Journal of Fractional Calculus and Applications,
Vol. 4(2) July 2013, pp. 260-266.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

CERTAIN SPECIAL DIFFERENTIAL SUPERORDINATIONS USING LINEAR OPERATOR

R. M. EL-ASHWAH, M. K. AOUF AND S. M. EL-DEEB

Abstract. In this paper, we obtain special differential superordinations by using linear operator $\aleph_{p,b}^{s}$.

1. Introduction

Let $\mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ denote the subclass of functions $f \in \mathcal{H}(U)$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C};\ n \in \mathbb{N} = \{1, 2, ...\}).$$

Also, let $\mathcal{A}(p, n)$ denote the subclass of functions $f \in \mathcal{H}(U)$ of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (n \in \mathbb{N}). \tag{1}$$

If f and g are analytic functions in U, we say that f is subordinate to g (g is superordinate to f), written $f \prec g$ if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [1] and [3]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\varphi(r, s; z) : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U, $p, h \in \mathcal{H}(U)$, let $p(z)$ satisfies the first order differential superordination

$$h(z) \prec \varphi(p(z), zp'(z); z), \tag{2}$$

then $p(z)$ is a solution of the differential superordination (2). The analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination , if $q(z) \prec p(z)$ for all the functions $p(z)$ satisfying (2). An univalent subordinant $\widetilde{q}(z)$ is said to be the best subordinant of (2) if $\widetilde{q}(z) \prec q(z)$ for all subordinant $q(z)$ (see [4]).

Key words and phrases. p-Valent functions, differential superordination.
2000 Mathematics Subject Classification: 30C45.
Submitted April 11, 2013.

El-Ashwah [2] defined the linear operator $\aleph_{p,b}^{s} f(z) : A(p,n) \to A(p,n)$ as follows:

$$\aleph_{p,b}^{s} f(z) = z^{p} + \sum_{k=n}^{\infty} \left(\frac{k+b+1}{b+1} \right)^{s} a_{k+p} z^{k+p} \ \ (b \in \mathbb{C} \backslash \mathbb{Z}^{-} = \{-1,-2,...\}; s \in \mathbb{C}; p,n \in \mathbb{N}; z \in U). \tag{3}$$

We can easily verify from (3) that (see [2]):

$$z \left(\aleph_{p,b}^{s} f(z) \right)' = (b+1) \aleph_{p,b}^{s+1} f(z) - (b+1-p) \aleph_{p,b}^{s} f(z). \tag{4}$$

We note that

(i) $\aleph_{p,b}^{0} f(z) = f(z)$;

(ii) $\aleph_{p,p-1}^{1} f(z) = z^{p} + \sum\limits_{n=1}^{\infty} \left(\frac{n+p}{p} \right) a_{n+p} z^{n+p} = \dfrac{zf'(z)}{p}$.

In order to prove our results, we shall need the following definition and lemmas.

Definition 1 [4]**.** *Let Q be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.*

Lemma 1 [3]. *Let h* be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} zp^{'}(z)$ is univalent in U and

$$h(z) \prec p(z) + \frac{1}{\gamma} zp^{'}(z),$$

then

$$q(z) \prec p(z),$$

where $q(z) = \dfrac{\gamma}{nz^{\frac{\gamma}{n}}} {}_{0}^{z} h(t) t^{\frac{\gamma}{n}-1} dt,\ z \in U$. The function q is convex and is the best subordinant.

