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EXISTENCE AND UNIQUENESS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY VIA RESOLVENT OPERATORS

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Abstract. In this paper, we establish sufficient conditions for existence and uniqueness of solutions for semilinear functional differential equations with infinite delay. Our approach is based on resolvent operators, the Banach contraction principle, the Leray-Schauder nonlinear alternative and Schaefer's fixed point theorem. For the illustration of the results, an example is also discussed.

1. Introduction

In the last few decades, the subject of fractional differential equations has become a hot topic for the researchers due to its intensive development and applications in the field of physics, mechanics, chemistry, engineering, etc. For a reader interested in the systematic development of the topic, we refer the books [16, 17, 19, 21, 22, 24]. Differential equations with fractional order have recently proved to be valuable tools for the description of hereditary properties of various materials and systems. For more details, see [18]. For some recent developments on the subject, see for instance [1, 3, 4, 15, 20] and references cited therein.

In the literature devoted to equations with finite delay, the phase space is much of time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory, a usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [12]. For detailed discussion on this topic, we refer the reader to the books by Hino *et al.* [14]. For some recent developments on the subject, see for instance [1, 5, 6, 8, 20] and references cited therein.

It is well known that one important way to introduce the concept of mild solutions for fractional evolution equations is based on some probability densities and Laplace transform. This method was initialed by El-Borai [11]. For some recent

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developments see the paper [26], [9], [2]. Another approach to treat abstract equations with fractional derivatives based on the well developed theory of resolvent operators for integral equations [13]. Motivated by the approach in [13], Ye *et al.* [25] studied the existence, uniqueness and continuous dependence of the mild solutions for a class of fractional neutral functional differential equations with infinite delay, by using the Krasnoselskki fixed point theorem and the theory of resolvent operators. The fractional derivative in [25] is understood in the Caputo sense.

Recently in [7], motivated by the approach in [13], we studied fractional order semilinear functional differential equations defined on a compact real interval with finite delay. Existence and uniqueness of solutions are proved, based on the theory of resolvent operators and Banach's contraction principle and Leray-Schauder nonlinear alternative. We emphasize that in [7] the fractional derivative is understood in the Riemann-Liouville sense.

In this paper we continue the study in [7] to cover the case of infinite delay. More precisely this paper, motivated by [13] and [25], is concerned with fractional order semilinear functional differential equations with infinite delay of the form

$$D^\alpha y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b],\ 0 < \alpha < 1 \tag{1}$$
$$y_0 = \phi \in \mathcal{B}, \tag{2}$$

where D^α is the standard Riemann-Liouville fractional derivative, $f : J \times \mathcal{B} \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : \mathcal{B} \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $-\infty$ up to the present time t and $\mathcal{B}$ is called a phase space.

The purpose of this paper is to study the existence and uniqueness of mild solutions for (1)-(2) by virtue of resolvent operators. In Section 2 we recall some definitions and preliminary facts which will be used in the sequel. In Section 3, we give our main existence and uniqueness results by using Banach's contraction principle, the Leray-Schauder nonlinear alternative and Schaefer's fixed point theorem. An example is presented in Section 4 illustrating the abstract theory.

2. Preliminaries

In this section, we recall some definitions and propositions of fractional calculus, phase space and resolvent operators. Let E be a Banach space. By $C(J, E)$ we denote the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|T\|_{B(E)} = \sup\{|T(y)| : |y| = 1\}.$$

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| dt.$$

Definition 2.1. [16, 22] *The Riemann-Liouville fractional primitive of order $\alpha \in \mathbb{R}^+$ of a function $h : (0,b] \to E$ is defined by*

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

provided the right hand side exists pointwise on $(0,b]$, where Γ is the gamma function.

Definition 2.2. [16, 22] *The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a continuous function $h : (0,b] \to E$ is defined by*

$$\begin{aligned} \frac{d^\alpha h(t)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds \\ &= \frac{d}{dt} I_0^{1-\alpha} h(t). \end{aligned}$$

In all this paper, we assume that the phase space $(\mathcal{B}, |\cdot|)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms introduced at first by Hale and Kato in [12]:

- (A_1) If $y : (-\infty, b] \to E$, $b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
  - (i) $y_t \in \mathcal{B}$,
  - (ii) $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$,
  - (iii) $\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}$,

  where $H > 0$ is a constant, $K, M : \mathbb{R}^+ \to \mathbb{R}^+$ with K is continuous and M is locally bounded and H, K, M are independent of $y(\cdot)$.
- (A_2) For the function $y(\cdot)$ in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.
- (A_3) The space $\mathcal{B}$ is complete.

