NUMERICAL APPROXIMATION FOR SPACE FRACTIONAL DIFFUSION EQUATIONS VIA CHEBYSHEV FINITE DIFFERENCE METHOD

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ABSTRACT. In this paper, we discuss the numerical solution of space fractional diffusion equations. The method of solution is based on using Chebyshev polynomials and finite difference with Gauss-Lobatto points. The validity and reliability of this scheme is tested by its application in various space fractional diffusion equations. The obtained results reveal that the proposed method is more accurate and efficient.

1. INTRODUCTION

In recent years, there has been a growing interest in the field of fractional calculus. Oldham and Spanier [9], Miller and Ross [8], Samko et al. [13] and Podlubny [10] provided the history and an extensive treatment of this subject. Many phenomena in physics, chemistry, engineering and other sciences can be explained very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integral of fractional order.

Also, the use of fractional partial differential equations in mathematical models has become increasingly popular in recent years. Different models using fractional partial differential equation have been suggested and there has been important interest in developing numerical methods for their solution.

Roughly speaking, fractional partial differential equations can be arranged into two important types: space-fractional partial differential equations (SFPDEs) and time-fractional partial differential equations (TFPDEs). One of the simplest examples of the former is fractional order diffusion equation, which is the generalization of classical diffusion equations, treating super diffusive flow processes. Much of the work published to date has been concerned with this kind of fractional partial differential equations, for example see [5], [7] and [16].

We describe some necessary definition and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1 The fractional derivative of $f(x)$ in the Caputo sense is defined as [10]

$$D_*^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) ds,$$

for $m-1 < \alpha \leq m, m \in N, x > 0$.

We have the following properties when $m-1 < \alpha \leq m, x > 0$:

$D_*^\alpha k = 0$, (k is a constant),

$$D_*^\alpha x^n = \begin{cases} 0 & for\, n \in \mathbb{W}\ and\ n < [\alpha], \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} & for\, n \in \mathbb{W}, and\, n \geq [\alpha], \end{cases}$$

where function $[\alpha]$ to denote the smallest integer greater than or equal to α and $\mathbb{W} = \{0, 1, 2, ...\}$. Note that for $\alpha \in \mathbb{W}$, the Caputo differential operator agrees with the usual differential operator of integer order.

In this paper, we consider the one-dimensional space fractional diffusion equation of the form

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + p(x,t),\ 0 < x < 1,\ 0 \leq t \leq T,\ 1 < \alpha \leq 2, \qquad (1)$$

with initial condition

$$u(x,0) = f(x),\quad 0 < x < 1, \qquad (2)$$

and boundary conditions

$$u(0,t) = g_0(t),\ 0 < t \leq T, \qquad (3)$$

$$u(1,t) = g_1(t),\ 0 < t \leq T. \qquad (4)$$

The function $p(x,t)$ is a source term and note that for $\alpha = 2$, Eq.(1) is the classical diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t).$$

In [4] the author used Chebyshev collocation method to discretize Eq.(1) to obtain a linear system of ordinary differential equations and used the finite difference method to solve the resulting system. Saadatmandi and Dehghan used tau approach to solve Eq.(1) [11]. Also in [14] Sousa applied splines and finite difference to solve space fractional diffusion equation.

The main idea of the current work is to apply non-uniform finite difference method with Chebyshev polynomials and Gauss-Lobato points. Application of this method for Eq.(1) leads to solve an algebraic system.

Cebyshev finite difference methd (ChFDM) has been used in the numerical solution of Fredholm integro-differential equations, boundary value problems, boundary layer equations and nonlinear system of second-order boundary value problems [1, 2, 3, 12].

2. Chebyshev series expansion

Definition 2 [6] The well-known Chebyshev polynomials of the first kind of degree n are defined on the interval $[-1, 1]$ as

$$T_n(x) = \cos(n \arccos(x)),$$

obviously $T_0(x) = 1$, $T_1(x) = x$ and they satisfy the recurrence relations:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\quad n = 1, 2, \cdots.$$

In order to use these polynomials on the interval $x \in [0,1]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2x - 1$. The shifted Chebyshev polynomials are defined as: $T_n^*(x) = T_n(2x - 1)$.

