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ON A DISCRETIZATION PROCESS OF FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION

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Abstract. In this work we introduce a discretization process to discretize the fractional order differential equations. First of all, we consider the fractional order Riccati differential equation then, we consider the corresponding fractional order Riccati differential equation with piecewise constant arguments and we apply the proposed discretization on it. The stability of the fixed points of the resultant dynamical system and the Lyapunov exponent are investigated. Finally, we study some dynamic behavior of the resultant systems such as bifurcation and chaos.

1. Introduction

In recent years differential equations with fractional order have attracted many researchers because of their applications in many areas of science and engineering. The need for fractional order differential equations stems in part from the fact that many phenomena cannot be modeled by differential equations with integer derivatives. Analytical and numerical techniques have been implemented to study such equations. The fractional calculus has allowed the operations of integration and differentiation to be applied upon any fractional order. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on [4],[13],[14],[16]. For the existence of solutions for fractional differential equations, one can see [10],[11],[12].

About the development of existence theorems for fractional functional differential equations, many contributions exist and can be referred to [17],[18],[19]. Many applications of fractional calculus amounting to replace the time derivative in a given evolution equation by a derivative of fractional order.

Recalling the basic definitions (Caputo) and properties of fractional order differentiation and integration

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Definition 1. *The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t > 0$ is defined by*

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds,$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt}.$$

In addition, the following results are the main in fractional calculus. Let $\beta, \gamma \in \mathbb{R}^+$, $\alpha \in (0,1)$,

- $I_a^\beta : L^1 \to L^1$, and if $f(x) \in L^1$, then$I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.
- $\lim_{\beta \to n} I_a^\beta f(x) = I_a^n f(x)$ uniformly on $[a,b]$, $n = 1,2,3,...$, where $I_a^1 f(x) = \int_a^x f(s) ds$.
- $\lim_{\beta \to 0} I_a^\beta f(x) = f(x)$ weakly.
- If $f(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 1} D_a^\alpha f(x) = \frac{df(x)}{dx}$.

On the other hand, some examples of dynamical systems generated by piecewise constant arguments have been studied in [1]-[2],[8],[9]. Here we propose a discretization process to obtain the discrete version of fractional order differential equations. Mean while, we apply the same discretization process to discretize the fractional order Riccati differential equation.

2. Discretization process

Consider the differential equation of Riccati type

$$\frac{dx}{dt} = 1 - \rho x^2(t), \quad t \in (0, T], \tag{2.1}$$

$$x(0) = x_o. \tag{2.2}$$

Its solution is given by

$$x = \frac{1}{\sqrt{\rho}} tanh(t + tanh^{-1} \sqrt{\rho} x_o).$$

On the other hand, consider the corresponding equation with piecewise constant arguments given by

$$\frac{dx}{dt} = 1 - \rho x^2([\frac{t}{r}]r), \tag{2.3}$$

with $x(0) = x_o$.

Let $t \in [0, r)$, then $[\frac{t}{r}] = 0$, and the solution of (2.3) is given by

$$x_1(t) = x_o + t(1 - \rho x_o^2), \quad t \in [0, r)$$

Let $t \in [r, 2r)$, then $[\frac{t}{r}] = 1$, and the solution of (2.3) is given by

$$x_2(t) = x_1(r) + (t - r)(1 - \rho x_1^2), \quad t \in [r, 2r).$$

Repeating the process we get

$$x_{n+1}(t) = x_n(nr) + r(1 - \rho x_n^2(nr)), \quad t \in [nr, (n+1)r),$$

as $t \to (n+1)r$, we obtain

$$x_{n+1}((n+1)r) = x_n(nr) + r(1 - \rho x_n^2(nr)),$$

i.e.

$$x_{n+1} = x_n + r(1 - \rho x_n^2). \tag{2.4}$$

It should be noticed that the discretization (2.4) can be obtained by applying Euler method [5].

