{(\lambda_i)_{n_i} (-1)^{n_i}}{(\beta_i+\rho_i+1)_{n_i}}.$$

Proof. In Theorem 3.2 (i), if we take $\frac{t_i}{1-y_i}$ instead of t_i $(i = 1, 2, ..., r)$, multiply $y_1^{\gamma_1-1} ... y_r^{\gamma_r-1}$ and apply the operator $D_{y_1}^{\gamma_1-\delta_1} ... D_{y_r}^{\gamma_r-\delta_r}$, we obtain the desired mixed generating function.

If we take $r = 1$ in (11), the multivariable Jacobi polynomials $P^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}_{\mathbf{n}}(x_1, ..., x_r)$ reduce to classical Jacobi polynomials $P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)$ defined by

$$P_n^{*(\alpha-\rho-n,\beta+\rho)}(x) = \frac{(-1)^n \Gamma(\beta+\rho+n+1)\, x^{-(\beta+\rho)}}{n!\Gamma(\alpha+\beta+1)} D_x^{\alpha-\rho} \left\{ x^{\alpha+\beta} (1-x)^n \right\}$$

$$(\mathbb{R}(\beta+\rho) > -1\ ,\ \mathbb{R}(\alpha+\beta) > -1).$$

If we get $r = 1$ in Theorem 3.2 (i) and Corollary 3.3, we have the next results for the Jacobi polynomials $P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)$.

Remark 3.7. Jacobi polynomials $P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)$ hold that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n (-1)^n}{(\beta+\rho+1)_n} P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)\, t^n \qquad (18) \\ = &(1-t)^{-\lambda}\ {}_2F_1\left[\alpha+\beta+1, \lambda; \beta+\rho+1; -\frac{xt}{1-t}\right] \end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{k}\frac{(-k)_n(-1)^n}{(\beta+\rho+1)_n}P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)\,t^n\\ =\;&(1-t)^k\;{}_2F_1\left[\alpha+\beta+1,-k;\beta+\rho+1;-\frac{xt}{1-t}\right]\end{aligned}$$

or equivalently,

$$\begin{aligned}&\sum_{n=0}^{k}\frac{(-k)_n(-1)^n}{(\beta+\rho+1)_n}P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)\,t^n\\ =\;&\frac{(t-1)^k\,k!}{(\beta+\rho+1)_k}P_k^{*(\alpha-\rho-k,\beta+\rho)}\left(-\frac{xt}{1-t}\right)\end{aligned}$$

for $\left|\frac{xt}{1-t}\right|<1$.

Formula (18) gives a special case of the relation (Srivastava [17], p. 108, Eq. (16)).

As a consequence of theorems obtained above, for the Jacobi polynomials $P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)$, we can give as follows:

Remark 3.8. Jacobi polynomials $P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)$ have the following bilateral generating functions:

$$\begin{aligned}&\sum_{n=0}^{\infty}P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)\,F_1(\gamma,\rho,-n;\delta;wy,y)\,(-t)^n\\ =\;&(1-t)^{\alpha-\rho}(1-t+xt)^{-(\alpha+\beta+1)}\\ &\times F_D^{(3)}\left(\gamma,\rho,-\alpha+\rho,\alpha+\beta+1;\delta;wy,\frac{ty}{t-1},\frac{(x-1)\,ty}{1-t+xt}\right)\\ &\left(\max\left\{|y|,|wy|,\left|\frac{ty}{t-1}\right|,\left|\frac{(x-1)\,ty}{1-t+xt}\right|\right\}<1\right)\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n(-1)^n}{(\beta+\rho+1)_n}P_n^{*(\alpha-\rho-n,\beta+\rho)}(x)\;{}_2F_1(\gamma,\lambda+n;\delta;y)\;t^n\\ =\;&(1-t)^{-\lambda}F_2\left(\lambda,\alpha+\beta+1,\gamma;\beta+\rho+1,\delta;\frac{-xt}{1-t},\frac{y}{1-t}\right)\\ &\left(|y|<1\;,\;\left|\frac{xt}{1-t}\right|+\left|\frac{y}{1-t}\right|<1\right).\end{aligned}$$

4. Multilinear and Multilateral Generating Functions

In this section, we derive several families of multilinear and multilateral generating functions for the multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha_1,\ldots,\alpha_r;\;\beta_1,\ldots,\beta_r)}(\mathbf{x})$ defined via fractional derivative operators.

