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SOME FRACTIONAL INTEGRAL INEQUALITIES IN QUANTUM CALCULUS

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Abstract. In this paper, using the Riemann-Liouville fractional q-integral, we establish some new fractional integral inequalities by using two parameters of deformation q_1 and q_2.

1. introduction

The field of fractional calculus is almost as old as calculus itself. In particular, the fractional integral inequalities have been studied extensively by several researchers either in classical analysis or in the quantum one (see [1, 2, 3, 4, 6]).
The main objective of this paper is to establish the q-analogue of some inequalities proved in [1, 3, 4, 5] by using two parameters of deformation q_1 and q_2.

2. Basic Definitions

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [7, 9, 12]). We write for $a, b \in \mathbb{C}$ and $q \in (0,1)$,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a-qb)^{(\alpha)} = a^\alpha \frac{(\frac{qb}{a};q)_\infty}{(q^{\alpha+1}\frac{b}{a};q)_\infty}.$$

The q-Jackson integral from 0 to a is defined by (see [8])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{1}$$

provided the sum converges absolutely.

The fractional q-integral of the Riemann-Liouville type is (see [12])

$$\left(J_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)}f(t)d_qt; \quad \alpha > 0 \tag{2}$$

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where

$$\Gamma_q(\alpha) = \frac{1}{1-q}\int_0^1 \left(\frac{u}{1-q}\right)^{\alpha-1} e_q(qu)d_qu, \qquad and \quad e_q(t) = \prod_{k=0}^{\infty}(1-q^k t).$$

In the sequel, let q_1, q_2 be two real numbers in$(0,1)$.

3. Main results

Definition 1 Let f and g be two functions defined on I. The functions f and g are said synchronous on I if

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \quad \forall x, y \in I.$$

Theorem 1 Let f and g be two synchronous functions on $[0,+\infty[$ and v be a positive function on $[0,+\infty[$. Then for all $b>0$ and for all $\alpha, \beta > 0$, we have

$$J_{q_1}^{\alpha} v(b) J_{q_2}^{\beta} vfg(b) + J_{q_2}^{\beta} v(b) J_{q_1}^{\alpha} vfg(b) \geq J_{q_1}^{\alpha} vf(b) J_{q_2}^{\beta} vg(b) + J_{q_2}^{\beta} vf(b) J_{q_1}^{\alpha} vg(b). \quad (3)$$

Proof. Since f and g are two synchronous functions on $[0,+\infty[$, we get

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0, \quad \tau, \rho \in [0,+\infty[, \quad (4)$$

which implies that

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + g(\tau)f(\rho). \quad (5)$$

Multiplying both sides of (5) by $\frac{(b-q_1\tau)^{(\alpha-1)}}{\Gamma_{q_1}(\alpha)} v(\tau)$ and integrating with respect to τ from 0 to b, we get

$$J_{q_1}^{\alpha}(vfg)(b) + f(\rho)g(\rho) J_{q_1}^{\alpha} v(b) \geq g(\rho) J_{q_1}^{\alpha}(vf)(b) + f(\rho) J_{q_1}^{\alpha}(vg)(b). \quad (6)$$

Multiplying now both sides of(6) by $\frac{(b-q_2\rho)^{(\beta-1)}}{\Gamma_{q_2}(\beta)} v(\rho)$ and integrating the resulting identity with respect to ρ from 0 to b, we obtain

$$J_{q_2}^{\beta}(v)(b) J_{q_1}^{\alpha}(vfg)(b) + J_{q_1}^{\alpha}(v)(b) J_{q_2}^{\beta}(vfg)(b) \geq J_{q_2}^{\beta}(vg)(b) J_{q_1}^{\alpha}(vf)(b) + J_{q_2}^{\beta}(vf)(b) J_{q_1}^{\alpha}(vg)(b). \quad (7)$$

The result is proved.

The previous result can be generalized to the following

Theorem 2 Let f and g be two synchronous functions on $[0,+\infty[$ and v, w be two positive functions on $[0,+\infty[$. Then for all $b>0$ and for all $\alpha, \beta > 0$, we have

$$J_{q_1}^{\alpha} v(b) J_{q_2}^{\beta} wfg(b) + J_{q_2}^{\beta} w(b) J_{q_1}^{\alpha} vfg(b) \geq J_{q_1}^{\alpha} vf(b) J_{q_2}^{\beta} wg(b) + J_{q_2}^{\beta} wf(b) J_{q_1}^{\alpha} vg(b). \quad (8)$$

Proof. By multiplying both sides of(6) by $\frac{(b-q_2\rho)^{(\beta-1)}}{\Gamma_{q_2}(\beta)} w(\rho)$ and integrating the resulting identity with respect to ρ from 0 to b the result follows.