Moreover, if we consider the equation

$$\frac{dx}{dt} = 1 - \rho x([\frac{t}{r}]r)x([\frac{t-r}{r}]r), \tag{2.5}$$

with $x(0) = x_o$.
We can apply our procedure to obtain the discretization of the second order difference equation

$$x_{n+1}((n+1)r) = x_n(nr) + r(1 - \rho x_n(nr)x_{(n-1)}(nr)). \tag{2.6}$$

i.e.

$$x_{n+1} = x_n + r(1 - \rho x_n x_{n-1}). \tag{2.7}$$

The main purpose of this section is to introduce a discretization process of the fractional order differential equation of Riccati type given by

$$D^\alpha x(t) = 1 - \rho x^2(t), \quad t > 0, \tag{2.8}$$

with the initial condition $x(0) = x_o$.

Consider the counterpart of (2.8) with piecewise constant arguments

$$D^\alpha x(t) = 1 - \rho x^2([\frac{t}{r}]r), \tag{2.9}$$

with the initial condition $x(0) = x_o$.

The steps of the discretization process is as follows

1) Let $t \in [0, r)$, then $\frac{t}{r} \in [0, 1)$. So, we get

$$D^\alpha x(t) = 1 - \rho x_o^2, \quad t \in [0, r),$$

and the solution of (2.9) is given by

$$\begin{aligned} x_1(t) &= x_o + I^\alpha(1 - \rho x_0^2) \\ &= x_o + (1 - \rho x_0^2)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &= x_o + (1 - \rho x_0^2)\frac{t^\alpha}{\Gamma(1+\alpha)}. \end{aligned}$$

2) Let $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$. So, we get

$$D^{\alpha}x(t) = 1 - \rho x_1^2, \quad t \in [r, 2r),$$

and the solution of (2.9) is given by

$$\begin{aligned} x_2(t) &= x_1(r) + I_r^{\alpha}(1 - \rho x_1^2) \\ &= x_1(r) + (1 - \rho x_0^2)\int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &= x_1(r) + (1 - \rho x_1^2(r))\frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)} \end{aligned}$$

Repeating the process we can easily deduce that the solution of (2.9) is given by

$$x_{n+1}(t) = x_n(nr) + \frac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)}(1 - \rho x_n^2(nr)), \quad t \in [nr, (n+1)r).$$

Let $t \to (n+1)r$, we obtain the discretization

$$x_{n+1}((n+1)r) = x_n(nr) + \frac{r^{\alpha}}{\Gamma(1+\alpha)}(1 - \rho x_n^2(nr)),$$

we get

$$x_{n+1} = x_n + \frac{r^{\alpha}}{\Gamma(1+\alpha)}(1 - \rho x_n^2). \tag{2.10}$$

On a similar manner, consider the corresponding equation of (2.8) with piecewise constant arguments

$$D^{\alpha}x(t) = 1 - \rho x([\frac{t}{r}]r)x([\frac{t-r}{r}]r), \tag{2.11}$$

with the initial condition $x(0) = x_o$.
So, we obtain the second order discretization

$$x_{n+1} = x_n + \frac{r^{\alpha}}{\Gamma(1+\alpha)}(1 - \rho x_n x_{n-1}). \tag{2.12}$$

3. Approximate solution

In this part we give the error of the approximate solutions of (2.1) with the two equations (2.4) and (2.7) where $t = nr$. The following table gives the absolute error $= ||exact - approximate||$ for some different values of n and r between the exact solution of (2.1) and (2.4).

n	r=0.1	r=0.2	r=0.3
10	7.5392e-003	0.0030	4.0000e-004
30	2.0000e-004	1.0000e-004	2.9086e-005
50	1.0000e-004	1.8527e-007	1.5981e-009

Table 1: Absolute error of equation(2.4)

Similarly, The following table gives the absolute error $= ||exact - approximate||$ for some different values of n and r between the exact solution of (2.1) and (2.7).

n	r=0.1	r=0.2	r=0.3
10	0.0232	0.0061	3.0000e-004
30	4.0000e-004	1.0000e-004	2.8788e-005
50	1.0000e-004	1.1772e-007	1.5984e-009

Table 2: Absolute error of equation(2.7)

4. Fixed points and their asymptotic stability

Now we study the fixed points of the system (2.10)which has two fixed points namely, $\pm\frac{1}{\sqrt{\rho}}$ given by solving the equation

$$x = x + \frac{r^{\alpha}}{\Gamma(1+\alpha)}(1-\rho x^2).$$

We deduce the stability analysis of these fixed points as follows [5]
$x_{fix1} = \frac{1}{\sqrt{\rho}}$ is stable if

$$0 < \rho < \frac{(\Gamma(1+\alpha))^2}{r^{2\alpha}}, \tag{4.1}$$

and the second fixed point $x_{fix2} = \frac{-1}{\sqrt{\rho}}$ is unstable.

