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AN APPLICATION OF HOMOTOPY PERTURBATION TRANSFORM METHOD TO FRACTIONAL HEAT AND WAVE-LIKE EQUATIONS

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Abstract. In this paper, we present an algorithm of the homotopy perturbation transform method (HPTM) to solve fractional heat and wave-like equations. The fractional derivatives are described by Caputo sense. The HPTM is combined form of the Laplace transform and homotopy perturbation method. The proposed method finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

1. Introduction

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. The fractional differential equations are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering [4,5,8,9,19,21,22,23,24]. The importance of obtaining the exact and approximate solutions of fractional differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. In recent years, many research workers have paid attention to study the solutions of fractional differential equations by using various methods. Among these are Adomian decomposition method (ADM) [2,5], homotopy analysis method (HAM) [17,18], variational iteration method (VIM) [11,12], Laplace decomposition method (LDM) [14] and homotopy perturbation method (HPM) [10,24]. Most of these methods have their inbuilt

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deficiencies like the calculation of Adomian's polynomials, the Lagrange multipliers, divergent results, and huge computational work. Very recently, the homotopy perturbation transform method (HPTM) is proposed by Khan and Wu [13] for handling many linear and nonlinear problems. The homotopy perturbation transform method (HPTM) has been successfully applied to solve fractional Black-Scholes European option pricing equation [16].
In this paper, we will consider the fractional heat and wave-like equations of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz},$$

$$0 < x < a,\ 0 < y < b,\ 0 < z < c,\ t > 0, \tag{1}$$

subject to the Neumann boundary conditions:

$$u_x(0,y,z,t) = f_1(y,z,t),\ u_x(a,y,z,t) = f_2(y,z,t),$$

$$u_y(x,0,z,t) = g_1(x,z,t),\ u_y(x,b,z,t) = g_2(x,z,t), \tag{2}$$

$$u_z(x,y,0,t) = h_1(x,y,t),\ u_z(x,y,c,t) = h_2(x,y,t),$$

and initial conditions:

$$u(x,y,z,0) = \psi(x,y,z),\ u_t(x,y,z,0) = \theta(x,y,z), \tag{3}$$

where α is a parameter describing the fractional derivative. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $0 < \alpha \leq 1$, then Eq. (1) reduces to a fractional heat-like equation with variable coefficients, and to a wave-like equation with variable coefficients for $0 < \alpha \leq 2$. In this paper, further we apply the homotopy perturbation transform method (HPTM) to solve fractional heat and wave-like equations. It is worth mentioning that this method is an elegant combination of the Laplace transformation, the homotopy perturbation method and He's polynomials and is mainly due to Ghorbani [6,7]. The HPTM provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear equations. The plan of our paper is as follows: Basic definitions of the fractional calculus and Laplace transform are given in Section 2. The HPTM is presented in Section 3. In Section 4, six numerical examples are solved to illustrate the applicability of the considered method. Conclusions are presented in Section 5.

2. Basic Definitions of fractional calculus and Laplace transform

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper:
Definition 1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu, \mu \geq -1$is defined as [22]:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \tag{4}$$

$$J^0 f(t) = f(t). \tag{5}$$

For the Riemann-Liouville fractional integral we have:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \tag{6}$$

Definition 2 The fractional derivative of $f(t)$ in the Caputo sense is defined as [3]:

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{7}$$

for $n-1 < \alpha \leq n,\ n \in N, \quad x > 0$.

Definition 3 The Laplace transform of a function $f(t), t > 0$ is defined as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt, \tag{8}$$

where $f(t)$ is piecewise continuous and of the expontiantial order (i.e. $|e^{-at} f(t)| < M$) for some constants $a,\ M$and complex parameter s.

Definition 4 The Laplace transform of the Caputo derivative is given by Caputo [3]; see also Kilbas et al. [15] in the form

$$\mathrm{L}\,[\mathrm{D}^\alpha \mathrm{f(t)}] \;=\; \mathrm{s}^\alpha\, L[\mathrm{f(t)}] \;-\; \sum_{\mathrm{r}=0}^{n-1} \mathrm{s}^{\alpha-\mathrm{r}-1}\, \mathrm{f}^{(\mathrm{r})}(0+),\ (\mathrm{n}\; -1\; <\; \alpha\; \leq\; \mathrm{n}). \tag{9}$$

Definition 5 The Mittag-Leffler is defined as [20]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)},\ \ (\alpha \in C,\ Re(\alpha) > 0). \tag{10}$$