Theorem 3 Let v, w be two positive functions on $[0,+\infty[$ and let f and g be two functions defined on $[0,+\infty[$ satisfying the condition

$$\varphi \leq f(x) \leq \Phi, \qquad \psi \leq g(x) \leq \Psi, \qquad \varphi, \Psi, \Phi, \psi \in \mathbb{R}, x \in [0,+\infty[. \quad (9)$$

Then

$$|J_{q_2}^{\beta} w(b) J_{q_1}^{\alpha} vfg(b) + J_{q_1}^{\alpha} v(b) J_{q_2}^{\beta} wfg(b) - J_{q_1}^{\alpha} vf(b) J_{q_2}^{\beta} wg(b) - J_{q_2}^{\beta} wf(b) J_{q_1}^{\alpha} vg(b)|$$
$$\leq J_{q_1}^{\alpha} v(b) J_{q_2}^{\beta} w(b) (\Phi-\varphi)(\Psi-\psi) \quad (10)$$

Proof. From the condition (9), we have

$$|f(\tau)-f(\rho)| \leq \Phi-\varphi, \quad |g(\tau)-g(\rho)| \leq \Psi-\psi, \quad \tau, \rho \in [0,+\infty[, \quad (11)$$

which implies that

$$|\,(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\,| \leq (\Phi-\varphi)(\Psi-\psi). \tag{12}$$

Define

$$H(\tau,\rho)=f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau), \quad \tau,\rho\in[0,+\infty[. \tag{13}$$

Multiplying (13) by $\frac{(b-q_1\tau)^{(\alpha-1)}}{\Gamma_{q_1}(\alpha)}v(\tau)$ and integrating with respect to τ from 0 to b, we get

$$\frac{1}{\Gamma_{q_1}(\alpha)}\int_0^b (b-q_1\tau)^{(\alpha-1)}v(\tau)H(\tau,\rho)d_{q_1}\tau$$
$$= J_{q_1}^{\alpha}vfg(b)+f(\rho)g(\rho)J_{q_1}^{\alpha}v(b)-g(\rho)J_{q_1}^{\alpha}vf(b)-f(\rho)J_{q_1}^{\alpha}vg(b). \tag{14}$$

Now, multiplying (14) by $\frac{(b-q_2\rho)^{(\beta-1)}}{\Gamma_{q_2}(\beta)}w(\rho)$ and integrating with respect to ρ from 0 to b, we can state that

$$\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_{q_1}\tau d_{q_2}\rho$$
$$= J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}vfg(b)+J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}wfg(b)-J_{q_1}^{\alpha}vf(b)J_{q_2}^{\beta}wg(b)-J_{q_2}^{\beta}wf(b)J_{q_1}^{\alpha}vg(b). \tag{15}$$

Using (12), we can estimate (15) as follows

$$\left|\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_{q_1}\tau d_{q_2}\rho\right|$$
$$\leq \frac{(\Phi-\varphi)(\Psi-\psi)}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{\alpha-1}(b-q_2\rho)^{\beta-1}v(\tau)w(\rho)d_{q_1}\tau d_{q_2}\rho. \tag{16}$$

Consequently,

$$\left|\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_{q_1}\tau d_{q_2}\rho\right|$$
$$\leq J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}w(b)(\Phi-\varphi)(\Psi-\psi).$$

The result is thus proved.
In the particular case $\beta=\alpha$, we have the following result.

