

## ALTERNATIVE DERIVATION OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

R.K. SAXENA, J. RAM AND D. KUMAR

**ABSTRACT.** In view of the usefulness and a great importance of the kinetic equation in certain astrophysical problems the authors develop a new and further generalized form of the fractional kinetic equation involving Mittag-Leffler function and G-function. This new generalization can be used for the computation of the change of chemical composition in stars like the sun. The manifold generality of the Mittag-Leffler function and G-function is discussed in terms of the solution of the above fractional kinetic equation. Saxena et al. [23, 24] derived the solutions of generalized fractional kinetic equations in terms of Mittag-Leffler functions by the application of Laplace transform [8, 25]. The present work is extension of earlier work done by Saxena et al. [24], and Chaurasia and Pandey [6].

### 1. INTRODUCTION AND PRELIMINARIES

#### **Definition 1:- Sumudu Transform:**

An integral transform, called the Sumudu transform was defined and studied by G.K. Watugala [28] to facilitate the process of solving differential and integral equations in the time domain, and for the use in various applications of system engineering and applied physics. In [3, 4, 5, 28], some fundamental properties of the Sumudu transform are established. It turns out that the Sumudu transform has very special and useful properties and it is useful in solving problems of science and engineering governing kinetic equations. The Sumudu transform has been shown to be the theoretical dual of the Laplace transform. The Laplace transform is defined by

$$F(p) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-pt} f(t) dt, \operatorname{Re}(p) > 0. \quad (1)$$

The Sumudu transform is defined over the set of functions,

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j} \text{ if } t \in (-1)^j \times [0, \infty) \right\}, \quad (2)$$

---

2000 *Mathematics Subject Classification.* 44A10, 44A15, 44A35, 44A05, 26A33.

*Key words and phrases.* Kinetic equations, Functional calculus, Special functions, Sumudu transforms, Nuclear reactions, G -function, Mellin-Barnes contour integral.

Submitted Jan. 19, 2013.

by

$$G(u) = S(f(t)) = \int_0^\infty e^{-t} f(ut) dt, \quad u \in (-\tau_1, \tau_2); \tag{3}$$

where  $M$  is a real finite number and  $\tau_1$  and  $\tau_2$  can be finite or infinite [28]. Hence,  $G(u)$  is called as the Sumudu transform of  $f(t)$ . It is obvious that this is a linear operator. It can be easily verified that in (3) the function  $G(u)$  keeps the same units as  $f(t)$ , for any real or complex number  $\lambda$  it gives that  $S[f(\lambda t)] = G(\lambda u)$ . The Sumudu and Laplace transforms exhibit a duality relation that may be expressed either as

$$(i) \ G\left(\frac{1}{u}\right) = u F(u) \quad \text{or} \quad G(u) = \frac{1}{u} F\left(\frac{1}{u}\right), \tag{4}$$

$$(ii) \ F\left(\frac{1}{p}\right) = p G(p) \quad \text{or} \quad F(p) = \frac{1}{p} G\left(\frac{1}{p}\right). \tag{5}$$

The Sumudu transform is connected to the p-multiplied Laplace transform (see [18]).

**Fractional kinetic equations:**

Fractional kinetic equations have gained popularity during the past decade mainly due to the discovery of their relation with the CTRW-theory in [15].

Hilfer [13, 14] have investigated fractional kinetic equations in order to determine and deduce certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on.

Saxena and Kalla [22] considered the following fractional kinetic equation:

$$N(t) - N_0 f(t) = -c^v ({}_0D_t^{-v} N)(t) \quad (Re(v) > 0), \tag{6}$$

where  $N(t)$  denotes the number density of a given species at time  $t$ ,  $N_0 = N(0)$  is the number density of that species at time  $t = 0$ ,  $c$  is a constant and  $f \in L(0, \infty)$ . By applying the Laplace transform to (6), we have

$$\mathcal{L}[N(t)](p) = N_0 \frac{F(p)}{1 + c^v p^{-v}} = N_0 \left( \sum_{n=0}^\infty (-c^v)^n p^{-nv} \right) F(p) \left( n \in N_0, \left| \frac{c}{p} \right| < 1 \right). \tag{7}$$

Tomovski et al. [27] provided the corrected version of the obviously erroneous solution of the fractional kinetic equation (6) given by Saxena and Kalla [[22], p. 508, Eqn. (3.2)] as follows:

$$N(t) = N_0 \left( f(t) + \sum_{n=1}^\infty \frac{(-c^v)^n}{\Gamma(nv)} (t^{nv-1} * f(t)) \right), \tag{8}$$

or

$$N(t) = N_0 \left( f(t) + \sum_{n=1}^\infty (-c^v)^n ({}_0D_t^{-nv} f)(t) \right), \tag{9}$$

where the relationship between the Laplace convolution and the Riemann-Liouville fractional integral operator  $({}_0D_t^{-v} f)$  with  $a = 0$ , given as following :

$$t^{nv-1} * f(t) = \int_0^t (t-p)^{nv-1} f(p) dp = \Gamma(nv) ({}_0D_t^{-nv} f)(t), \quad (n \in N, Re(v) > 0). \tag{10}$$

The solution (9) provides the new version of the equation (6) by applying a technique which was employed earlier by Al. Saqabi and Tuan [2] for solving fractional differintegral equations.