Hereafter are some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al.* [14].

Example 2.3. *The spaces $\mathcal{BC}$, $\mathcal{BUC}$, $\mathcal{C}^\infty$ and $\mathcal{C}^0$. Let*

- *$\mathcal{BC}$ the space of bounded continuous functions defined from $(-\infty, 0]$ to E,*
- *$\mathcal{BUC}$ the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E,*
- $\mathcal{C}^\infty = \{\phi \in \mathcal{BC} : \lim_{\theta \to -\infty} \phi(\theta)$ *exists in* $E\}$,
- $\mathcal{C}^0 = \{\phi \in \mathcal{BC} : \lim_{\theta \to -\infty} \phi(\theta) = 0\}$, *endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}.$$

We have that the spaces $\mathcal{BUC}$, $\mathcal{C}^\infty$ and $\mathcal{C}^0$ satisfy conditions $(A_1) - (A_3)$. $\mathcal{BC}$ satisfies (A_2), (A_3) but (A_1) is not satisfied.

Example 2.4. *The spaces $\mathcal{C}_g$, $\mathcal{UC}_g$, $\mathcal{C}_g^0$ and $\mathcal{C}_g^\infty$. Let g be a positive continuous function on $(-\infty, 0]$. We define:*

- $\mathcal{C}_g = \{\phi \in \mathcal{C}((-\infty, 0], E) : (\phi(\theta)/g(\theta))$ *is bounded on* $(-\infty, 0]\}$,
- $\mathcal{C}_g^0 = \{\phi \in \mathcal{C}_g : \lim_{\theta \to -\infty}(\phi(\theta)/g(\theta)) = 0\}$ *endowed with the uniform norm*

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : -\infty < \theta \leq 0\right\}.$$

We consider the following condition on the function g.

$$(\mathcal{G}): \sup_{0\le t\le a} \sup\left\{\frac{g(\theta+t)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty \text{ for all } a > 0.$$

Then we have that the spaces $\mathcal{C}_g$ and $\mathcal{C}_g^0$ satisfy conditions (A_3). They satisfy conditions (A_1) and (A_2) if $(\mathcal{G})$ holds.

Consider the fractional differential equation

$$D^\alpha y(t) = Ay(t) + f(t), \quad t \in J,\ 0 < \alpha < 1,\ y(0) = 0, \tag{3}$$

where A is a closed linear unbounded operator in E and $f \in C(J, E)$. Equation (3) is equivalent to the following integral equation [16]

$$y(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,\ t \in J. \tag{4}$$

This equation can be written in the following form of integral equation

$$y(t) = h(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} Ay(s)ds,\ t \ge 0, \tag{5}$$

where

$$h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{6}$$

Examples where the exact solution of (3) and the integral equation (4) are the same, are given in [4]. Let us assume that the integral equation (5) has an associated resolvent operator $(S(t))_{t\ge0}$ on E.

Next we define the resolvent operator of the integral equation (5).

Definition 2.5. [23, Definition 1.1.3] *A one parameter family of bounded linear operators $(S(t))_{t\ge0}$ on E is called a resolvent operator for (4) if the following conditions hold:*

(a) *$S(\cdot)x \in C([0,\infty), E)$ and $S(0)x = x$ for all $x \in E$;*
(b) *$S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every $t \ge 0$;*
(c) *for every $x \in D(A)$ and $t \ge 0$,*

$$S(t)x = x + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} AS(s)xds. \tag{7}$$

Here and hereafter we assume that the resolvent operator $(S(t))_{t\ge0}$ is analytic [23, Chapter 2], and there exist a function $\varphi_A \in L^1_{loc}([0,\infty), \mathbb{R}^+)$ such that $\|S'(t)x\| \le \varphi_A(t)\|x\|_{[D(A)]}$ for all $t > 0$ and each $x \in D(A)$.

We have the following concept of solution using Definition 1.1.1 in [23].