Lemma 2 [5]. For real or complex parameters $\alpha_1, \alpha_2, \alpha_3$ $\left(\alpha_3 \notin \mathbb{Z}_0^{-} = \{0,-1,-2,...\}\right)$,

$${}_{0}^{1} t^{\alpha_2 - 1} (1-t)^{\alpha_3 - \alpha_2 - 1} (1-tz)^{-\alpha_1} dt = \frac{\Gamma(\alpha_2) \Gamma(\alpha_3 - \alpha_2)}{\Gamma(\alpha_3)} {}_2F_1(\alpha_1, \alpha_2; \alpha_3; z) \ \ (Re(\alpha_3) > Re(\alpha_2) > 0) \tag{5}$$

and

$${}_2F_1(\alpha_1, \alpha_2; \alpha_3; z) = (1-z)^{-\alpha_1} \ {}_2F_1\left(\alpha_1, \alpha_3 - \alpha_2; \alpha_3; \frac{z}{z-1}\right). \tag{6}$$

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $b \in \mathbb{C} \backslash \mathbb{Z}^{-}$, $s \in \mathbb{C}$, $p,n \in \mathbb{N}$ and $z \in U$ and the powers are understood as principle values.

Theorem 1. *Let h* be a convex function in U with $h(0) = 1$. Let $f \in \mathcal{A}(p,n)$, $F(z) = I_{c,p}(f)(z) = \frac{c+1}{z^{c-p+1}} {}_{0}^{z} t^{c-p} f(t) dt$, $z \in U$, $Rec > -1$ and suppose that $\dfrac{\left(\aleph_{p,b}^{s} f(z)\right)^{'}}{pz^{p-1}}$ is univalent in U, $\frac{\left(\aleph_{p,b}^{s}F(z)\right)^{'}}{pz^{p-1}} \in \mathcal{H}[1,n] \cap Q$ and

$$h(z) \prec \frac{\left(\aleph_{p,b}^{s}f(z)\right)^{'}}{pz^{p-1}}, \tag{7}$$

then

$$q(z) \prec \frac{\left(\aleph_{p,b}^{s}F(z)\right)^{'}}{pz^{p-1}},$$

where $q(z) = \frac{c+1}{nz^{\frac{c+1}{n}}} \int_0^z h(t)t^{\frac{c+1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. We have

$$z^{c-p+1}F(z) = (c+1)\int_0^z t^{c-p}f(t)dt, \tag{8}$$

by differentiating (8) with respect to z, we obtain that

$$z^{c-p+1}F^{'}(z) + (c-p+1)z^{c-p}F(z) = (c+1)z^{c-p}f(z)$$

that is, that

$$zF^{'}(z) + (c-p+1)F(z) = (c+1)f(z)$$

and

$$z\left(\aleph_{p,b}^{s}F(z)\right)^{'} + (c-p+1)\left(\aleph_{p,b}^{s}F(z)\right) = (c+1)\left(\aleph_{p,b}^{s}f(z)\right) \ \ (z \in U). \tag{9}$$

Differentiating (9) with respect to z, we have

$$z\left(\aleph_{p,b}^{s}F(z)\right)^{''} + \left(\aleph_{p,b}^{s}F(z)\right)^{'} + (c-p+1)\left(\aleph_{p,b}^{s}F(z)\right)^{'} = (c+1)\left(\aleph_{p,b}^{s}f(z)\right)^{'}$$

then

$$z\left(\aleph_{p,b}^{s}F(z)\right)^{''} + (c-p+2)\left(\aleph_{p,b}^{s}F(z)\right)^{'} = (c+1)\left(\aleph_{p,b}^{s}f(z)\right)^{'}. \tag{10}$$

Denote

$$\phi(z) = \frac{\left(\aleph_{p,b}^{s}F(z)\right)^{'}}{pz^{p-1}} \ \ (z \in U),$$

then

$$pz^{p-1}\phi(z) = \left(\aleph_{p,b}^{s}F(z)\right)^{'} \tag{11}$$

and differentiating (11) with respect to z, we obtain that

$$p(p-1)z^{p-1}\phi(z) + pz^{p}\phi^{'}(z) = z\left(\aleph_{p,b}^{s}F(z)\right)^{''} \tag{12}$$

using (10), (11) and (12), the differential superordination (2.1) becomes

$$h(z) \prec \phi(z) + \frac{1}{c+1}z\phi^{'}(z),$$

by using Lemma 1 for $\gamma = c+1$, we have

$$q(z) \prec \phi(z),$$

i.e.