On the other hand, to study the stability of the fixed points of equation (2.12) we first split it into two equations as follows

$$y_{n+1} = x_n \tag{4.2}$$

$$x_{n+1} = x_n + \frac{r^{\alpha}}{\Gamma(1+\alpha)}(1-\rho x_n y_n). \tag{4.3}$$

This system has two fixed points namely, $(x, y)_{fix1} = (\frac{1}{\sqrt{\rho}}, \frac{1}{\sqrt{\rho}})$ and $(x, y)_{fix2} = (\frac{-1}{\sqrt{\rho}}, \frac{-1}{\sqrt{\rho}})$.

We deduce the stability analysis of these fixed points as follows [15].

The Jacobian matrix is given by

$$J = \begin{pmatrix} 1 & 0 \\ 1 - \frac{r^{\alpha}\rho}{\Gamma(1+\alpha)}y & -\frac{r^{\alpha}\rho}{\Gamma(1+\alpha)}x \end{pmatrix},$$

which has the eigenvalues

$$\lambda_{1,2} = 0.5(1 - \frac{r^{\alpha}\sqrt{\rho}}{\Gamma(1+\alpha)}) \pm 0.5\sqrt{(1 - \frac{r^{\alpha}\sqrt{\rho}}{\Gamma(1+\alpha)})^2 + 4\frac{r^{\alpha}\sqrt{\rho}}{\Gamma(1+\alpha)}},$$

If we take for instance $r = 0.2$, $\alpha = 0.85$, and $\rho = 12$, we get $\lambda_1 = 1$ and $\lambda_2 = -0.6021$. This means that this fixed point is unstable.
While the eigenvalues corresponding to the second fixed point are

$$\lambda_{1,2} = 0.5(1 + \frac{r^{\alpha}\sqrt{\rho}}{\Gamma(1+\alpha)}) \pm 0.5\sqrt{(1 + \frac{r^{\alpha}\sqrt{\rho}}{\Gamma(1+\alpha)})^2 - 4\frac{r^{\alpha}\sqrt{\rho}}{\Gamma(1+\alpha)}}.$$

Similarly, if we take $r = 0.3$, $\alpha = 0.95$, and $\rho = 8$, we get $\lambda_1 = 1$ and $\lambda_2 = 0.7616$. This means that this fixed point is unstable.
In the next section, we assure our analytical results obtained above by numerical experiments.

It's worth to mention here that Lyapunov exponent for (2.10) is given by

$$Lya.exp = \lim_{n\to\infty} \Sigma log_2(1 - 2\frac{r^\alpha}{\Gamma(1+\alpha)}\rho x_n).$$

When $\alpha \to 1$ the same Lyapunov exponent for the original discrete system ($x_{n+1} = x_n + r(1 - \rho x_n^2)$) is obtained.

5. Bifurcation and chaos

In this section we show by numerical experiments that the dynamical behavior of the dynamical systems (2.10) and (2.12) is affected by the change in both r and α.
Take $r = 0.3$ and $\alpha = 0.85$ in (2.10) (Figure (1)).
Take $r = 0.3$ and $\alpha = 0.95$ in (2.10) (Figure (2)).
Take $r = 0.4$ and $\alpha = 0.90$ in (2.10) (Figure (3)).
Take $r = 0.5$ and $\alpha = 0.90$ in (2.10) (Figure (4)).
Take $r = 0.2$ and $\alpha = 0.85$ in (2.12) (Figure (5)).
Take $r = 0.2$ and $\alpha = 0.90$ in (2.12) (Figure (6)).
Take $r = 0.4$ and $\alpha = 0.90$ in (2.12) (Figure (7)).
Take $r = 0.3$ and $\alpha = 0.95$ in (2.12) (Figure (8)).