Definition 2.6. *A function $u \in C(J, E)$ is called a mild solution of the integral equation (5) on J if $\int_0^t (t-s)^{\alpha-1}u(s)ds \in D(A)$ for all $t \in J$, $h(t) \in C(J, E)$ and*

$$u(t) = \frac{A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds + h(t), \quad \forall t \in J.$$

The next result follows from [23, Proposition I.1.2, Theorem II.2.4, Corollary II.2.6].

Lemma 2.7. *Under the above conditions the following properties are valid.*

(i) *If $u(\cdot)$ is a mild solution of (5) on J, then the function $t \to \int_0^t S(t-s)h(s)ds$ is continuously differentiable on J, and*

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)h(s)ds, \quad \forall t \in J.$$

(ii) *If $h \in C^\beta(J,E)$ for some $\beta \in (0,1)$, then the function defined by*

$$u(t) = S(t)(h(t)-h(0)) + \int_0^t S'(t-s)[h(s)-h(t)]ds + S(t)h(0), \quad t \in J,$$

is a mild solution of (5) on J.

(iii) *If $h \in C(J,[D(A)])$ then the function $u : J \to E$ defined by*

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \quad t \in J,$$

is a mild solution of (5) on J.

3. Main Results

Consider the following space

$$\Omega = \{y : (-\infty, b] \to E : y|_J \in C(J,E) \text{ and } y_0 \in \mathcal{B}\}$$

where $y|_J$ is the restriction of y to J. Let $\|\cdot\|_b$ be the seminorm in Ω defined by:

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(s)| : 0 \le s \le b\},\ y \in \Omega.$$

In this section we give our main existence results for problem (1)-(2). This problem is equivalent to the following integral equation

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{A}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}y(s)ds + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y_s)ds, & t \in J. \end{cases}$$

Motivated by Lemma 2.7 and the above representation, we introduce the concept of mild solution.

Definition 3.1. *One says that a function $y \in \Omega$ is a mild solution of problem (1)-(2) if:*

(1) $\displaystyle\int_0^t (t-s)^{\alpha-1}y(s)ds \in D(A)$ *for $t \in J$,*

(2) $y_0 = \phi \in \mathcal{B}$ *and*

(3) $y(t) = \dfrac{A}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}y(s)ds + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y_s)ds$, *$t \in J$.*

Suppose that there exists a resolvent $(S(t))_{t\ge 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.7 (iii), if $y : \Omega \to \Omega$ is a mild solution of (1)-(2), then

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y_s)ds & \\ + \displaystyle\int_0^t S'(t-s)\left(\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^s (s-\tau)^{\alpha-1}f(\tau,y_\tau)d\tau\right)ds, & t \in J. \end{cases}$$

Our first existence result for problem (1)-(2) is based on the Banach's contraction principle.

Theorem 3.2. *Let $f : J \times \mathcal{B} \to E$ be continuous and there exists a constant $L > 0$ such that*

$$|f(t,u) - f(t,v)| \leq L\|u - v\|_{\mathcal{B}}, \quad for \quad t \in J \quad and \ u, v \in \mathcal{B}.$$

If

$$\frac{LK_b b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1}) < 1, \tag{8}$$

where $K_b = \sup\{|K(t)| : t \in [0,b]\}$, then the problem (1)-(2) has a unique mild solution on $(-\infty, b]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $\mathcal{A} : \Omega \to \Omega$ defined by:

$$\mathcal{A}(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds \\ + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds, & t \in J. \end{cases}$$

Let $x(\cdot) : (-\infty, b] \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ 0, & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. We denote by $\overline{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0]; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\cdot)$ satisfies

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds$$

we can decompose it as $y(t) = \bar{z}(t) + x(t), t \in J$ which implies $y_t = \bar{z}_t + x_t, t \in J$ and the function $z(\cdot)$ satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) \quad &= \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \\ &\quad + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau) d\tau \right) ds. \end{aligned}$$

Let

$$\Omega_0 = \{z \in \Omega \text{ such that } z_0 = 0\},$$

and let $\|\cdot\|_b$ be the seminorm in Ω_0 defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s \leq b\} = \sup\{|z(s)| : 0 \leq s \leq b\}, \quad z \in \Omega_0.$$