$$q(z) \prec \frac{\left(\aleph_{p,b}^{s}F(z)\right)^{'}}{pz^{p-1}},$$

where $q(z)=\dfrac{c+1}{nz^{\frac{c+1}{n}}}\int_0^z h(t)t^{\frac{c+1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Putting $h(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\leq\beta<1)$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let* $h(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\leq\beta<1)$. Let $f\in\mathcal{A}(p,n)$, $F(z)=I_{c,p}(f)(z)=\frac{c+1}{z^{c-p+1}}\int_0^z t^{c-p}f(t)dt,\ Rec>-1,\ z\in U$ and suppose that $\dfrac{\left(\aleph_{p,b}^s f(z)\right)^{'}}{pz^{p-1}}$ is univalent in U, $\dfrac{\left(\aleph_{p,b}^s F(z)\right)^{'}}{pz^{p-1}}\in\mathcal{H}[1,n]\cap Q$ and

$$\frac{1+(1-2\beta)z}{1-z}\prec\frac{\left(\aleph_{p,b}^s f(z)\right)^{'}}{pz^{p-1}},\tag{13}$$

then

$$q(z)\prec\frac{\left(\aleph_{p,b}^s F(z)\right)^{'}}{pz^{p-1}},$$

where q is given by $q(z)=(2\beta-1)+2(1-\beta)\ {}_2F_1\left(1,\frac{c+1}{n};\frac{c+1}{n}+1;z\right)$. The function q is convex and it is the best subordinant.

Theorem 2. *Let* h be a convex function in U with $h(0)=1$. Let $f\in\mathcal{A}(p,n)$, suppose that $\dfrac{\left(\aleph_{p,b}^s f(z)\right)^{'}}{pz^{p-1}}$ is univalent in U, $\dfrac{\aleph_{p,b}^s f(z)}{z^p}\in\mathcal{H}[1,n]\cap Q$. If

$$h(z)\prec\frac{\left(\aleph_{p,b}^s f(z)\right)^{'}}{pz^{p-1}},\tag{14}$$

then

$$q(z)\prec\frac{\aleph_{p,b}^s f(z)}{z^p},\tag{15}$$

where $q(z)=\dfrac{p}{nz^{\frac{p}{n}}}\int_0^z h(t)t^{\frac{p}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. consider

$$\phi(z)=\frac{\aleph_{p,b}^s f(z)}{z^p}=\frac{z^p+\sum\limits_{k=n}^{\infty}\left(\frac{k+b+1}{b+1}\right)^s a_{k+p}z^{k+p}}{z^p}=1+\phi_n z^n+\phi_{n+1}z^{n+1}+..\ (z\in U).\tag{16}$$

Differentiating (16) with respect to z, we obtain

$$\left(\aleph_{p,b}^s f(z)\right)^{'}=pz^{p-1}\phi(z)+z^p\phi^{'}(z),$$

that is, that

$$\frac{\left(\aleph_{p,b}^s f(z)\right)^{'}}{pz^{p-1}}=\phi(z)+\frac{1}{p}z\phi^{'}(z).$$

Then, the differential superordination (14) becomes

$$h(z) \prec \phi(z) + \frac{1}{p} z\phi'(z).$$

By using Lemma 1 for $\gamma = p$, we have

$$q(z) \prec \phi(z),$$

i.e.

$$q(z) \prec \frac{\aleph_{p,b}^{s} f(z)}{z^{p}},$$

where $q(z) = \frac{p}{nz^{\frac{p}{n}}} \int_0^z h(t) t^{\frac{p}{n}-1} dt$. The function q is convex and it is the best subordinant.

Putting $h(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$ in Theorem 2, we obtain the following corollary.