3. Homotopy Perturbation Transform Method (HPTM)

To illustrate the basic idea of this method, we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial conditions of the form:

$$D_t^\alpha\, u(x,t) + R\, u(x,t) + N\, u(x,t) = g(x,t), \tag{11}$$

$$u(x,0) = h(x),\ \ u_t(x,0) = f(x), \tag{12}$$

where $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of the function u(x,t), R is the linear differential operator, N represents the general nonlinear differential operator and g(x,t) is the source term. Taking the Laplace transform (denoted in this paper by$L$) on both sides of Eq. (11), we get

$$L\,[D_t^\alpha\, u(x,t)] + L\,[R\, u(x,t)] + L\,[N\, u(x,t)] = L\,[g(x,t)]. \tag{13}$$

Using the property of the Laplace transform, we have

$$L\,[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^\alpha} L\,[g(x,t)] - \frac{1}{s^\alpha} L\,[R\, u(x,t)] - \frac{1}{s^\alpha} L\,[N\, u(x,t)]. \tag{14}$$

Operating with the Laplace inverse on both sides of Eq. (14) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^{\alpha}} L\left[R\,u(x,t) + N\,u(x,t)\right]\right], \tag{15}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n\, u_n(x,t) \tag{16}$$

and the nonlinear term can be decomposed as

$$N\,u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{17}$$

for some He's polynomials $H_n(u)$[7] that are given by

$$H_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0, 1, 2, 3, ... \tag{18}$$

The first few components He's polynomials are given by

$$\begin{gathered} H_0 = N(u_0), \\ H_1 = N^{'}(u_0)u_1, \\ H_2 = N^{'}(u_0)u_2 + N^{''}(u_0)\frac{u_1^2}{2!}, \\ H_3 = N^{'}(u_0)u_3 + N^{''}(u_0)u_1u_2 + N^{(3)}(u_0)\frac{u_1^3}{3!}, \\ \vdots \end{gathered} \tag{19}$$

Substituting Eqs. (16) and (17) in Eq. (15), we get

$$\sum_{n=0}^{\infty} p^n\, u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[R\sum_{n=0}^{\infty} p^n\, u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \tag{20}$$

which is the coupling of the Laplace transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained.

$$\begin{gathered} p^0 :\ u_0(x,t) = G(x,t), \\ p^1 :\ u_1(x,t) = -L^{-1}\left[\frac{1}{s^{\alpha}} L\ \left[R\,u_0(x,t) + H_0(u)\right]\right], \\ p^2 :\ u_2(x,t) = -L^{-1}\left[\frac{1}{s^{\alpha}} L\ \left[R\,u_1(x,t) + H_1(u)\right]\right], \end{gathered} \tag{21}$$

$$p^3:\ u_3(x,t) = -L^{-1}\left[\frac{1}{s^\alpha}L\ [R\,u_2(x,t) + H_2(u)]\right],$$

$$\vdots$$

Proceeding in this same manner, the rest of the components $u_n(x,t)$ can be completely obtained and the series solution is thus entirely determined.
Finally, we approximate the analytical solution $u(x,t)$ by truncated series

$$u(x,t) = \underset{N\to\infty}{Lim}\sum_{n=0}^{N} u_n(x,t). \tag{22}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [1].

4. Examples

In this section, we apply the homotopy perturbation transform method (HPTM) for solving fractional heat and wave-like equations.
Example 1. Consider the following one-dimensional fractional heat-like equation:

$$D_t^\alpha u = \frac{1}{2}x^2 u_{xx}, \quad 0 < x < 1,\ 0 < \alpha \le 1, \quad t > 0, \tag{23}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^t, \tag{24}$$

and the initial condition

$$u(x,\,0) = x^2. \tag{25}$$

Applying the Laplace transform on both sides of Eq. (23) subject to the initial condition, we have

$$L[u(x,t)] = \frac{x^2}{s} + \frac{1}{2s^\alpha}x^2 L[u_{xx}]. \tag{26}$$

The inverse of Laplace transform implies that

$$u(x,t) = x^2 + \frac{1}{2}x^2 L^{-1}\left[\frac{1}{s^\alpha}\,L[u_{xx}]\right]. \tag{27}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n\,u_n(x,t) = x^2 + p\left(\frac{1}{2}x^2 L^{-1}\left[\frac{1}{s^\alpha}\,L\left[\left(\sum_{n=0}^{\infty} p^n\,u_n(x,t)\right)_{xx}\right]\right]\right). \tag{28}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,t) = x^2,$$

$$p^1 : u_1(x,t) = \frac{1}{2}x^2 L^{-1}\left[\frac{1}{s^\alpha}[(u_0)_{xx}]\right] = x^2\,\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,t) = \frac{1}{2}x^2 L^{-1}\left[\frac{1}{s^\alpha}[(u_1)_{xx}]\right] = x^2\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{29}$$

$$\vdots$$

$$p^n : u_n(x,t) = \frac{1}{2}x^2 L^{-1}\left[\frac{1}{s^\alpha}[(u_{n-1})_{xx}]\right] = x^2\,\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$\vdots$$