Then $(\Omega_0, \|\cdot\|_b)$ is a Banach space. Let the operator $F:\Omega_0\to\Omega_0$ be defined by

$$F(z)(t)=\begin{cases} 0, & t\in(-\infty,0],\\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,\bar z_s+x_s)ds & \\ +\displaystyle\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}f(\tau,\bar z_\tau+x_\tau)d\tau\right)ds, & t\in J.\end{cases}$$

We need to prove that F has a fixed point, which is a unique mild solution of (1)-(2) on $(-\infty,b]$. We shall show that F is a contraction. Let $z,z^*\in\Omega_0$. Then we have for each $t\in J$

$$\begin{aligned}
&|F(z)(t)-F(z^*)(t)|\\
=\;& \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s,\bar z_s+x_s)-f(s,\bar z^*_s+x_s)]ds\right.\\
&\left.+\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}\big[f(\tau,\bar z_\tau+x_\tau)-f(\tau,\bar z^*_\tau+x_\tau)\big]d\tau\right)ds\right|\\
\le\;& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,\bar z_s+x_s)-f(s,\bar z^*_s+x_s)|ds\\
&+\int_0^t \varphi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}|f(\tau,\bar z_\tau+x_\tau)-f(\tau,\bar z^*_\tau+x_\tau)|d\tau ds\\
\le\;& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L\|z_s-z^*_s\|_{\mathcal B}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^\tau (s-\tau)^{\alpha-1}L\|z_\tau-z^*_\tau\|_{\mathcal B}d\tau ds\\
\le\;& \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_b\sup_{s\in[0,t]}|z(s)-z^*(s)|ds\\
&+\frac{L}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^\tau (s-\tau)^{\alpha-1}d\tau K_b\sup_{s\in[0,t]}|z(s)-z^*(s)|ds\\
\le\;& \frac{LK_bt^\alpha}{\Gamma(\alpha+1)}\|z-z^*\|_b+\frac{\|\varphi_A\|_{L^1}LK_bt^\alpha}{\Gamma(\alpha+1)}\|z-z^*\|_b.
\end{aligned}$$

Taking the supremum over t we get

$$\|F(z)-F(z^*)\|_b \le \frac{LK_bb^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\|z-z^*\|_b.$$

By (8) F is a contraction and thus, by the contraction mapping theorem, we deduce that F has a unique fixed point z. Then $y(t)=\bar z(t)+x(t),\ t\in(-\infty,b]$ is a fixed point of the operator $\mathcal A$, which gives rise to a unique mild solution of (1)-(2). □

Our second existence result is based on Leray-Schauder nonlinear alternative.

Lemma 3.3. *(Nonlinear alternative for single valued maps)*[10, p.135]. *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline U\to C$ is a continuous, compact (that is, $F(\overline U)$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline U$, or*
(ii) *there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

Theorem 3.4. *Let $f : J \times \mathcal{B} \to E$ be continuous. Assume that:*

(A_1) *$S(t)$ is compact for all $t > 0$;*

(A_2) *there exist a function $p \in C(J, \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f(t,x)| \le p(t)\psi(\|x\|_{\mathcal{B}}),\ \forall (t,x) \in J \times \mathcal{B};$$

(A_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{K_b\|p\|_\infty \psi(M)\dfrac{b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) + M_b\|\phi\|_{\mathcal{B}}} > 1.$$

Then, the problem (1)-(2) has at least one mild solution on $(-\infty, b]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $F : \Omega_0 \to \Omega_0$ defined in Theorem 3.2, namely,

$$F(z)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds & \\ +\displaystyle\int_0^t S'(t-s)\left(\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds, & t \in [0,b]. \end{cases}$$

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(z)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds,$$

and

$$F_2(z)(t) = \int_0^t S'(t-s)F_1(z)(s)ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (A_1) implies that $S'(t)$ is compact for all $t > 0$ (see [13, Lemma 2.2]).