The function $u(x,t)$ may be expressed in terms of shifted Chebyshev polynomials as:

$$u(x,t) = \sum_{n=0}^{N} \sum_{m=0}^{N} r_{nm} T_n^*(x) T_m^*(t), \tag{5}$$

where the coefficients r_{nm} are given by

$$r_{nm} = \frac{4}{N^2 c_n c_m} \sum_{k=0}^{N}{}'' \sum_{l=0}^{N}{}'' u(x_k, t_l) T_n^*(x_k) T_m^*(t_l), \ \ n, m = 0, 1, ..., N, \tag{6}$$

$x_k = \frac{\cos(\frac{k\pi}{N}) + 1}{2}, k = 0, 1, ..., N$, $t_l = \frac{T(\cos(\frac{l\pi}{N}) + 1)}{2}, l = 0, 1, ..., N$ are Gauss-Lobatto points where shifted to interval $[0, 1]$, $[0, T]$ and $c_0 = 2$, $c_i = 1$ for $i = 1, 2, ..., N$. The summation symbol with double primes denotes a sum with both the first and last terms halved.

The derivatives of the Chebyshev function are formed as the following[6]:

$$T_n'(t) = \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{2n}{c_k} T_k(t), \tag{7}$$

$$T_n''(t) = \sum_{\substack{k=0 \\ (n+k)even}}^{n-2} \frac{n}{c_k}(n^2 - k^2) T_k(t). \tag{8}$$

From equations (7) and (8) we get

$$\frac{\partial u(x,t)}{\partial t} = \frac{4}{N^2} \sum_{n=0}^{N} \sum_{m=0}^{N} \sum_{k=0}^{N}{}'' \sum_{l=0}^{N}{}'' \sum_{\substack{i=0 \\ (m+i)odd}}^{m-1} \frac{2m}{c_i c_n c_m} u(x_k, t_l) T_n^*(x_k) T_m^*(t_l) T_n^*(x) T_i^*(t), \tag{9}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{4}{N^2} \sum_{n=0}^{N} \sum_{m=0}^{N} \sum_{k=0}^{N}{}'' \sum_{l=0}^{N}{}'' \sum_{\substack{i=0 \\ (n+i)even}}^{n-2} \frac{n(n^2 - i^2)}{c_i c_n c_m} u(x_k, t_l) T_n^*(x_k) T_m^*(t_l) T_n^*(x) T_i^*(t). \tag{10}$$

We see from equations (9) and (10) the derivatives of the function $u(x,t)$ at any point from shifted Guass-Lobatto nodes are expanded as linear combination of the values of the function at these points.

3. NUMERICAL SCHEME

In this section, the space fractional diffusion equation (1) is solved. In order to find the solution $u(x,t)$ in equation (1), we first calculated Eq.(1) in shifted Gauss-Lobatto nods (x_j, t_h) for $j = 1, 2, ..., N - 1$ and $h = 1, 2, ..., N$ and using equations (9) and (10) and definition 1 we obtain

$$\frac{4}{N^2} \sum_{n=0}^{N} \sum_{m=0}^{N} \sum_{k=0}^{N}{}'' \sum_{l=0}^{N}{}'' \sum_{\substack{i=0 \\ (m+i)odd}}^{m-1} \frac{2m}{c_i c_n c_m} u(x_k, t_l) T_n^*(x_k) T_m^*(t_l) T_n^*(x_j) T_i^*(t_h)$$

$$= d(x_j)\frac{1}{\Gamma(2-\alpha)}\frac{4}{N^2}\sum_{n=0}^{N}\sum_{m=0}^{N}\sideset{}{''}\sum_{k=0}^{N}\sideset{}{''}\sum_{l=0}^{N}\sum_{\substack{i=0\\(n+i)even}}^{n-2}\frac{n(n^2-i^2)}{c_ic_nc_m}u(x_k,t_l)T_n^*(x_k)$$

$$T_m^*(t_l)T_i^*(t_h)\int_0^{x_j}(x_j-s)^{1-\alpha}\,T_n^*(s)\,ds+p(x_j,t_h). \tag{11}$$