Using the above iterations, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(x,t)$$

$$= x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) = x^2 E_\alpha[t^\alpha]. \tag{30}$$

If we select $\alpha = 1$, then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,t)$ converges to the exact solution $u(x,t) = x^2 e^t$. It is quite important to notice that higher number of iteration and higher orders of p are needed to gain more accuracy.

Example 2. Consider the following two-dimensional fractional heat-like equation:

$$D_t^\alpha u = u_{xx} + u_{yy}, \qquad 0 < x,\ y < 2\pi,\ 0 < \alpha \le 1, \quad t > 0, \tag{31}$$

subject to the boundary conditions

$$\begin{aligned} u(0,y,t) &= 0, \qquad u(2\pi,y,t) = 0,\\ u(x,0,t) &= 0, \qquad u(x,2\pi,t) = 0, \end{aligned} \tag{32}$$

and the initial condition

$$u(x,y,\,0) = \sin x \sin y. \tag{33}$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,y,t) &= \sin x \sin y + p\left(L^{-1}\left[\frac{1}{s^\alpha}\,L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{xx}\right]\right]\right.\\ &\left. + L^{-1}\left[\frac{1}{s^\alpha}\,L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{yy}\right]\right]\right). \end{aligned} \tag{34}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,y,t) = \sin x \sin y,$$

$$p^1 : u_1(x,y,t) = -2\sin x \sin y \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,y,t) = 4\sin x \sin y \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{35}$$

$$\vdots$$

$$p^n : u_n(x,y,t) = (-2)^n \sin x \sin y \frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$\vdots$$

Using the above iterations, the solution $u(x,y,t)$ is given by

$$\begin{aligned} u(x,y,t) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n(x,y,t) \\ &= \sin x \sin y \left(1 - 2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + (-2)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \\ &= \sin x \sin y E_{\alpha}[-2t^{\alpha}]. \end{aligned} \tag{36}$$

If we select $\alpha = 1$, then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,y,t)$ converges to the exact solution $u(x,y,t) = e^{-2t} \sin x \sin y$. It is quite important to notice that higher number of iteration and higher orders of p are needed to gain more accuracy.

Example 3. Consider the following three-dimensional inhomogeneous fractional heat-like equation:

$$D_t^{\alpha} u = x^4 y^4 z^4 + \frac{1}{36}(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}),$$

$$0 < x,\ y,\ z < 1,\ \ 0 < \alpha \le 1,\ \ t > 0, \tag{37}$$

subject to the boundary conditions

$$\begin{aligned} u(0,y,z,t) &= 0, \quad & u(1,y,z,t) &= y^4 z^4 (e^t - 1), \\ u(x,0,z,t) &= 0, \quad & u(x,1,z,t) &= x^4 z^4 (e^t - 1), \\ u(x,y,0,t) &= 0, \quad & u(x,y,1,t) &= x^4 y^4 (e^t - 1), \end{aligned} \tag{38}$$

and the initial condition

$$u(x,y,\ z,0) = 0. \tag{39}$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,y,z,t) &= x^4 y^4 z^4 \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &+p\left(\frac{1}{36} x^2 L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right)_{xx}\right]\right]\right. \\ &+\frac{1}{36} y^2 L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right)_{yy}\right]\right] \\ &\left.+\frac{1}{36} z^2 L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right)_{zz}\right]\right]\right). \end{aligned} \tag{40}$$

Comparing the coefficients of like powers of p, we have

$$
\begin{aligned}
p^0 &: u_0(x,y,z,t) = x^4y^4z^4\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
p^1 &: u_1(x,y,z,t) = x^4y^4z^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
p^2 &: u_2(x,y,z,t) = x^4y^4z^4\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots\\
p^n &: u_n(x,y,z,t) = x^4y^4z^4\frac{t^{(n+1)\alpha}}{\Gamma\{(n+1)\alpha+1\}},\\
&\vdots
\end{aligned}
\tag{41}
$$