The general fractional kinetic differintegral equation given as following:

$$a \left( D_{0+}^{\alpha, \beta} N \right) (t) - N_0 f(t) = b \left( {}_0D_t^{-\nu} N \right) (t), \quad (11)$$

under the initial condition

$$\left( {}_0D_t^{-[(1-\beta)(1-\alpha)]} f \right) (0+) = c, \quad (12)$$

where  $a$ ,  $b$  and  $c$  are constant and  $f \in L(0, \infty)$ .

Tomovski et al. [27] provided explicit solution of the fractional kinetic differintegral equation (11) with the initial condition (12) as follows:

$$\begin{aligned} N(t) &= \frac{N_0}{a} \sum_{n=0}^{\infty} \left( \frac{b}{a} \right)^n \frac{(t^{\alpha+n(v+\alpha)-1} * f(t))}{\Gamma(\alpha+n(v+\alpha))} \\ &+ c \sum_{n=0}^{\infty} \left( \frac{b}{a} \right)^n \frac{(t^{\alpha-\beta(1-\alpha)+n(v+\alpha)-1})}{\Gamma(\alpha-\beta(1-\alpha)+n(v+\alpha))} \quad (\alpha \neq 0), \end{aligned} \quad (13)$$

or, equivalently, by

$$\begin{aligned} N(t) &= \frac{N_0}{a} \sum_{n=0}^{\infty} \left( \frac{b}{a} \right)^n \left( {}_0D_t^{-[\alpha+n(v+\alpha)]} f \right) (t) \\ &+ c \sum_{n=0}^{\infty} \left( \frac{b}{a} \right)^n \frac{(t^{\alpha-\beta(1-\alpha)+n(v+\alpha)-1})}{\Gamma(\alpha-\beta(1-\alpha)+n(v+\alpha))} \quad (\alpha \neq 0), \end{aligned} \quad (14)$$

where  $a$ ,  $b$  and  $c$  are constant and  $f \in L(0, \infty)$ .

## 2. GENERALIZED MITTAG-LEFFLER FUNCTION

In 1903, the Swedish mathematician Gosta Mittag-Leffler [19, 20] introduced the function  $E_{\alpha}(z)$ , defined by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in C, \operatorname{Re}(\alpha) > 0). \quad (15)$$

The Mittag-Leffler function  $E_{\alpha}(z)$  was studied by Wiman [29] who defined the function  $E_{\alpha, \beta}(z)$  as follows

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in C, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0). \quad (16)$$

The function  $E_{\alpha, \beta}(z)$  is now known as Wiman function, which was later studied by Agarwal [1] and others. The generalization of (16) was introduced by Prabhakar [21] in terms of the series representation

$$E_{\alpha, \beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!}, \quad (\alpha, \beta, \gamma \in C, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0). \quad (17)$$

where  $(\gamma)_n$  is Pochhammer's symbol, defined by

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1, & (n = 0, \gamma \neq 0) \\ \gamma(\gamma + 1) \dots (\gamma + n - 1), & (n \in N, \gamma \in C) \end{cases}. \quad (18)$$

It is an entire function of order [21]  $\rho = [Re(\alpha)]^{-1}$ . Some special cases of (17) are

$$E_\alpha(z) = E_{\alpha,1}^1(z), E_{\alpha,\beta}(z) = E_{\alpha,\beta}^1(z), \phi(\beta, \gamma; z) = {}_1F_1(\beta, \gamma; z) = \Gamma(\gamma) E_{1,\gamma}^\beta(z), \tag{19}$$

where  $\phi(\beta, \gamma; z)$  is Kummer’s confluent hypergeometric function [12]. Mellin-Barnes integral representation for the function (17) is given by [24]:

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{2\pi i \Gamma(\gamma)} \int_\Omega \frac{\Gamma(-s) \Gamma(\gamma + s) (-z)^s}{\Gamma(\beta + s\alpha)} ds, \tag{20}$$

where  $i = \sqrt{-1}$ .

**Remark 1** A detailed account of Mittag-Leffler functions and their Applications can be found in the monograph by Haubold, Mathai and Saxena [12].

The following integral gives the Sumudu transform of  $E_{\alpha,\beta}^\gamma(z)$ :

$$\int_0^\infty e^{-t} (ut)^{\beta-1} E_{\alpha,\beta}^\gamma(w(ut)^\alpha) dt = u^{\beta-1} (1 - wu^\alpha)^{-\gamma}, \tag{21}$$

where  $Re(u) > |w|^{-1/Re(\alpha)}$ ,  $Re(\beta) > 0$ ,  $Re(u) > 0$ ,  $u \in (-\tau_1, \tau_2)$ ,  $|f(t)| < Me^{|t|/\tau_j}$  which can be established by means of the Gamma function

$$\int_0^\infty e^{-t} t^{x-1} = \Gamma(x), \quad Re(x) > 0, \tag{22}$$

and the binomial formula

$$(1 - z)^{-a} = \sum_{n=0}^\infty \frac{(a)_n z^n}{n!} \tag{23}$$

The Laplace transform of  $E_{\alpha,\beta}^\gamma(at^\alpha)$  is as follows [24]:

$$\int_0^\infty e^{-pt} t^{\beta-1} E_{\alpha,\beta}^\gamma(at^\alpha) dt = p^{-\beta} (1 - ap^{-\alpha})^{-\gamma}. \tag{24}$$