Initial condition (2) for $i = 1, 2, ..., N-1$ and from boundary conditions (3) and (4) for $j = 0, 1, ..., N$ is used to obtain

$$u(x_i,0) = \sum_{n=0}^{N}\sum_{m=0}^{N}\sideset{}{''}\sum_{k=0}^{N}\sideset{}{''}\sum_{l=0}^{N}\frac{4}{N^2c_nc_m}u(x_k,t_l)T_n^*(x_k)T_m^*(t_l)T_n^*(x_i)T_m^*(0)$$

$$= f(x_i), \tag{12}$$

$$u(0,t_j) = \sum_{n=0}^{N}\sum_{m=0}^{N}\sideset{}{''}\sum_{k=0}^{N}\sideset{}{''}\sum_{l=0}^{N}\frac{4}{N^2c_nc_m}u(x_k,t_l)T_n^*(x_k)T_m^*(t_l)T_n^*(0)T_m^*(t_j)$$

$$= g_0(t_j), \tag{13}$$

$$u(1,t_j) = \sum_{n=0}^{N}\sum_{m=0}^{N}\sideset{}{''}\sum_{k=0}^{N}\sideset{}{''}\sum_{l=0}^{N}\frac{4}{N^2c_nc_m}u(x_k,t_l)T_n^*(x_k)T_m^*(t_l)T_n^*(1)T_m^*(t_j)$$

$$= g_1(t_j). \tag{14}$$

Thus equations (11)-(14) create a set of $(N + 1)^2$ algebraic equations, which the unknowns $u(x_k, t_l)$ for $k = 0, 1, ..., N$ and $l = 0, 1, ..., N$ obtain by solving it. Therefore $u(x, t)$ in equation (5) can be calculated.

4. Stability and convergent

The system that created in previous section can be written as the following matrix form:
$$A[u] = [b].$$
The square matrix A is usually dense matrix but for solve this system we used maple 13 and in test problems that use this paper, A is non-singular matrix and the spectral radius of matrix A^{-1} is the less than one therefore the proposed method has unique solution and is unconditionally stable [15]. Unconditionally stable for the general case is open problem.

5. Numerical examples

In this section, for the sake of comparison, we have selected some examples where the exact solutions already exist, which will ultimately show the simplicity, effectiveness and exactness of the proposed method.

Example 1 Consider the following space fractional differential equation
$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} + p(x,t),$$
on a finite domain $0 < x < 1, 0 \leq t \leq 1$, with the diffusion coefficient
$d(x) = \frac{\Gamma(0.5)}{4}x^{0.5}$,

the source function
$p(x,t) = 2t + x$,
the initial condition
$u(x,0) = x^2$, $0 < x < 1$,
and the boundary conditions
$u(0,t) = t^2, u(1,t) = 1 + t^2$.
The exact solution of this problem is
$u(x,t) = x^2 + t^2$.
We applied present method with $N = 2$ for this problem and we obtained the exact solution.

Example 2 [11] In this example, we consider (1) with $\alpha = 1.8$, of the form:

$$\frac{\partial u(x,t)}{\partial t} = \Gamma(1.2)x^{1.8}\,\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + 3x^2(2x - 1)e^{-t},$$

with the initial condition
$u(x,0) = x^2 - x^3$,
and zero Dirichlet conditions.
The exact solution of this problem is $u(x,t) = x^2(1 - x)e^{-t}$.
We solved this problem by applying the present method. In Table 1 the absolute errors between the exact solution and the approximate solution of the new method with the Chebyshev collocation method given in [4] and the Tau method by Legendre polynomials given in [11] are compared.
From Table 1, can be seen our results are in good agreement with the methods introduced in [4] and [11].