Step 1: *F_1 is completely continuous.*

At first, we prove that F_1 is continuous. Let $\{z_n\}$ be a sequence such that $z_n \to z$ in Ω_0 as $n \to \infty$. Then for $t \in [0,b]$ we have

$$\begin{aligned} |F_1(z_n)(t) - F_1(z)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big|f(s, \bar{z}_{ns} + x_s) - f(s, \bar{z}_s + x_s)\Big|ds \\ &\le \frac{1}{\Gamma(\alpha)}\|f(\cdot, \bar{z}_{n\cdot} + x_\cdot) - f(\cdot, \bar{z}_\cdot + x_\cdot)\|_\infty \int_0^t (t-s)^{\alpha-1}ds \\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)}\|f(\cdot, \bar{z}_{n\cdot} + x_\cdot) - f(\cdot, \bar{z}_\cdot + x_\cdot)\|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(z_n) - F_1(z)\|_b \to 0 \ \ as \ \ n \to \infty.$$

Thus F_1 is continuous.

Next, we prove that F_1 maps bounded sets into bounded sets in Ω_0. Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $z \in B_\rho = \{z \in \Omega_0 : \|z\|_b \le \rho\}$ one has $F_1(z) \in B_\delta$. Let $z \in B_\rho$. Since f is a continuous function, we have for each $t \in [0, b]$

$$\begin{aligned}
|F_1(z)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s, \bar{z}_s + x_s)\big|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(s)\psi(\|\bar{z}_s + x_s\|_{\mathcal{B}})ds \\
&\le \frac{b^\alpha \psi(\rho^*)\|p\|_\infty}{\Gamma(\alpha+1)} = \delta < \infty,
\end{aligned}$$

where

$$\begin{aligned}
\|\bar{z}_s + x_s\|_{\mathcal{B}} &\le \|\bar{z}_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\
&\le K(t)\sup\{|z(t)| : 0 \le s \le t\} + M(t)\|z_0\|_{\mathcal{B}} \\
&\quad + K(t)\sup\{|x(t)| : 0 \le s \le t\} + M(t)\|x_0\|_{\mathcal{B}} \\
&\le K(t)\sup\{|z(t)| : 0 \le s \le t\} + M(t)\|x_0\|_{\mathcal{B}} \\
&\le K_b\rho + M_b\|\phi\|_{\mathcal{B}} = \rho^*,
\end{aligned}$$

and $M_b = \sup\{|M(t)| : t \in [0, b]\}$.

Then, $\|F_1(z)\|_b \le \delta$, and hence $F_1(z) \in B_\delta$.

Now, we prove that F_1 maps bounded sets into equicontinuous sets of Ω_0. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set. Let $z \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \le \tau_1 \le \tau_2$ we have

$$\begin{aligned}
&|F_1(z)(\tau_2) - F_1(z)(\tau_1)| \\
= &\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds\right| \\
\le &\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] f(s, \bar{z}_s + x_s)ds\right| \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] f(s, \bar{z}_s + x_s)ds\right| \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds\right| \\
\le &\frac{\|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha)}\Big(\int_0^{\tau_1-\epsilon} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}ds\Big).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$F_{1\epsilon}(z)(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon} (t - s - \epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s)ds.$$

Note that the set

$$\left\{\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}f(s,\bar{z}_s+x_s)ds : z\in B_\rho\right\}$$

is bounded since

$$\begin{aligned} &\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}f(s,\bar{z}_s+x_s)ds\right| \\ \leq\ & \|p\|_\infty\psi(\rho^*)\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}ds\right| \\ \leq\ & \frac{\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha+1)}(t-\epsilon)^\alpha. \end{aligned}$$

Then for $t>0$, the set

$$Z_\epsilon(t)=\{F_{1\epsilon}(z)(t) : z\in B_\rho\}$$

is precompact in E for every ϵ, $0<\epsilon<t$. Moreover

$$\begin{aligned} &\Big|F_1(z)(t)-F_{1\epsilon}(z)(t)\Big| \\ \leq\ & \frac{\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha)}\left(\int_0^{t-\epsilon}[(t-s)^{\alpha-1}-(t-s-\epsilon)^{\alpha-1}]ds+\int_{t-\epsilon}^t(t-s)^{\alpha-1}ds\right) \\ \leq\ & \frac{\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha+1)}\Big(t^\alpha-(t-\epsilon)^\alpha\Big). \end{aligned}$$

Therefore, the set $Z(t)=\{F_1(z)(t) : z\in B_\rho\}$ is precompact in E. Hence the operator F_1 is completely continuous.