If we set  $\gamma = 1$ , (21) reduces to

$$\int_0^\infty e^{-t} (ut)^{\beta-1} E_{\alpha,\beta}(w(ut)^\alpha) dt = u^{\beta-1} (1 - wu^\alpha)^{-1}, \tag{25}$$

where  $|u| > |w|^{-1/Re(\alpha)}$ ,  $Re(\beta) > 0$ ,  $Re(u) > 0$ ,  $u \in (-\tau_1, \tau_2)$ ,  $|f(t)| < Me^{|t|/\tau_j}$ .

Now, we recall the definition of Riemann-Liouville integral operator [26] of order  $\alpha \in C$ .

$${}_a D_x^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \quad Re(\alpha) > 0, \tag{26}$$

in the form

$${}_0 D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} f(s) ds, \quad t > 0, \quad Re(v) > 0, \tag{27}$$

with  ${}_a D_t^0 f(t) = f(t)$ .

Fractional derivative for  $\alpha > 0$  is defined as

$${}_0 D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u) du}{(t-u)^{\alpha-n+1}}, \quad n = [Re(\alpha)] + 1;$$

where  $[Re(\alpha)]$  stand for the integral part of  $Re(\alpha)$ .

If we apply convolution theorem for Sumudu transform [3, 4, 5], we observe that (27) can be written in the following form:

$$S \{ {}_0D_t^{-v} f(t) \} = S \left\{ \frac{t^{v-1}}{\Gamma(v)} \right\} . S \{ f(t) \} = u^v G(u) \quad (28)$$

In the following, we present the solution of generalized fractional kinetic equations. The results are obtained in terms of generalized Mittag-Leffler functions in a compact form and can be worked out easily. A detailed account of the fractional integral operators and their applications is available in Ref. [26].

### 3. SOLUTION OF FRACTIONAL KINETIC EQUATIONS

**Theorem 1** If  $c > 0, v > 0, \mu > 0, \gamma > 0$ , for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{v,\mu}^\gamma(-c^v t^v) = -c^v {}_0D_t^{-v} N(t), \quad (29)$$

there holds the formula

$$N(t) = N_0 t^{\mu-2} E_{v,\mu-1}^{\gamma+1}(-c^v t^v), \quad (30)$$

where  $E_{v,\mu}^\gamma(z)$  is the generalized Mittag-Leffler function.

**Proof** Applying the Sumudu transform to the both sides of (29) and using (28), we get

$$S[N(t)] = N_0 S[t^{\mu-1} E_{v,\mu}^\gamma(-c^v t^v)] - c^v S[{}_0D_t^{-v} N(t)],$$

$$N^*(u) = N_0 \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^\gamma} \right] - c^v u^v N^*(u),$$

and we have the following:

$$N^*(u) = N_0 \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^{\gamma+1}} \right]. \quad (31)$$

Using the relation  $S^{-1}\{u^v\} = \frac{t^{v-1}}{\Gamma(v)}$ ,  $Re(v) > 0, Re(u) > 0$ , and taking the Sumudu inverse of (31), we have,

$$S^{-1}\{N^*(u)\} = N_0 S^{-1} \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^{\gamma+1}} \right] = N_0 S^{-1} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma+1)_n u^{\mu-1} (uc)^{vn}}{n!} \right]$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma+1)_n c^{vn}}{n!} S^{-1}\{u^{\mu+vn-1}\},$$

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma+1)_n c^{vn}}{n!} \frac{t^{\mu+vn-2}}{\Gamma(\mu+vn-1)} = N_0 t^{\mu-2} E_{v,\mu-1}^{\gamma+1}(-c^v t^v).$$

This completes the proof of Theorem 1.

Now, if we follow the definition of  $E_{\alpha,\beta}^\gamma(z)$  given by (17) and set  $\gamma = 1$ , then, we arrive at the following result:

**Corollary 1.1** If  $c > 0, v > 0, \mu > 0$  then for the solution of

$$N(t) - N_0 t^{\mu-1} E_{v,\mu}(-c^v t^v) = -c^v {}_0D_t^{-v} N(t), \quad (32)$$

there holds the relation

$$N(t) = \frac{N_0 t^{\mu-2}}{v} [E_{v,\mu-2}(-c^v t^v) + (2+v-\mu) E_{v,\mu-1}(-c^v t^v)]. \quad (33)$$

**Proof** Applying the Sumudu transform to the both sides of (32) and using (28), we get

$$S[N(t)] = N_0 S[t^{\mu-1} E_{v,\mu}(-c^v t^v)] - c^v S[{}_0D_t^{-v} N(t)]$$

$$N^*(u) = N_0 \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^2} \right] - c^v u^v N^*(u),$$

or

$$N^*(u) = N_0 \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^2} \right]. \tag{34}$$