We begin by stating the following theorem.

Theorem 4.1. Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of s complex variables $y_1, ..., y_s$ $(s \in \mathbb{N})$ and of complex order μ, let

$$\Lambda_{\mu,\nu}(y_1, ..., y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, ..., y_s) z^k \tag{19}$$

$$(a_k \neq 0, \ \ \mu, \nu \in \mathbb{C}).$$

Then, we have

$$\begin{aligned} &\sum_{n_1=0}^{\infty} \sum_{n_2,...,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} a_k \ \sigma(n_1 - pk, n_2, ..., n_r)\, x_1^{-(\beta_1+\rho_1)} ... x_r^{-(\beta_r+\rho_r)} \\ &\times D_{x_1}^{\alpha_1-\rho_1} ... D_{x_r}^{\alpha_r-\rho_r} \left\{ x_1^{\alpha_1+\beta_1} (1-x_1)^{n_1-pk} \prod_{i=2}^{r} x_i^{\alpha_i+\beta_i} (1-x_i)^{n_i} \right\} \\ &\times t_2^{n_2} ... t_r^{n_r} \Omega_{\mu+\nu k}(y_1, ..., y_s) \eta^k t_1^{n_1-pk} \\ = \ &\Lambda_{\mu,\nu}(y_1, ..., y_s; \eta) \prod_{i=1}^{r} (1-t_i)^{-\lambda_i} \ {}_2F_1\left[\alpha_i + \beta_i + 1, \lambda_i; \beta_i + \rho_i + 1; -\frac{x_i t_i}{1-t_i}\right] \end{aligned} \tag{20}$$

provided that each member of (20) exists. Here

$$\sigma(n_1, ..., n_r) = \prod_{i=1}^{r} \frac{\Gamma(\beta_i + \rho_i + n_i + 1)(\lambda_i)_{n_i}}{n_i! \Gamma(\alpha_i + \beta_i + 1)(\beta_i + \rho_i + 1)_{n_i}}.$$

Proof. For convenience, let S denote the first member of the assertion (20) of Theorem 4.1. A straightforward calculation gives

$$\begin{aligned} S \ = \ &\sum_{n_1,...,n_r=0}^{\infty} \sum_{k=0}^{\infty} a_k \ \sigma(n_1, ..., n_r)\, x_1^{-(\beta_1+\rho_1)} ... x_r^{-(\beta_r+\rho_r)} \\ &\times D_{x_1}^{\alpha_1-\rho_1} ... D_{x_r}^{\alpha_r-\rho_r} \left\{ \prod_{i=1}^{r} x_i^{\alpha_i+\beta_i} (1-x_i)^{n_i} \right\} \\ &\times t_1^{n_1} ... t_r^{n_r} \Omega_{\mu+\nu k}(y_1, ..., y_s) \eta^k \\ = \ &\sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, ..., y_s) \eta^k \\ &\times \sum_{n_1,...,n_r=0}^{\infty} \sigma(n_1, ..., n_r)\, x_1^{-(\beta_1+\rho_1)} ... x_r^{-(\beta_r+\rho_r)} \\ &\times D_{x_1}^{\alpha_1-\rho_1} ... D_{x_r}^{\alpha_r-\rho_r} \left\{ \prod_{i=1}^{r} x_i^{\alpha_i+\beta_i} (1-x_i)^{n_i} \right\} t_1^{n_1} ... t_r^{n_r} \end{aligned}$$

where

$$\sigma(n_1, ..., n_r) = \prod_{i=1}^{r} \frac{\Gamma(\beta_i + \rho_i + n_i + 1)(\lambda_i)_{n_i}}{n_i! \Gamma(\alpha_i + \beta_i + 1)(\beta_i + \rho_i + 1)_{n_i}}.$$

If we use (12), then the proof of Theorem 4.1 is completed.