Step 2: F_2 *is completely continuous.*

The operator F_2 is continuous, since $S'(\cdot)\in C(J,B(E))$ and F_1 is continuous as proved in Step 1.
Now, let B_ρ be a bounded set as in Step 1. For $z\in B_\rho$ we have

$$\begin{aligned} |F_2(z)(t)| &\leq \int_0^t|S'(t-s)||F_1(z)(s)|ds \\ &\leq \int_0^t\varphi_A(t-s)\|F_1(z)(s)\|_{[D(A)]}ds \\ &\leq \frac{\|\varphi_A\|_{L^1}b^\alpha\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha+1)}=\delta'. \end{aligned}$$

Thus, there exists a positive number δ' such that $\|F_2(z)\|_b\leq\delta'$. This means that $F_2(z)\in B_{\delta'}$.
Next, we shall show that F_2 maps bounded sets into equicontinuous sets in Ω_0. Let $\tau_1,\tau_2\in J$, $\tau_2>\tau_1$ and let B_ρ be a bounded set as in Step 1. Let $z\in B_\rho$. Then if $\epsilon>0$ and $\epsilon\leq\tau_1\leq\tau_2$ we have

$$\begin{aligned} &|F_2(z)(\tau_2)-F_2(z)(\tau_1)| \\ =\ & \left|\int_0^{\tau_2}S'(\tau_2-s)F_1(z)(\tau_2)ds-\int_0^{\tau_1}S'(\tau_1-s)F_1(z)(\tau_1)ds\right| \\ \leq\ & \frac{b^\alpha\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha+1)}\Bigg(\int_0^{\tau_1-\epsilon}|S'(\tau_2-s)-S'(\tau_1-s)|\,ds \end{aligned}$$

$$+\int_{\tau_1-\epsilon}^{\tau_1}|S'(\tau_2-s)-S'(\tau_1-s)|\,ds+\int_{\tau_1}^{\tau_2}|S'(\tau_2-s)|ds\Big).$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$F_{2\epsilon}(z)(t) = S'(\epsilon)\int_0^{t-\epsilon} S'(t-s-\epsilon)F_1(z)(s)ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Z_\epsilon(t) = \{F_{2\epsilon}(z)(t) : z \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\Big|F_2(z)(t) - F_{2\epsilon}(z)(t)\Big| \le \frac{\|\varphi_A\|_{L^1}\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha+1)}\Big(t^\alpha-(t-\epsilon)^\alpha\Big).$$

Then $Z(t) = \{F_2(z)(t) : z \in B_\rho\}$ is precompact in E. Hence the operator F_2 is completely continuous.

Step 3: *We show there exists an open set $U \subset C(J,E)$ with $z \notin \lambda F(z)$ for $\lambda \in (0,1)$ and $y \in \partial U$.*

Let $\lambda \in (0,1)$ and

$$\begin{aligned} z(t)=\lambda F(z)(t) &= \lambda\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\bar z_s+x_s)ds \\ &\quad +\lambda\int_0^t S'(t-s)\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,\bar z_\tau+x_\tau)d\tau\Big)ds. \end{aligned}$$

Then

$$\begin{aligned} |z(t)| &\le \Big|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\bar z_s+x_s)ds \\ &\quad +\int_0^t S'(t-s)\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,\bar z_\tau+x_\tau)d\tau\Big)ds\Big| \\ &\le \int_0^t\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}|f(s,\bar z_s+x_s)|ds \\ &\quad +\int_0^t\frac{\varphi_A(t-s)}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}|f(\tau,\bar z_\tau+x_\tau)|d\tau ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)\psi(\|\bar z_s+x_s\|_{\mathcal B})ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t\varphi_A(t-s)\int_0^s(s-\tau)^{\alpha-1}p(s)\psi(\|\bar z_s+x_s\|_{\mathcal B})d\tau ds. \end{aligned} \tag{9}$$

But

$$\|\bar z_s+x_s\|_{\mathcal B} \le K_b\sup\{|z(s)| : 0\le s\le t\}+M_b\|\phi\|_{\mathcal B}$$

as proved in Step 1. If we let $w(t)$ be the right-hand side of the above inequality then we have that

$$\|\bar z_s+x_s\|_{\mathcal B} \le w(t), \quad t\in J,$$

and therefore (9) becomes

$$\begin{aligned} |z(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\psi(w(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}p(s)\psi(w(s))d\tau ds. \end{aligned} \tag{10}$$