Using the relation  $S^{-1}\{u^v\} = \frac{t^{v-1}}{\Gamma(v)}$ ,  $Re(v) > 0$ ,  $Re(u) > 0$ , and taking the Sumudu inverse of (34), we have,

$$S^{-1}\{N^*(u)\} = N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n (2)_n c^{vn}}{n!} S^{-1}\{u^{\mu+vn-1}\}$$

$$= N_0 t^{\mu-2} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(ct)^{vn}}{\Gamma(\mu+vn-1)}$$

$$= N_0 t^{\mu-2} \sum_{n=0}^{\infty} \frac{(-1)^n \left[ \frac{1}{v} \{(nv + \mu - 2) + (2 + v - \mu)\} \right] (ct)^{vn}}{\Gamma(\mu+vn-1)}$$

$$= N_0 \frac{t^{\mu-2}}{v} \left[ \sum_{n=0}^{\infty} \frac{(-c^v t^v)^n}{\Gamma(\mu+vn-2)} + (2+v-\mu) \sum_{n=0}^{\infty} \frac{(-c^v t^v)^n}{\Gamma(\mu+vn-1)} \right]$$

$$= \frac{N_0 t^{\mu-2}}{v} [E_{v,\mu-2}(-c^v t^v) + (2+v-\mu) E_{v,\mu-1}(-c^v t^v)].$$

This completes the proof of (33).

If we set  $\gamma = 2$  in Theorem 1, then we obtain the following:

**Corollary 1.2** If  $c > 0, v > 0, \mu > 0$  then for the solution of

$$N(t) - N_0 t^{\mu-1} E_{v,\mu}^2(-c^v t^v) = -c^v {}_0D_t^{-v} N(t), \tag{35}$$

there holds the relation

$$N(t) = \frac{N_0 t^{\mu-2}}{2v^2} [E_{v,\mu-3}(-c^v t^v) + \{3v - 2\mu + 5\} E_{v,\mu-2}(-c^v t^v)$$

$$+ \{2(v^2 + \mu^2 - 2\mu + 1) + 6v - 2\mu - 3v\mu + 3\} E_{v,\mu-1}(-c^v t^v)]. \tag{36}$$

**Proof** Applying the Sumudu transform to the both sides of (35) and using (28), we get

$$S[N(t)] = N_0 S[t^{\mu-1} E_{v,\mu}^2(-c^v t^v)] - c^v S[{}_0D_t^{-v} N(t)]$$

$$N^*(u) = N_0 \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^2} \right] - c^v u^v N^*(u),$$

or

$$N^*(u) = N_0 \left[ \frac{u^{\mu-1}}{[1+(uc)^v]^3} \right]. \tag{37}$$

Using the relation  $S^{-1}\{u^v\} = \frac{t^{v-1}}{\Gamma(v)}$ ,  $Re(v) > 0$ ,  $Re(u) > 0$ , and taking the Sumudu inverse of (37), we have,

$$S^{-1}\{N^*(u)\} = N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n (3)_n c^{vn}}{n!} S^{-1}\{u^{\mu+vn-1}\}$$

$$\begin{aligned}
&= N_0 \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)c^{vn}}{2(n)!} \frac{t^{\mu+vn-1}}{\Gamma(\mu+vn-1)} \\
&= \frac{N_0 t^{\mu-2}}{2v^2} [E_{v,\mu-3}(-c^v t^v) + \{3v-2\mu+5\} E_{v,\mu-2}(-c^v t^v) \\
&\quad + \{2(v^2+\mu^2-2\mu+1)+6v-2\mu-3v\mu+3\} E_{v,\mu-1}(-c^v t^v)].
\end{aligned}$$

This completes the proof of (36).

**Theorem 2** If  $c > 0, d > 0, v > 0, \mu > 0, Re(u) > |d|^{v/\alpha}, c \neq d$ , for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{v,\mu}(-d^v t^v) = -c^v {}_0D_t^{-v} N(t), \quad (38)$$

there holds the formula

$$N(t) = \frac{N_0}{c^v - d^v} t^{\mu-v-2} [E_{v,\mu-v-1}(-d^v t^v) - E_{v,\mu-v-1}(-c^v t^v)], \quad (39)$$

where  $E_{v,\mu}(z)$  is the generalized Mittag-Leffler function (also known as Agarwal function [1]).