Theorem 4.2. Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let

$$\Lambda_{\mu,\psi}(y_1,...,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_s)z^k \tag{21}$$

$$(a_k\neq 0\,,\ \ \mu,\psi\in\mathbb{C})$$

and

$$\Theta_{n,p,\mu,\psi}(x_1,...,x_r;y_1,...,y_s;\zeta):=\sum_{k=0}^{[n/p]}a_k y_{n-pk}\left(x_1,...,x_r\right)\Omega_{\mu+\psi k}(y_1,...,y_s)\zeta^k \tag{22}$$

where $n,p\in\mathbb{N}$. Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\psi}(x_1,...,x_r;y_1,...,y_s;\frac{\eta}{t^p})t^n \\ =\ &\prod_{i=1}^{r}(1-t)^{-\lambda_i}\ {}_2F_1\left[\alpha_i+\beta_i+1,\lambda_i;\beta_i+\rho_i+1;-\frac{x_it}{1-t}\right] \\ &\times\Lambda_{\mu,\psi}(y_1,...,y_s;\eta)\end{aligned} \tag{23}$$

provided that each member of (23) exists. Here

$$\begin{aligned}&y_n\left(x_1,...,x_r\right) \\ =\ &\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}...\sum_{n_{r-1}=0}^{n-n_1-...-n_{r-2}}\delta_n\left(n_1,...,n_{r-1}\right) \\ &\times P^{*(\alpha_1-\rho_1-n+n_1+...+n_{r-1},\alpha_2-\rho_2-n_1,...,\alpha_r-\rho_r-n_{r-1};\ \beta_1+\rho_1,...,\beta_r+\rho_r)}_{n-(n_1+...+n_{r-1}),n_1,...,n_{r-1}}\left(x_1,...,x_r\right)\end{aligned}$$

and

$$\delta_n\left(n_1,...,n_{r-1}\right)=\frac{(-1)^n\left(\lambda_1\right)_{n-(n_1+...+n_{r-1})}\left(\lambda_2\right)_{n_1}...\left(\lambda_r\right)_{n_{r-1}}}{\left(\beta_1+\rho_1+1\right)_{n-(n_1+...+n_{r-1})}\left(\beta_2+\rho_2+1\right)_{n_1}...\left(\beta_r+\rho_r+1\right)_{n_{r-1}}}.$$

Proof. For convenience, let S denote the first member of the assertion (23). Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\psi}(x_1,...,x_r;y_1,...,y_s;\frac{\eta}{t^p})$$

from the definition (22) into the left-hand side of (23), we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k y_{n-pk}\left(x_1,...,x_r\right)\Omega_{\mu+\psi k}(y_1,...,y_s)\eta^k t^{n-pk}\,. \tag{24}$$

Upon inverting the order of summation in (24), if we replace n by $n+pk$, we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\, y_n\,(x_1,...,x_r)\,\Omega_{\mu+\psi k}(y_1,...,y_s\,)\eta^k t^n \\ &= \sum_{n=0}^{\infty} y_n\,(x_1,...,x_r)\,t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,...,y_s\,)\eta^k \\ &= \prod_{i=1}^{r}(1-t)^{-\lambda_i}\ {}_2F_1\left[\alpha_i+\beta_i+1,\lambda_i;\beta_i+\rho_i+1;-\frac{x_i t}{1-t}\right] \\ &\quad \times\Lambda_{\mu,\psi}(y_1,...,y_s;\eta), \end{aligned}$$

which completes the proof of Theorem 4.2.

5. Further Consequences and Miscellaneous Properties

By expressing the multivariable function

$$\Omega_{\mu+\nu k}(y_1,...,y_s\,)\ (k\in\mathbb{N}_0\,,\ s\in\mathbb{N})$$

in terms of a simple function of one and more variables, we can have further applications of Theorem 4.1. For example, if we set

$$s=1\,\text{and}\ \ \Omega_{\mu+\nu k}(y\,)=H_{\mu+\nu k}(\zeta,\tau,y)$$

in Theorem 4.1, where Rice polynomials

$$H_n(\zeta,\tau,y)=\ {}_3F_2\,(-n,n+1,\zeta;1,\tau;y)$$

are generated by [12] (see also [11])

$$\sum_{n=0}^{\infty} H_n(\zeta,\tau,y)t^n=(1-t)^{-1}\ \ {}_2F_1\left(\zeta,\tfrac{1}{2};\tau;\tfrac{-4yt}{(1-t)^2}\right), \tag{25}$$

then we obtain the following result which provides a class of bilateral generating functions for the Rice polynomials and the multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(\mathbf{x})$ defined by (11).