Using the above iterations, the solution $u(x,y,z,t)$ is given by

$$
\begin{aligned}
u(x,y,z,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\\
&= x^4y^4z^4\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{t^{(n+1)\alpha}}{\Gamma\{(n+1)\alpha+1\}}+\cdots\right)\\
&= x^4y^4z^4[E_{\alpha}(t^{\alpha})-1].
\end{aligned}
\tag{42}
$$

If we select $\alpha = 1$, then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,y,z,t)$ converges to the exact solution $u(x,y,z,t) = x^4y^4z^4\,(e^t-1)$. It is quite important to notice that higher number of iteration and higher orders of p are needed to gain more accuracy.

Example 4. Consider the following one-dimensional fractional wave-like equation:

$$
D_t^{\alpha}u = \frac{1}{2}x^2u_{xx}, \quad 0<x<1, \;\; 0<\alpha\le 2, \; t>0, \tag{43}
$$

subject to the boundary conditions

$$
u(0,t)=0, \quad u(1,t)=1+\sinh t, \tag{44}
$$

and the initial conditions

$$
u(x,0)=x, \qquad u_t(x,0)=x^2. \tag{45}
$$

In a similar way as above, we have

$$
\sum_{n=0}^{\infty} p^n\,u_n(x,t) = x+x^2t+p\left(\frac{1}{2}x^2L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(\sum_{n=0}^{\infty}p^n\,u_n(x,t)\right)_{xx}\right]\right]\right). \tag{46}
$$

Comparing the coefficients of like powers of p, we have

$$
p^0 : u_0(x,t) = x+x^2t,
$$

$$
\begin{aligned}
p^1 &: u_1(x,t) = x^2\,\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\\
p^2 &: u_2(x,t) = x^2\,\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},\\
&\vdots\\
p^n &: u_n(x,t) = x^2\,\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)},\\
&\vdots
\end{aligned}
\tag{47}
$$

Using the above iterations, the solution $u(x,t)$ is given by

$$
\begin{aligned}
u(x,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(x,t)\\
&= x + x^2\left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots + \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} + \cdots\right)\\
&= x + x^2 t E_{\alpha,2}[t^\alpha].
\end{aligned}
\tag{48}
$$

If we select $\alpha = 2$, then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,t)$ converges to the exact solution $u(x,t) = x + x^2\sinh t$. It is quite important to notice that higher number of iteration and higher orders of p are needed to gain more accuracy.

Example 5. Consider the following two-dimensional fractional wave-like equation:

$$
D_t^\alpha u = \frac{1}{12}(x^2 u_{xx} + y^2 u_{yy}), \qquad 0 < x,\, y < 1, \quad 0 < \alpha \le 2,\ t > 0, \tag{49}
$$

subject to the boundary conditions

$$
\begin{aligned}
u(0,y,t) &= 0, \qquad u(1,y,t) = 4\cosh t,\\
u(x,0,t) &= 0, \qquad u(x,1,t) = 4\sinh t,
\end{aligned}
\tag{50}
$$

and the initial conditions

$$
u(x,y,\,0) = x^4, \qquad u_t(x,y,\,0) = y^4\,. \tag{51}
$$

In a similar way as above, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n u_n(x,y,t) &= x^4 + y^4 t + p\left(\frac{1}{12}x^2 L^{-1}\left[\frac{1}{s^\alpha} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{xx}\right]\right]\right.\\
&\left. + \frac{1}{12}y^2 L^{-1}\left[\frac{1}{s^\alpha} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{yy}\right]\right]\right).
\end{aligned}
\tag{52}
$$

Comparing the coefficients of like powers of p, we have

$$
p^0 : u_0(x,y,t) = x^4 + y^4 t,
$$

$$
\begin{aligned}
p^1 : u_1(x,y,t) &= x^4 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\\
p^2 : u_2(x,y,t) &= x^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + y^4 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},\\
&\vdots\\
p^n : u_n(x,y,t) &= x^4 \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + y^4 \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)},\\
&\vdots
\end{aligned}
\tag{53}
$$

Using the above iterations, the solution $u(x,y,t)$ is given by

$$
\begin{aligned}
u(x,y,t) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n(x,y,t)\\
&= x^4 \left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right)\\
&\quad + y^4 \left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots + \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} + \cdots\right)\\
&= x^4 E_{\alpha}[t^{\alpha}] + y^4 t E_{\alpha,2}[t^{\alpha}].
\end{aligned}
\tag{54}
$$

If we select $\alpha = 2$, then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,y,t)$ converges to the exact solution $u(x,y,t) = x^4 \cosh t + y^4 \sinh t$. It is quite important to notice that higher number of iteration and higher orders of p are needed to gain more accuracy.