**Proof** Applying the Sumudu transform to the both sides of (38) and using (28), we get

$$\begin{aligned}
S[N(t)] &= N_0 S[t^{\mu-1} E_{v,\mu}(-d^v t^v)] - c^v S[{}_0D_t^{-v} N(t)], \\
N^*(u) &= N_0 \left[ \frac{u^{\mu-1}}{[1+(du)^v][1+(cu)^v]} \right] \\
&= \frac{N_0}{c^v - d^v} u^{\mu-v-1} \left[ \sum_{n=0}^{\infty} (-1)^n (du)^{vn} - \sum_{n=0}^{\infty} (-1)^n (cu)^{vn} \right]. \quad (40)
\end{aligned}$$

Using the relation  $S^{-1}\{u^v\} = \frac{t^{v-1}}{\Gamma(v)}$ ,  $Re(v) > 0, Re(u) > 0$ , and taking the Sumudu inverse of (40), we have,

$$\begin{aligned}
S^{-1}\{N^*(u)\} &= \frac{N_0}{c^v - d^v} \left[ \sum_{n=0}^{\infty} (-1)^n (d)^{vn} S^{-1}\{u^{\mu-v+vn-1}\} - \sum_{n=0}^{\infty} (-1)^n (c)^{vn} S^{-1}\{u^{\mu-v+vn-1}\} \right] \\
&= \frac{N_0}{c^v - d^v} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n d^{vn} t^{\mu-v+vn-2}}{\Gamma(\mu+vn-v-1)} - \sum_{n=0}^{\infty} \frac{(-1)^n c^{vn} t^{\mu-v+vn-2}}{\Gamma(\mu+vn-v-1)} \right] \\
N(t) &= \frac{N_0}{c^v - d^v} t^{\mu-v-2} [E_{v,\mu-v-1}(-d^v t^v) - E_{v,\mu-v-1}(-c^v t^v)].
\end{aligned}$$

This completes the proof of Theorem 2.

When  $\mu = v + 2$ , Theorem 2 reduces to

**Corollary 2.1** If  $c > 0, d > 0, v > 0, \mu > 0, Re(u) > |d|^{v/\alpha}, c \neq d$ , for the solution of

$$N(t) - N_0 t^{v+1} E_{v,v+2}(-d^v t^v) = -c^v {}_0D_t^{-v} N(t), \quad (41)$$

the following result holds

$$N(t) = \frac{N_0}{c^v - d^v} [E_v(-d^v t^v) - E_v(-c^v t^v)]. \quad (42)$$

On the other hand if  $d \rightarrow 0$  in Theorem 2, we arrive at the following result:

**Corollary 2.2** If  $c > 0, v > 0, \mu > 0, Re(u) > |d|^{v/\alpha}$ , then for the solution of

$$N(t) - \frac{N_0 t^{\mu-1}}{\Gamma(\mu)} = -c^v {}_0D_t^{-v} N(t), \quad (43)$$

the following result holds

$$N(t) = \frac{N_0 t^{\mu-v-2}}{c^v} \left[ \frac{1}{\Gamma(\mu-v-1)} - E_{v,\mu-v-1}(-c^v t^v) \right]. \quad (44)$$

#### 4. THE $G$ -FUNCTION AND ITS RELATIONSHIP WITH OTHER SPECIAL FUNCTIONS

The generalized function for the fractional calculus  $G_{v,\mu,\delta}(a, c, t)$  was introduced by Lorenzo and Hartley [16], defined as

$$G_{v,\mu,\delta}(a, c, t) = \sum_{n=0}^{\infty} \frac{(\delta)_n a^n (t-c)^{(n+\delta)v-\mu-1}}{n! \Gamma((n+\delta)v-\mu)}, \quad [Re(v\delta-\mu) > 0], \quad Re(s) > 0, \quad \left| \frac{a}{s^v} \right| < 1. \quad (45)$$

Particularly at  $c = 0$ , the above  $G$ -function reduces in to the following useful form:

$$G_{v,\mu,\delta}(a, 0, t) = G_{v,\mu,\delta}(a, t) = \sum_{n=0}^{\infty} \frac{(\delta)_n a^n t^{(n+\delta)v-\mu-1}}{n! \Gamma((n+\delta)v-\mu)}. \quad (46)$$

The  $G$ -function readily yields the following relationships with various special functions.

*Mittag-Leffler function* [19, 20]:

$$G_{v,v-1,1}(-a, t) = E_v(-at^v) = \sum_{n=0}^{\infty} \frac{(-a)^n t^{nv}}{\Gamma(nv+1)}. \quad (47)$$

*Agarwal's function* [1]:

$$G_{v,v-\mu,1}(1, t) = E_{v,\mu}(t^v) = \sum_{n=0}^{\infty} \frac{t^{nv+\mu-1}}{\Gamma(nv+\mu)}. \quad (48)$$

*Erdélyi's function* [26]:

$$G_{v,v-\mu,1}(1, t) = t^{\mu-1} E_{v,\mu}(t^v) = t^{\mu-1} \sum_{n=0}^{\infty} \frac{t^{nv}}{\Gamma(nv+\mu)}, \quad v > 0, \quad \mu > 0. \quad (49)$$

*Robotnov and Hartley function* [16]:

$$G_{v,0,1}(-a, t) = F_v(-a, t) = \sum_{n=0}^{\infty} \frac{(-a)^n t^{(n+1)v-1}}{\Gamma((n+1)v)}. \quad (50)$$

*Miller and Ross's function*:

$$G_{1,-\mu,1}(a, t) = E_t(\mu, a) = \sum_{n=0}^{\infty} \frac{a^n t^{n+\mu}}{\Gamma(n+\mu+1)}. \quad (51)$$

*Generalized Mittag-Leffler function* [12]:

$$G_{v,\mu,1}(a, t) = t^{v-\mu-1} E_{v,v-\mu}(at^v) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)v-\mu-1}}{\Gamma((n+1)v-\mu)}. \quad (52)$$