Corollary 5.1. If $\Lambda_{\mu,\nu}(y;z):=\sum\limits_{k=0}^{\infty} a_k H_{\mu+\nu k}(\zeta,\tau,y)z^k$, $(a_k\neq 0\,,\ \mu,\nu\in\mathbb{C})$. Then, we have

$$\begin{aligned} &\sum_{n_1=0}^{\infty}\ \sum_{n_2,...,n_r=0}^{\infty}\ \sum_{k=0}^{[n_1/p]} a_k\ \sigma\,(n_1-pk,n_2,...,n_r)\,x_1^{-(\beta_1+\rho_1)}...x_r^{-(\beta_r+\rho_r)} \qquad (26)\\ &\times D_{x_1}^{\alpha_1-\rho_1}...D_{x_r}^{\alpha_r-\rho_r}\left\{x_1^{\alpha_1+\beta_1}\,(1-x_1)^{n_1-pk}\prod_{i=2}^{r}x_i^{\alpha_i+\beta_i}\,(1-x_i)^{n_i}\right\}\\ &\times t_2^{n_2}...t_r^{n_r}H_{\mu+\nu k}(\zeta,\tau,y)\eta^k t_1^{n_1-pk}\\ =\ &\Lambda_{\mu,\nu}(y;\eta)\prod_{i=1}^{r}(1-t_i)^{-\lambda_i}\ \ {}_2F_1\left[\alpha_i+\beta_i+1,\lambda_i;\beta_i+\rho_i+1;-\frac{x_i t_i}{1-t_i}\right] \end{aligned}$$

provided that each member of (26) exists. Where

$$\sigma\,(n_1,...,n_r)=\prod_{i=1}^{r}\frac{\Gamma\,(\beta_i+\rho_i+n_i+1)\,(\lambda_i)_{n_i}}{n_i!\Gamma\,(\alpha_i+\beta_i+1)\,(\beta_i+\rho_i+1)_{n_i}}.$$

Remark 5.2. Using (25) and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\begin{aligned}
&\sum_{n_1=0}^{\infty} \sum_{n_2,\ldots,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} \sigma\left(n_1 - pk, n_2, \ldots, n_r\right) x_1^{-(\beta_1+\rho_1)} \ldots x_r^{-(\beta_r+\rho_r)} \\
&\times D_{x_1}^{\alpha_1-\rho_1} \ldots D_{x_r}^{\alpha_r-\rho_r} \left\{ x_1^{\alpha_1+\beta_1} \left(1 - x_1\right)^{n_1-pk} \prod_{i=2}^{r} x_i^{\alpha_i+\beta_i} \left(1 - x_i\right)^{n_i} \right\} \\
&\times t_2^{n_2} \ldots t_r^{n_r} H_k(\zeta, \tau, y) \eta^k t_1^{n_1-pk} \\
= \ & (1-\eta)^{-1} \ {}_2F_1\left(\zeta, \tfrac{1}{2}; \tau; \tfrac{-4y\eta}{(1-\eta)^2}\right) \\
&\times \prod_{i=1}^{r} (1-t_i)^{-\lambda_i} \ {}_2F_1\left[\alpha_i + \beta_i + 1, \lambda_i; \beta_i + \rho_i + 1; -\frac{x_i t_i}{1 - t_i}\right].
\end{aligned}$$

If we set $s = 2$ and $\Omega_{\mu+\psi k}(y_1, y_2) = g_{\mu+\psi k}^{(\gamma_1,\gamma_2)}(y_1, y_2)$, $(\mu, \psi \in \mathbb{C})$ in Theorem 4.2, where the classical Lagrange polynomials $g_n^{(\alpha,\beta)}(x, y)$ [4, p.267] are generated by

$$\frac{1}{(1-xt)^{\alpha} (1-yt)^{\beta}} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x, y) \ t^n \tag{27}$$

where $|t| < \min\left\{|x|^{-1}, |y|^{-1}\right\}$, we have the following result which provides a class of bilateral generating functions for the classical Lagrange polynomials and the multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(\mathbf{x})$.