Corollary 2. *Let* $h(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$. Let $f \in \mathcal{A}(p,n)$, suppose that $\frac{\left(\aleph_{p,b}^{s} f(z)\right)'}{pz^{p-1}}$ is univalent in U, $\frac{\aleph_{p,b}^{s} f(z)}{z^{p}} \in \mathcal{H}[1,n] \cap Q$. If

$$\frac{1+(1-2\beta)z}{1-z} \prec \frac{\left(\aleph_{p,b}^{s} f(z)\right)'}{pz^{p-1}}, \tag{17}$$

then

$$q(z) \prec \frac{\aleph_{p,b}^{s} f(z)}{z^{p}}, \tag{18}$$

where q is given by $q(z) = (2\beta - 1) + 2(1-\beta)\ {}_2F_1\left(1, \frac{p}{n}; \frac{p+n}{n}; z\right)$. The function q is convex and it is the best subordinant.

Theorem 3. *Let* h be a convex function in U with $h(0) = 1$. Let $f \in \mathcal{A}(p,n)$, suppose that $\frac{1}{pz^{p-1}} \left(\frac{z^{p} \aleph_{p,b}^{s+1} f(z)}{\aleph_{p,b}^{s} f(z)} \right)'$ is univalent in U and $\frac{\aleph_{p,b}^{s+1} f(z)}{\aleph_{p,b}^{s} f(z)} \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec \frac{1}{pz^{p-1}} \left(\frac{z^{p} \aleph_{p,b}^{s+1} f(z)}{\aleph_{p,b}^{s} f(z)} \right), \tag{19}$$

then

$$q(z) \prec \frac{\aleph_{p,b}^{s+1} f(z)}{\aleph_{p,b}^{s} f(z)}, \tag{20}$$

where $q(z) = \frac{p}{nz^{\frac{p}{n}}} \int_0^z h(t) t^{\frac{p}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. consider

$$\phi(z) = \frac{\aleph_{p,b}^{s+1} f(z)}{\aleph_{p,b}^{s} f(z)} = \frac{z^{p} + \sum\limits_{k=n}^{\infty} \left(\frac{k+b+1}{b+1}\right)^{s+1} a_{k+p} z^{k+p}}{z^{p} + \sum\limits_{k=n}^{\infty} \left(\frac{k+b+1}{b+1}\right)^{s} a_{k+p} z^{k+p}}$$

we have $\frac{z}{p}\phi^{'}(z)+\phi(z)=\frac{1}{pz^{p-1}}\left(\frac{z^p\aleph_{p,b}^{s+1}f(z)}{\aleph_{p,b}^{s}f(z)}\right)^{'}$. Then, the differential superordination (19) becomes

$$h(z)\prec\phi(z)+\frac{z}{p}\phi^{'}(z).$$

By using Lemma 1 for $\gamma=p$, we have

$$q(z)\prec\phi(z),$$

i.e.

$$q(z)\prec\frac{\aleph_{p,b}^{s+1}f(z)}{\aleph_{p,b}^{s}f(z)},$$

where $q(z)=\frac{p}{nz^{\frac{p}{n}}}\int_0^z h(t)t^{\frac{p}{n}-1}dt$. The function q is convex and it is the best subordinant.

Putting $h(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$ in Theorem 3, we obtain the following corollary.

Corollary 3. *Let* $h(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$. Let $f\in\mathcal{A}(p,n)$, suppose that $\frac{1}{pz^{p-1}}\left(\frac{z^p\aleph_{p,b}^{s+1}f(z)}{\aleph_{p,b}^{s}f(z)}\right)^{'}$ is univalent in U and $\frac{\aleph_{p,b}^{s+1}f(z)}{\aleph_{p,b}^{s}f(z)}\in\mathcal{H}[1,n]\cap Q$. If

$$\frac{1+(1-2\beta)z}{1-z}\prec\frac{1}{pz^{p-1}}\left(\frac{z^p\aleph_{p,b}^{s+1}f(z)}{\aleph_{p,b}^{s}f(z)}\right)^{'},$$

then

$$q(z)\prec\frac{\aleph_{p,b}^{s+1}f(z)}{\aleph_{p,b}^{s}f(z)},$$

where q is given by $q(z)=(2\beta-1)+2(1-\beta)\ {}_2F_1\left(1,\frac{p}{n};\frac{p+n}{n};z\right)$. The function q is convex and it is the best subordinant.