*New generalized Mittag-Leffler function* [21]:

$$G_{v,\mu,\delta}(a, t) = t^{v\delta-\mu-1} E_{v,v\delta-\mu}^{\delta}(at^v) = t^{v\delta-\mu-1} \sum_{n=0}^{\infty} \frac{(\delta)_n a^n t^{vn}}{n! \Gamma((n+\delta)v-\mu)}. \quad (53)$$



*R-function* [16]:

$$G_{v,\mu,1}(a, t) = R_{v,\mu}(a, t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)v-\mu-1}}{\Gamma((n+1)v-\mu)}, \quad v > 0, \mu > 0, (v-\mu) > 0. \quad (54)$$

*Wright function* [17]:

$$G_{v,\mu,\delta}(a, t) = \frac{t^{v\delta-\mu-1}}{\Gamma(\delta)} {}_1\Psi_1 \left[ \begin{matrix} (\delta, 1) \\ (v\delta - \mu, v) \end{matrix}; at^v \right], \quad (55)$$

where  ${}_1\Psi_1(t)$  is special case of the Wright's generalized hypergeometric function  ${}_p\Psi_q(t)$ .

*H function* [17]:

$$G_{v,\mu,\delta}(a, t) = \frac{t^{v\delta-\mu-1}}{\Gamma(\delta)} H_{1,2}^{1,1} \left[ -at^v \left| \begin{matrix} (1-\delta, 1) \\ (0, 1), (1-v\delta + \mu, v) \end{matrix} \right. \right]. \quad (56)$$

*H̄ function* [17]:

$$G_{v,\mu,\delta}(a, t) = \frac{t^{v\delta-\mu-1}}{\Gamma(\delta)} \bar{H}_{1,2}^{1,1} \left[ -at^v \left| \begin{matrix} (1-\delta, 1; 1) \\ (0, 1), (1-v\delta + \mu, v; 1) \end{matrix} \right. \right], \quad (57)$$

Next, we obtain the solution of generalized fractional kinetic equations in terms of  $G$ -functions in a compact form and can be worked out easily.

## 5. SOLUTION OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

First of all we give the Sumudu transform of  $G$ -function.

$$\begin{aligned} S[G_{v,\mu,\delta}(a, c, t)] &= \sum_{n=0}^{\infty} \frac{(\delta)_n}{n! \Gamma((n+\delta)v-\mu)} \int_0^{\infty} e^{-t}(ut-c)^{(n+\delta)v-\mu-1} \\ &= \sum_{n=0}^{\infty} \frac{(\delta)_n a^n}{n! \Gamma((n+\delta)v-\mu)} \int_0^{\infty} e^{-tu} u^{(n+\delta)v-\mu-1} \left(t - \frac{c}{u}\right)^{(n+\delta)v-\mu-1} dt \\ &= \sum_{n=0}^{\infty} \frac{(\delta)_n a^n e^{-\frac{c}{u}} u^{(n+\delta)v-\mu-1}}{n! \Gamma((n+\delta)v-\mu)} \int_0^{\infty} e^{-(t-\frac{c}{u})} \left(t - \frac{c}{u}\right)^{(n+\delta)v-\mu-1} dt. \end{aligned}$$

Now, by using Gamma function formula  $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$ ,

$$S[G_{v,\mu,\delta}(a, c, t)] = \frac{e^{-\frac{c}{u}} u^{v\delta-\mu-1}}{(1-au^v)^\delta}. \quad (58)$$

Taking  $c = 0$  in (58), then we have

$$S[G_{v,\mu,\delta}(a, 0, t)] = S[G_{v,\mu,\delta}(a, t)] = \frac{u^{v\delta-\mu-1}}{(1-au^v)^\delta}. \quad (59)$$

**Theorem 3** If  $c > 0, v > 0, b \geq 0, \mu > 0, \delta > 0, (\delta v - \mu) > 0$  then for the solution of the equation

$$N(t) - N_0 G_{v,\mu,\delta}(-c^v, b, t) = -c^v {}_0D_t^{-v} N(t), \quad (60)$$

where  ${}_0D_t^{-v}$  is well known standard Riemann-Liouville integral operator, there holds the formula

$$N(t) = N_0 G_{v,\mu+v+1,\delta+1}(-c^v, b, t). \tag{61}$$

**Proof** Taking Sumudu transform to the both sides of (60) and using (28), we get

$$S[N(t)] = N_0 S[G_{v,\mu,\delta}(-c^v, b, t)] - c^v S[{}_0D_t^{-v}N(t)], \tag{62}$$

by the application of the convolution theorem of the Sumudu transform in (62), we have

$$\begin{aligned} N^*(u) &= N_0 \frac{e^{-\frac{b}{u}} u^{v\delta-\mu-1}}{(1+c^v u^v)^\delta} - c^v u^v N^*(u) \\ &= N_0 \frac{e^{-\frac{b}{u}} u^{v\delta-\mu-1}}{(1+c^v u^v)^{\delta+1}}. \end{aligned} \tag{63}$$