Corollary 5.3. If $\Lambda_{\mu,\psi}(y_1, y_2; z) := \sum_{k=0}^{\infty} a_k g_{\mu+\psi k}^{(\gamma_1,\gamma_2)}(y_1, y_2) z^k$ where $a_k \neq 0$, $\psi, \mu \in \mathbb{C}$; and

$$\begin{aligned}
&\Theta_{n,p,\mu,\psi}(x_1, \ldots, x_r; y_1, y_2; \zeta) \\
:\ = &\sum_{k=0}^{[n/p]} a_k y_{n-pk}(x_1, \ldots, x_r) g_{\mu+\psi k}^{(\gamma_1,\gamma_2)}(y_1, y_2) \zeta^k
\end{aligned}$$

where $n, p \in \mathbb{N}$ and

$$\begin{aligned}
y_n\left(x_1, \ldots, x_r\right) = &\sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \ldots \sum_{n_{r-1}=0}^{n-n_1-\ldots-n_{r-2}} \delta_n\left(n_1, \ldots, n_{r-1}\right) \\
&\times P_{n-(n_1+\ldots+n_{r-1}),n_1,\ldots,n_{r-1}}^{*(\alpha_1-\rho_1-n+n_1+\ldots+n_{r-1},\alpha_2-\rho_2-n_1,\ldots,\alpha_r-\rho_r-n_{r-1};\ \beta_1+\rho_1,\ldots,\beta_r+\rho_r)}\left(x_1, \ldots, x_r\right) .
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\psi}(x_1, \ldots, x_r; y_1, y_2; \frac{\eta}{t^p}) t^n \qquad (28) \\
= \ &\prod_{i=1}^{r} (1-t)^{-\lambda_i} \ {}_2F_1\left[\alpha_i + \beta_i + 1, \lambda_i; \beta_i + \rho_i + 1; -\frac{x_i t}{1 - t}\right] \\
&\times \Lambda_{\mu,\psi}(y_1, y_2; \eta)
\end{aligned}$$

provided that each member of (28) exists.

Remark 5.4. Using the generating relation (27) for the classical Lagrange polynomials and taking $a_k = 1$, $\mu = 0$, $\psi = 1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} y_{n-pk}(x_1,...,x_r)\mathrm{g}_k^{(\gamma_1,\gamma_2)}(y_1,y_2)\eta^k t^{n-pk} \\
= &\prod_{i=1}^{r}(1-t)^{-\lambda_i}\ {}_2F_1\left[\alpha_i+\beta_i+1,\lambda_i;\beta_i+\rho_i+1;-\frac{x_i t}{1-t}\right] \\
&\times\frac{1}{(1-y_1\eta)^{\gamma_1}(1-y_2\eta)^{\gamma_2}},
\end{aligned}$$

where $|\eta| < \min\left\{|y_1|^{-1}, |y_2|^{-1}\right\}$.

Setting $s = r$ and $\Omega_{\mu+\psi k}(y_1,...,y_r) = y_{\mu+\psi k}(y_1,...,y_r)$, $(\mu,\psi\in\mathbb{N}_0)$, in Theorem 4.2, we obtain the following class of bilinear generating functions for the multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(\mathbf{x})$.

Corollary 5.5. If

$$\begin{aligned}
&\Lambda_{\mu,\psi}(y_1,...,y_r;z) \\
:\ &=\sum_{k=0}^{\infty} a_k y_{\mu+\psi k}(y_1,...,y_r)\, z^k,
\end{aligned}$$

where $a_k \neq 0$, $\mu,\psi\in\mathbb{C}$ and

$$\begin{aligned}
&\Theta_{n,p,\mu,\psi}(x_1,...,x_r;y_1,...,y_r;\zeta) \\
:\ &=\sum_{k=0}^{[n/p]} a_k y_{n-pk}(x_1,...,x_r) y_{\mu+\psi k}(y_1,...,y_r)\,\zeta^k,
\end{aligned}$$

for $n,p\in\mathbb{N}$. Then, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\psi}\left(x_1,...,x_r;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n \\
= &\prod_{i=1}^{r}(1-t)^{-\lambda_i}\ {}_2F_1\left[\alpha_i+\beta_i+1,\lambda_i;\beta_i+\rho_i+1;-\frac{x_i t}{1-t}\right] \\
&\times\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \qquad (29)
\end{aligned}$$

provided that each member of (29) exists.

We further note that for every suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu k}(y_1,...,y_s)$, $(s\in\mathbb{N})$, is expressed as an appropriate product of several simple functions, the assertions of Theorem 4.1 and Theorem 4.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the multivariable Jacobi polynomials $P_{\mathbf{n}}^{*(\alpha-\rho-\mathbf{n};\ \beta+\rho)}(\mathbf{x})$.

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