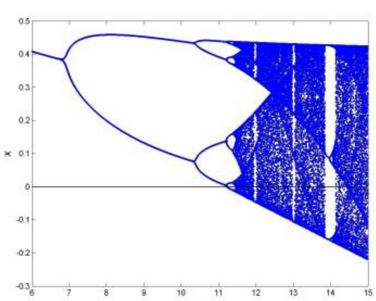

Figure 1. Bifurcation diagram of (2.10) when $r = 0.3$, $\alpha = 0.85$

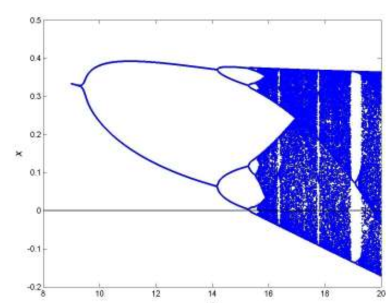

Figure 2. Bifurcation diagram of (2.10) when $r = 0.3$, $\alpha = 0.95$

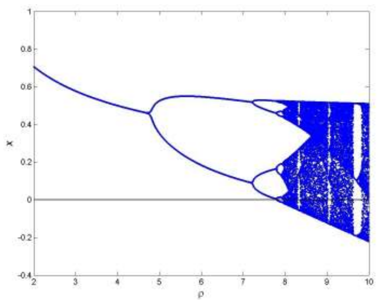

Figure 3. Bifurcation diagram of (2.10) when $r = 0.4$, $\alpha = 0.90$

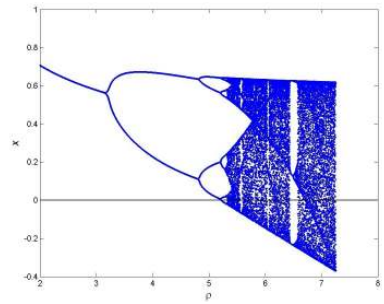

Figure 4. Bifurcation diagram of (2.10) when $r = 0.5$, $\alpha = 0.90$

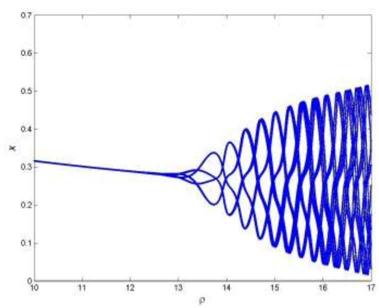

Figure 5. Bifurcation diagram of (2.12) when $r = 0.2$, $\alpha = 0.85$

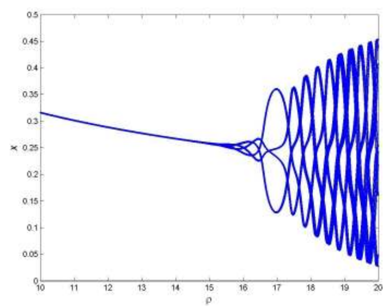

Figure 6. Bifurcation diagram of (2.12) when $r = 0.2$, $\alpha = 0.90$

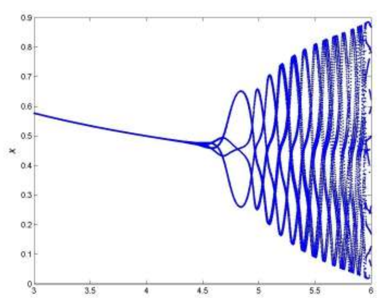

Figure 7. Bifurcation diagram of (2.12) when $r = 0.4$, $\alpha = 0.90$

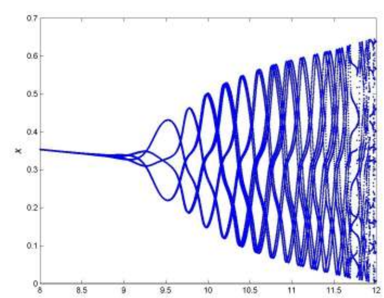

Figure 8. Bifurcation diagram of (2.12) when $r = 0.3$, $\alpha = 0.95$

6. Conclusion

We introduced a novel discretization process to discretize fractional order differential equations. We have noticed that when $\alpha \to 1$, the discretization will be Euler's method discretizartion [5]. Indeed, the parameter α plays as a brake for the stability of the resultant systems. Finally, all figures above agrees with our analytical results in (4.1).

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