Now taking inverse Sumudu transform both sides of (63), we obtain

$$\begin{aligned} N(t) &= S^{-1}[N^*(u)] = S^{-1}\left[N_0 \frac{e^{-\frac{b}{u}} u^{v\delta-\mu-1}}{(1+c^v u^v)^{\delta+1}}\right], \\ N(t) &= N_0 S^{-1}\left[\frac{e^{-\frac{b}{u}} u^{v\delta-\mu-1} (-c^v u^v)^n (\delta+1)_n}{n!}\right] \\ &= N_0 \sum_{n=0}^{\infty} \frac{(-c^v)^n (\delta+1)_n}{n!} S^{-1}\left[e^{-\frac{b}{u}} u^{v\delta+vn-\mu-1}\right]. \\ N(t) &= N_0 \sum_{n=0}^{\infty} \frac{(-c^v)^n (\delta+1)_n}{n!} \frac{(t-b)^{v\delta+vn-\mu-2}}{\Gamma(v\delta+vn-\mu-1)} \\ &= N_0 G_{v,\mu+v+1,\delta+1}(-c^v, b, t). \end{aligned}$$

This completes the proof of Theorem 3.

### 6. SPECIAL CASES OF THEOREM 3

**Corollary 3.1** If  $c > 0, v > 0, b = 0, \mu > 0, \delta > 0, (\delta v - \mu) > 0$  and using (59), then we have the solution of the equation

$$N(t) - N_0 G_{v,\mu,\delta}(-c^v, t) = -c^v {}_0D_t^{-v}N(t), \tag{64}$$

there holds the formula

$$N(t) = N_0 G_{v,\mu+v+1,\delta+1}(-c^v, t), \tag{65}$$

provided that each side of (65) exists.

**Remark 2** By using the relations (55)-(57), we can obtain the solution of (64) in terms of Wright function,  $H$ -function, and  $\overline{H}$ -function.

**Corollary 3.2** If  $\delta = 1, b \geq 0, c > 0, v > 0, \mu > 0$ , and  $(\delta v - \mu) > 0$ , then we have the solution of the equation

$$N(t) - N_0 G_{v,\mu,1}(-c^v, b, t) = -c^v {}_0D_t^{-v}N(t), \tag{66}$$

there holds the formula

$$N(t) = N_0 G_{v,\mu+v+1,2}(-c^v, b, t), \tag{67}$$

provided that each side of (67) exists.

If we set  $b = 0$  in Theorem 3 and use the relation (53), then we arrive at the following:

**Corollary 3.3** If  $v > 0, c > 0, \delta > 0, \mu > 0$ , and  $(\delta v - \mu) > 0$ , then for the solution of the equation

$$N(t) - N_0 t^{v\delta - \mu - 1} E_{v, v\delta - \mu}^{\delta}(-c^v t^v) = -c^v {}_0D_t^{-v} N(t), \quad (68)$$

where  $E_{v, v\delta - \mu}^{\delta}(at^v)$  is well known new generalized Mittag-Leffler function [21], there holds the formula

$$N(t) = N_0 t^{v\delta - \mu - 2} E_{v, v\delta - \mu - 1}^{\delta + 1}(-c^v t^v). \quad (69)$$

**Remark 3** If we set  $\mu = v\delta - \mu$  and  $b = 0$  in Theorem 3, then we obtain Theorem 1.

**Corollary 3.4** If  $c > 0, v > 0, b \geq 0, \mu > 0, v > \mu + 1$ , then for the solution of the equation

$$N(t) - N_0 R_{v, \mu}(-c^v, b, t) = -c^v {}_0D_t^{-v} N(t), \quad (70)$$

where  $R_{v, \mu}(-c^v, b, t)$  is a generalization of the  $F$ -function is presented by Lorenzo and Hartley [16, 24], there holds the formula

$$N(t) = \frac{N_0(t-b)^{v-\mu-2}}{v} [E_{v, v-\mu-2}[-c^v(t-b)^v] + (\mu+2)E_{v, v-\mu-1}[-c^v(t-b)^v]]. \quad (71)$$

For  $b = 0$ , it gives the following:

**Corollary 3.5** If  $c > 0, v > 0, b = 0, \mu > 0$ , and  $v > \mu + 1$ , then for the solution of the equation

$$N(t) - N_0 R_{v, \mu}(-c^v, t) = -c^v {}_0D_t^{-v} N(t), \quad (72)$$

there holds the formula

$$N(t) = \frac{N_0 t^{v-\mu-2}}{v} [E_{v, v-\mu-2}(-c^v t^v) + (\mu+2)E_{v, v-\mu-1}(-c^v t^v)]. \quad (73)$$

## 7. CONCLUSION

In this paper we have studied a new fractional generalization of the standard kinetic equation and derived solutions for the same. It is not difficult to obtain several further analogues fractional kinetic equations and their solutions as those exhibited here by Theorem 3 and its Corollaries, involving the generalized function for the fractional calculus  $G_{v, \mu, \delta}(-c^v, b, t)$ , by specializing the parameters  $v, \mu, \delta$  and  $b$ . Moreover, by the use of close relationships of the  $G$ -function with the  $R$ -function, the generalized Mittag-Leffler functions, the Robotnov and Hartley function etc., we can easily construct various known and new fractional kinetic equations.