Example 6. Consider the following three-dimensional fractional wave-like equation:

$$
\begin{gathered}
D_t^{\alpha} u = x^2 + y^2 + z^2 + \frac{1}{2}(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}),\\
0 < x,\ y,\ z < 1,\ 0 < \alpha \le 2, \quad t > 0,
\end{gathered}
\tag{55}
$$

subject to the boundary conditions

$$
\begin{array}{ll}
u(0,y,z,t) = y^2(e^t-1) + z^2(e^{-t}-1), & u(1,y,z,t) = (1+y^2)(e^t-1) + z^2(e^{-t}-1),\\
u(x,0,z,t) = x^2(e^t-1) + z^2(e^{-t}-1), & u(x,1,z,t) = (1+x^2)(e^t-1) + z^2(e^{-t}-1),\\
u(x,y,0,t) = (x^2+y^2)(e^t-1), & u(x,y,1,t) = (x^2+y^2)(e^t-1) + (e^{-t}-1),
\end{array}
\tag{56}
$$

and the initial conditions

$$
u(x,y,\ z,0) = 0, \quad u_t(x,y,\ z,0) = x^2 + y^2 - z^2. \tag{57}
$$

In a similar way as above, we have

$$
\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = (x^2+y^2+z^2)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (x^2+y^2-z^2)t
$$

$$+p\left(\frac{1}{2}x^2L^{-1}\left[\frac{1}{s^\alpha}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,y,z,t)\right)_{xx}\right]\right]\right.$$
$$+\frac{1}{2}y^2L^{-1}\left[\frac{1}{s^\alpha}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,y,z,t)\right)_{yy}\right]\right]$$
$$\left.+\frac{1}{2}z^2L^{-1}\left[\frac{1}{s^\alpha}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,y,z,t)\right)_{zz}\right]\right]\right). \tag{58}$$

Comparing the coefficients of like powers of p, we have

$$p^0: u_0(x,\mathrm{y},z,t) = (x^2+y^2+z^2)\frac{t^\alpha}{\Gamma(\alpha+1)} + (x^2+y^2-z^2)t,$$

$$p^1: u_1(x,\mathrm{y},z,t) = (x^2+y^2+z^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (x^2+y^2-z^2)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$p^2: u_2(x,\mathrm{y},z,t) = (x^2+y^2+z^2)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (x^2+y^2-z^2)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}, \tag{59}$$

$$\vdots$$

$$p^n: u_n(x,\mathrm{y},z,t) = (x^2+y^2+z^2)\frac{t^{(n+1)\alpha}}{\Gamma\{(n+1)\alpha+1\}} + (x^2+y^2-z^2)\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)},$$

$$\vdots$$

Using the above iterations, the solution $u(x,y,z,t)$ is given by

$$u(x,y,z,t) = \lim_{p\to 1}\sum_{n=0}^{\infty}p^n u_n(x,y,z,t)$$

$$= (x^2+y^2+z^2)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \frac{t^{(n+1)\alpha}}{\Gamma\{(n+1)\alpha+1\}} + \cdots\right)$$
$$+(x^2+y^2-z^2)\left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots + \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} + \cdots\right)$$
$$= (x^2+y^2+z^2)\left[E_\alpha(t^\alpha)-1\right] + (x^2+y^2-z^2)tE_{\alpha,2}(t^\alpha). \tag{60}$$

If we select $\alpha = 2$, then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,y,z,t)$ converges to the exact solution $u(x,y,z,t) = (x^2+y^2)\,e^t + z^2e^{-t}$ $-(x^2+y^2+z^2)$. It is quite important to notice that higher number of iteration and higher orders of p are needed to gain more accuracy.

5. Conclusions

In this paper, the homotopy perturbation transform method (HPTM) is applied to derive solutions of the fractional differential equations. We choose the fractional heat and wave-like equations with initial and boundary conditions to illustrate our method. As results, we obtain the exact solutions of fractional heat and wave-like equations. The obtained results demonstrate the reliability of the algorithm and its wider applicability to linear and nonlinear fractional differential equations. It is obvious to see that the HPTM is a very powerful, easy and efficient technique for solving various kinds of fractional problems in science and engineering and without many assumptions and restrictions.

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