Using (10) in the definition of w, we have

$$\begin{aligned} w(t) &= K_b\sup\{|z(s)| : 0\leq s\leq t\} + M_b\|\phi\|_{\mathcal{B}} \\ &\leq K_b\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\psi(w(s))ds \\ &\quad +K_b\frac{1}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}p(s)\psi(w(s))d\tau ds + M_b\|\phi\|_{\mathcal{B}}. \end{aligned}$$

Then

$$\|w\| \leq K_b\|p\|_\infty\psi(\|w\|)\frac{b^\alpha}{\Gamma(\alpha+1)} + K_b\|p\|_\infty\psi(\|w\|)\frac{b^\alpha}{\Gamma(\alpha+1)}\|\varphi_A\|_{L^1} + M_b\|\phi\|_{\mathcal{B}}$$

and consequently

$$\frac{\|w\|}{K_b\|p\|_\infty\psi(\|w\|)\dfrac{b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) + M_b\|\phi\|_{\mathcal{B}}} \leq 1.$$

Thus, by (A_3), there exists M such that $\|w\| \neq M$. Let us set

$$U = \{x\in C(J,E) : \|y\| < M\}.$$

From the choice of U, there is no $y\in\partial U$ such that $y=\lambda F(y)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that F has a fixed point $x\in\overline{U}$ which is a solution of (1)-(2) on $(-\infty,b]$. This completes the proof. □

Finally, we give an existence result based upon Schaefer's fixed point theorem.

Theorem 3.5. *(Schaefer's fixed point theorem)* [27, p.29]. *Let E a Banach space, and $F: E\to E$ be a completely continuous operator. Then either*

(a) *F has a fixed point, or*
(b) *The set $\mathcal{E} = \{x\in U : x=\lambda F(x),\ 0<\lambda<1\}$ is unbounded.*

Theorem 3.6. *Let $f: J\times\mathcal{B}\to E$ be continuous. Assume that:*

(B_1) *$S(t)$ is compact for all $t>0$;*
(B_2) *there exist functions $p,q\in C(J,\mathbb{R}_+)$ such that*

$$|f(t,u)| \leq p(t)+q(t)\|u\|_{\mathcal{B}},\ t\in J \text{ and } u\in\mathcal{B}.$$

Then, the problem (1)-(2) has at least one mild solution on $(-\infty,b]$, provident that

$$\frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) < 1.$$

Proof. Define F as in the proof of Theorem 3.2. As in Theorem 3.4 we can prove that F is completely continuous. Here we prove that the set

$$\mathcal{E} = \{z\in\Omega_0 : z=\lambda F(z),\ 0<\lambda<1\}$$

is bounded.
Let $z \in \mathcal{E}$ be any element. Then, for each $t \in [0, b]$,

$$
\begin{aligned}
|z(t)| &\leq \frac{1}{\Gamma(\alpha)}\bigg(\int_0^t p(s)(t-s)^{\alpha-1}ds + \int_0^t (t-s)^{\alpha-1}q(s)\Big[K_b\|z\|_b + M_b\|\phi\|_{\mathcal{B}}\Big]ds\bigg) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}\Bigg(p(s) + q(s)\Big[K_b\|z\|_b + M_b\|\phi\|_{\mathcal{B}}\Big]\Bigg)d\tau ds \\
&\leq \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\Big[K_b\|z\|_b + M_b\|\phi\|_{\mathcal{B}}\Big] + \frac{b^\alpha\|\varphi_A\|_{L^1}\|p\|_\infty}{\Gamma(\alpha+1)} \\
&\quad + \frac{b^\alpha\|\varphi_A\|_{L^1}\|q\|_\infty}{\Gamma(\alpha+1)}\Big[K_b\|z\|_b + M_b\|\phi\|_{\mathcal{B}}\Big] \\
&= \frac{b^\alpha}{\Gamma(\alpha+1)}\Bigg[\|p\|_\infty(1 + \|\varphi_A\|_{L^1}) + \|q\|_\infty\|\phi\|_{\mathcal{B}}M_b(1 + \|\varphi_A\|_{L^1})\Bigg] \\
&\quad + \frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1})\|z\|_b \\
&= \frac{b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1})(\|p\|_\infty + \|q\|_\infty\|\phi\|_{\mathcal{B}}M_b) \\
&\quad + \frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1})\|z\|_b
\end{aligned}
$$

and consequently

$$
\|z\|_b \leq \frac{b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})(\|p\|_\infty+\|q\|_\infty\|\phi\|_{\mathcal{B}}M_b)\left\{1-\frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\right\}^{-1}.
$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.5 we deduce that F has at least a fixed point, then the operator $\mathcal{A}$ has one, which gives rise to a mild solution of (1)-(2) on $(-\infty, b]$. □