Table 1: Comparison of absolute errors for $u(x, 2)$ from example 1.

x	Method [4] with m=5	Method [11] with m=5	present method(N=5)
0.0	2.74×10^{-5}	0.0	0.0
0.1	4.20×10^{-5}	4.47×10^{-6}	1.40×10^{-7}
0.2	3.76×10^{-5}	2.78×10^{-7}	9.06×10^{-7}
0.3	8.44×10^{-5}	5.81×10^{-6}	3.25×10^{-8}
0.4	3.27×10^{-5}	1.02×10^{-5}	6.55×10^{-8}
0.5	3.61×10^{-5}	1.17×10^{-5}	1.02×10^{-8}
0.6	1.94×10^{-5}	1.08×10^{-5}	7.38×10^{-9}
0.7	2.95×10^{-5}	8.54×10^{-6}	1.64×10^{-7}
0.8	4.92×10^{-5}	6.06×10^{-6}	2.75×10^{-8}
0.9	2.83×10^{-5}	3.67×10^{-6}	1.32×10^{-7}
1.0	7.73×10^{-5}	0.0	0.0

Example 3 [11] Consider the following space fractional differential equation

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} + p(x,t),$$

on a finite domain $0 < x < 1$, with the diffusion coefficient
$d(x) = \Gamma(1.5)x^{0.5}$,
the source function
$p(x,t) = (x^2 + 1)\cos(t + 1) - 2x\sin(t + 1)$,
with the initial condition

$u(x,0) = (x^2 + 1)\sin(1),$
and the boundary conditions
$u(0,t) = \sin(t+1),\ u(1,t) = 2\sin(t+1),$ for
$\quad t > 0.$
The exact solution of this problem is $u(x,t) = (x^2+1)\sin(t+1).$

We applied the proposed method and the comparison of our method with the method in [11] which are shown in Table 2. Also, figure 1 shows the absolute error function $|u(x,t) - u_{approx}(x,t)|$ obtained by the present method with $T = 1$ and $N = 7$. Note that from Table 2 and figure 1 can be seen our method achieve a good approximation for the above equation.

Table 2: Comparison of present method for $u(x,1)$ with the tau method [11] for exa. 3.

x	Method [11] with m=7	present method with N=7
0.1	4.66×10^{-5}	1.86×10^{-8}
0.2	7.74×10^{-5}	1.23×10^{-8}
0.3	5.00×10^{-5}	6.94×10^{-9}
0.4	2.30×10^{-5}	1.26×10^{-8}
0.5	2.74×10^{-5}	1.86×10^{-8}
0.6	4.38×10^{-5}	1.24×10^{-8}
0.7	3.87×10^{-5}	6.29×10^{-10}
0.8	1.01×10^{-5}	1.01×10^{-8}
0.9	3.35×10^{-6}	4.82×10^{-8}

Example 4 [14] Consider the problem (1) with initial condition
$u(x,0) = x^4,\ 0 < x < 1$
and boundary conditions
$u(0,t) = 0,\ u(1,t) = e^{-t}.$

Let $d(x) = \frac{1}{24}\Gamma(5-\alpha)x^\alpha$ and $p(x,t) = -2e^{-t}x^4$. The exact solution for this problem is $u(x,t) = e^{-t}x^4.$
We solved the problem by applying the technique described in section 3. In Table 3, the maximum errors between the exact solution and the approximate solution for different values of N and α in finite domain $0 \le x,t \le 1$ is obtained.
Figure 2 shows the absolute error function $|u(x,t) - u_{approx}(x,t)|$ obtained by the present method with $T = 1$ and $N = 6$ for $\alpha = 1.2$.

Table 3: The absolute errors for different values of N and α for example 4.

α	N=2	N=3	N=4	N=5	N=6
1.2	0.1	1.5×10^{-2}	3×10^{-5}	1.6×10^{-6}	7×10^{-8}
1.4	0.1	1.4×10^{-2}	3.2×10^{-5}	1.8×10^{-6}	8×10^{-8}
1.5	0.1	1.5×10^{-2}	3×10^{-5}	2×10^{-6}	9×10^{-8}