Theorem 4. *Let h be a convex function in U with* $h(0)=1$. Let $f\in\mathcal{A}(p,n)$, suppose that $(b+1)\frac{\aleph_{p,b}^{s+1}f(z)}{z^p}-b\frac{\aleph_{p,b}^{s}f(z)}{z^p}$ is univalent in U and $\frac{\aleph_{p,b}^{s}f(z)}{z^p}\in\mathcal{H}[1,n]\cap Q$. If

$$h(z)\prec(b+1)\frac{\aleph_{p,b}^{s+1}f(z)}{z^p}-b\frac{\aleph_{p,b}^{s}f(z)}{z^p},\tag{21}$$

then

$$q(z)\prec\frac{\aleph_{p,b}^{s}f(z)}{z^p},\tag{22}$$

where $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. consider

$$\phi(z)=\frac{\aleph_{p,b}^{s}f(z)}{z^p}=1+\sum_{k=n}^{\infty}\left(\frac{k+b+1}{b+1}\right)^s a_{k+p}z^k.$$

we obtain

$$\phi(z)+z\phi^{'}(z)=(b+1)\frac{\aleph_{p,b}^{s+1}f(z)}{z^p}-b\frac{\aleph_{p,b}^{s}f(z)}{z^p}.$$

Then, the differential superordination (21) becomes

$$h(z) \prec \phi(z) + z\phi^{'}(z).$$

By using Lemma 1 for $\gamma = 1$, we have

$$q(z) \prec \phi(z),$$

i.e.

$$q(z) \prec \frac{\aleph_{p,b}^{s} f(z)}{z^p},$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Putting $h(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$ in Theorem 4, we obtain the following corollary.

Corollary 4. *Let* $h(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$. Let $f \in \mathcal{A}(p,n)$, suppose that $(b+1)\frac{\aleph_{p,b}^{s+1} f(z)}{z^p} - b\frac{\aleph_{p,b}^{s} f(z)}{z^p}$ is univalent in U and $\frac{\aleph_{p,b}^{s} f(z)}{z^p} \in \mathcal{H}[1,n] \cap Q$. If

$$\frac{1+(1-2\beta)z}{1-z} \prec (b+1)\frac{\aleph_{p,b}^{s+1} f(z)}{z^p} - b\frac{\aleph_{p,b}^{s} f(z)}{z^p},$$

then

$$q(z) \prec \frac{\aleph_{p,b}^{s} f(z)}{z^p},$$

where q is given by $q(z) = (2\beta - 1) + 2(1-\beta)\ {}_2F_1\left(1, \frac{1}{n}; \frac{1+n}{n}; z\right)$. The function q is convex and it is the best subordinant.

References

[1] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[2] R. M. El-Ashwah, Majorization properties for subclass of analytic p-valent functions defined by linear operator, Int. J. Open Problems Complex Analysis, 4(2012), no. 3, 43-50.

[3] S. S. Miller and P. T. Mocanu, Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[4] S. S. Miller and P. T. Mocanu, Subordinates of differential superordinations, Complex Variables, 48 (2003), no. 10, 815-826.

[5] E. T. Whittaker and G. N. Watson, A course on Modern Analysis: An Introduction to the General Theory of infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, Fourth Edition, Cambridige Univ. pree, Cambridge, 1927.

R. M. El-Ashwah, Faculty of Science, Damietta University, New Damietta, Egypt
E-mail address: r_elashwah@yahoo.com

M. K. Aouf, Faculty of Science, Mansoura University, Mansoura, Egypt
E-mail address: mkaouf127@yahoo.com

S. M. El-Deeb, Faculty of Science, Damietta University, New Damietta, Egypt
E-mail address: shezaeldeeb@yahoo.com