4. An example

As an application of our results we consider the following fractional time partial functional differential equation of the form

$$
\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + \int_{-\infty}^0 P(\theta)g\big(t, u(t+\theta, x)\big)d\theta, \\
&\qquad x \in [0,\pi],\ t \in [0,b],\ 0 < \alpha < 1, &(11) \\
u(t,0) &= u(t,\pi) = 0,\ t \in [0,b], &(12) \\
u(t,x) &= u_0(t,x),\ x \in [0,\pi],\ t \in (-\infty, 0], &(13)
\end{aligned}
$$

where $P : (-\infty, 0] \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $u_0 : (-\infty, 0] \times [0, \pi] \to \mathbb{R}$ are continuous functions. To study this system, we take $E = L^2[0, \pi]$ and let A be the operator given by $Aw = w''$ with domain $D(A) = \{w \in E, w, w'$ are absolutely continuous, $w'' \in E, w(0) = w(\pi) = 0\}$.

Then

$$
Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \qquad w \in D(A),
$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx)$, $n = 1, 2, \dots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)w_n, \qquad w \in E.$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

From [23, Example 2.2.1] we know that the integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)ds, \ s \geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda^{\alpha} - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [3]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A)\, t > 0$.

For the phase space $\mathcal{B}$, we choose the well-known space $\mathcal{BUC}(\mathbb{R}^-, E)$ of uniformly bounded continuous functions equipped with the following norm:

$$\|\varphi\| \quad = \quad \sup_{\theta \leq 0} |\varphi(\theta)| \text{ for } \varphi \in \mathcal{B}.$$

To represent the system (11)-(13) in the abstract form (1)-(2) we consider $\varphi \in \mathcal{BUC}(\mathbb{R}^-, E)$, $x \in [0, \pi]$ and introduce the functions

$$\begin{aligned} y(t)(x) &= u(t, x), \quad t \in [0, b], x \in [0, \pi], \\ \phi(\theta)(x) &= u_0(\theta, x), \quad -\infty < \theta \leq 0, x \in [0, \pi], \\ f(t, \varphi)(x) &= \int_{-\infty}^0 P(\theta) g\big(t, \varphi(\theta)(x)\big) d\theta, \quad -\infty < \theta \leq 0, x \in [0, \pi]. \end{aligned}$$

Then the problem (11)-(13) takes the following abstract form:

$$\begin{cases} D^{\alpha}y(t) & = Ay(t) + f(t, y_t), \quad t \in J = [0, b], \ 0 < \alpha < 1; \\ y_0 & = \phi \in \mathcal{B}. \end{cases} \tag{14}$$

We assume the following assumptions:

(i) P is integrable on $(-\infty, 0]$.

(ii) There exist a continuous increasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$|g(t, v)| \leq \psi(|v|), \quad \text{for} \quad v \in \mathbb{R}.$$

By the dominated convergence theorem of Lebesgue, we can show that f is a continuous function of $\mathcal{B}$ in E. On the other hand, we have for $\varphi \in \mathcal{B}$ and $x \in [0, \pi]$

$$|f(t, \varphi)(x)| \quad \leq \quad \int_{-\infty}^0 |P(\theta)| g\big(t, |\varphi(\theta)(x)|\big) d\theta.$$

Since the function ψ is increasing, we have

$$|f(t,\varphi)| \quad \leq \quad \int_{-\infty}^{0} |P(\theta)| d\theta \psi\big(\|\varphi\|_{\mathcal{B}}\big) \text{ for } \varphi \in \mathcal{B}.$$

Choose b such that

$$\frac{Lb^{\alpha}}{\Gamma(\alpha+1)}(1+M) < 1.$$

Since the conditions of Theorem 3.2 are satisfied, there is a function $u \in C((-\infty, b], L^2[0,\pi])$ which is a mild solution of (11)-(13).

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