## REFERENCES

- [1] R.P. Agarwal, A propos d'une note de M. Pierre Humbert, C.R. Acad. Sci. Paris Vol. 236, 2031-2032, 1953.
- [2] B.N. Al-Saqabi and V.K. Tuan, Solution of a fractional differential equation, Integral Transform. Spec. Funct. Vol. 4, 321-326, 1996.
- [3] M.A. Asiru, Sumudu transform and the solution of integral equations of convolution type, International Journal of Mathematical Education in Science and Technology, Vol. 32(6), 906-910, 2001.

- [4] F.B.M. Belgacem and A.A. Karaballi, Sumudu transform fundamental properties investigations and applications, *International J. Appl. Math. Stoch. Anal.*, 1-23, 2005.
- [5] F.B.M. Belgacem, A.A. Karaballi and S.L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations *Mathematical Problems in Engineering*, Vol. 3, 103-118, 2003.
- [6] V.B.L. Chaurasia and S.C. Pandey, On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions, *Astrophys. Space Sci.* Vol. 317, 213-219, 2008.
- [7] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms*, McGraw-Hill, New York, Vol. 1, 1954.
- [8] C. Fox, Solving integral equations by  $L$  and  $L^{-1}$  operators, *American Mathematical Society*, Vol. 29(2), 1971.
- [9] R. Gorenflo, A.A. Kilbas, and S. Rogozin, On the generalized Mittag-Leffler type functions, *Integral Transforms and Special Functions*, Vol. 7, No. 3-4, 215-224, 1998.
- [10] V.G. Gupta and B. Sharma, On the solutions of generalized fractional kinetic equations, *Applied Mathematical Sciences*, Vol. 5(19), 899-910, 2011.
- [11] H.J. Haubold and A.M. Mathai, The fractional kinetic equation and thermonuclear functions, *Astrophys. Space Sci.*, Vol. 327, 53-63, 2000.
- [12] H.J. Haubold, A.M. Mathai and R.K. Saxena, Mittag-Leffler functions and their applications, *J. Appl. Math.* (Article ID 298628), 1-51, 2011.
- [13] R. Hilfer, in: R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
- [14] R. Hilfer, Fractional time evolution, in *Applications of Fractional Calculus in Physics*; R. Hilfer, ed., World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 87-130, 2000.
- [15] R. Hilfer and L. Anton, Fractional master equations and fractal time random walks, *Phys. Rev.*, E 51, R848-R851, 1995.
- [16] C.F. Lorenzo and T.T. Hartley, Generalized function for the fractional calculus, *NASA/TP-1999-209424*, 1999.
- [17] A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-function: Theory and Applications*, Springer, New York, 2010.
- [18] N.W. McLachlan, *Modern Operational Calculus with Applications in technical Mathematics*, Macmillan, London, 1948.
- [19] G.M. Mittag-Leffler, Sur la représentation analytique d'une branche uniforme d'une fonction monogène, *Acta Math.*, Vol. 29, 101-181, 1905.
- [20] G.M. Mittag-Leffler, Sur la nouvelle fonction  $E_\alpha(x)$ , *C.R. Acad. Sci. Paris*, Vol. 137, 554-558, 1903.
- [21] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the Kernel, *Yokohama Math. J.*, Vol. 19, 7-15, 1971.
- [22] R.K. Saxena and S.L. Kalla, On the solutions of certain fractional kinetic equations, *Appl. Math. Comput.*, Vol. 199, 504-511, 2008.
- [23] R.K. Saxena, A.M. Mathai and H.J. Haubold, On Fractional kinetic equations, *Astrophys. Space Sci.*, Vol. 282, 281-287, 2002.
- [24] R.K. Saxena, A.M. Mathai and H.J. Haubold, On generalized fractional kinetic equations, *Physica A*, Vol. 344, 657-664, 2004.
- [25] M.R. Spiegel, *Theory and Problems of Laplace Transforms*, Schaums Outline Series, McGraw-Hill, New York, 1965.
- [26] H.M. Srivastava and R.K. Saxena, Operators of fractional integration and their applications, *Appl. Math. Comput.*, Vol. 118, 1-52, 2001.
- [27] Z. Tomovski, R. Hilfer and H.M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Integral Transforms and Special Functions*, Vol. 21(11), 797-814, 2010.
- [28] G.K. Watugala, Sumudu transform- a new integral transform to solve differential equations and control engineering problems, *Math. Engrg. Indust.*, Vol. 4(6), 319-329, 1998.
- [29] A. Wiman, Über de fundamental satz in der theorie der funktionen  $E_\alpha(x)$ , *Acta Math.*, Vol. 29, 191-201, 1905.

R.K. SAXENA, J. RAM AND D. KUMAR  
DEPARTMENT OF MATHEMATICS AND STATISTICS,, JAI NARAIN VYAS UNIVERSITY, JODHPUR-342005,  
INDIA

*E-mail address:* ram.saxena@yahoo.com, bishnoi\_jr@yahoo.com., dinesh.dino03@yahoo.com