Note that in [14] this problem has been solved by finite difference method and splines. Maximum errors for $\alpha = 1.2$, $\alpha = 1.4$ and $\alpha = 1.5$ with $\Delta x = \frac{1}{30}$ are 0.3566×10^{-3}, 0.24616×10^{-3} and 0.2067×10^{-3}, respectively. Therefore from our method we obtain a good approximation for this problem.
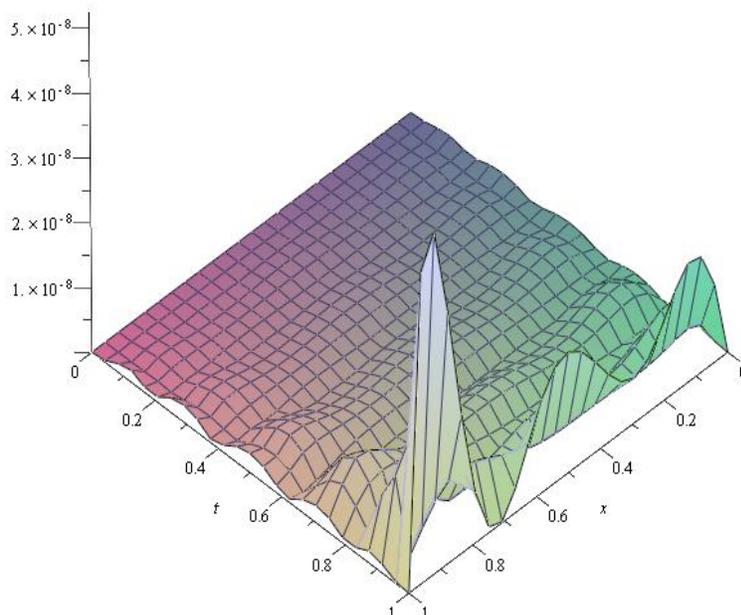
FIGURE 1. plot of error function $|u(x,t) - u_{approx}(x,t)|$ with $N = 7$ from example 3.

6. CONCLUSION

In this paper, we proposed an effective and convenient method to solve space fractional diffusion equations. The various examples were presented to numerically determine whether the new method leads to higher accuracy and simplicity, which in all cases was in an excellent performance. From the comparison with other methods we found that our method achieved a good approximation for space fractional diffusion equations.

For example in [4] fractional diffusion equation reduce to a system of ordinary differential equations, which solved by the finite difference method and if compared with proposed method, ChFD method is more accurate because the approximation of the derivatives is defined over the whole domain while the finite difference method produce a second-order accurate derivative with the error decreasing as $\dfrac{1}{m^2}$ (m being the number of grid points).

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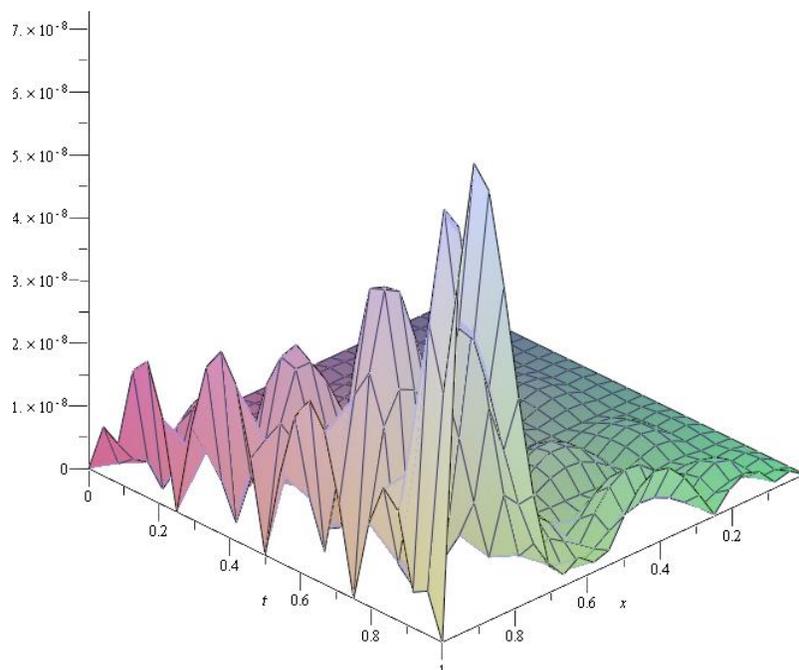FIGURE 2. plot of error function $|u(x,t) - u_{approx}(x,t)|$ with $N = 6$
for $\alpha = 1.2$ from example 4.

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