Corollary 1 Under the assumptions of Theorem 3, we have

$$|J_{q_2}^{\alpha}w(b)J_{q_1}^{\alpha}vfg(b)+J_{q_1}^{\alpha}v(b)J_{q_2}^{\alpha}wfg(b)-J_{q_1}^{\alpha}vf(b)J_{q_2}^{\alpha}wg(b)-J_{q_2}^{\alpha}wf(b)J_{q_1}^{\alpha}vg(b)|$$
$$\leq J_{q_1}^{\alpha}v(b)J_{q_2}^{\alpha}w(b)(\Phi-\varphi)(\Psi-\psi) \tag{17}$$

Theorem 4 Let v,w be two positive functions on $[0,+\infty[$ and let f and g be two functions defined on $[0,+\infty[$ satisfying the condition

$$|f(x)-f(y)|\leq M|g(x)-g(y)|; M>0, x,y\in[0,+\infty[. \tag{18}$$

Then the inequality

$$|J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}vfg(b)+J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}wfg(b)-J_{q_1}^{\alpha}vf(b)J_{q_2}^{\beta}wg(b)-J_{q_2}^{\beta}wf(b)J_{q_1}^{\alpha}vg(b)|$$
$$\leq M[J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}wg^2(b)+J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}vg^2(b)-2J_{q_1}^{\alpha}vg(b)J_{q_2}^{\beta}wg(b)] \tag{19}$$

is valid.

Proof. Multiplying (13) by $\frac{(b-q_1\tau)^{(\alpha-1)}v(\tau)}{\Gamma_{q_1}(\alpha)}$ and integrating the resulting identity with respect to τ from 0 to b, we obtain

$$\frac{1}{\Gamma_{q_1}(\alpha)}\int_0^b (b-q_1\tau)^{(\alpha-1)}v(\tau)H(\tau,\rho)d_{q_1}\tau$$
$$= J_{q_1}^{\alpha}vfg(b) - f(\rho)J_{q_1}^{\alpha}vg(b) - g(\rho)J_{q_1}^{\alpha}vf(b) + f(\rho)g(\rho)J_{q_1}^{\alpha}v(b). \quad (20)$$

Multiplying (20) by $\frac{(b-q_2\rho)^{(\beta-1)}w(\rho)}{\Gamma_{q_2}(\beta)}$ and integrating the resulting identity with respect to ρ from 0 to b, we get

$$\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)H(\tau,\rho)d_{q_1}\tau d_{q_2}\rho$$
$$= J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}vfg(b) - J_{q_2}^{\beta}wf(b)J_{q_1}^{\alpha}vg(b) - J_{q_1}^{\alpha}vf(b)J_{q_2}^{\beta}wg(b) + J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}wfg(b). \quad (21)$$

On the other hand, we have

$$|f(\tau)-f(\rho)| \leq M|g(\tau)-g(\rho)|. \quad (22)$$

This implies that

$$|H(\tau,\rho)| \leq M(g(\tau)-g(\rho))^2, \quad \tau,\rho \in [0,+\infty[. \quad (23)$$

Hence, it follows that

$$\frac{1}{\Gamma_{q_1}(\alpha)}\int_0^b (b-q_1\tau)^{(\alpha-1)}v(\tau)|H(\tau,\rho)|d_{q_1}\tau$$
$$\leq M\left(J_{q_1}^{\alpha}vg^2(b) - 2g(\rho)J_{q_1}^{\alpha}vg(b) + g^2(\rho)J_{q_1}^{\alpha}v(b)\right). \quad (24)$$

Consequently,

$$\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)|H(\tau,\rho)|d_{q_1}\tau d_{q_2}\rho$$
$$\leq \frac{M}{\Gamma_{q_2}(\beta)}\int_0^b \left((b-q_2\rho)^{\beta-1}w(\rho)\left[J_{q_1}^{\alpha}vg^2(b) - 2g(\rho)J_{q_1}^{\alpha}vg(b) + g^2(\rho)J_{q_1}^{\alpha}v(b)\right]\right)d_{q_1}\rho. \quad (25)$$

So,

$$\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b (b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)|H(\tau,\rho)|d_{q_1}\tau d_{q_2}\rho$$
$$\leq M[J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}wg^2(b) + J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}vg^2(b) - 2J_{q_1}^{\alpha}vg(b)J_{q_2}^{\beta}wg(b)]. \quad (26)$$

Theorem 4 is thus proved.

In the particular case $\beta=\alpha$, we have the following result.

Corollary 2 Under the assumptions of Theorem 4, we have

$$|J_{q_2}^{\alpha}w(b)J_{q_1}^{\alpha}vfg(b) + J_{q_1}^{\alpha}v(b)J_{q_2}^{\alpha}wfg(b) - J_{q_1}^{\alpha}vf(b)J_{q_2}^{\alpha}wg(b) - J_{q_2}^{\alpha}wf(b)J_{q_1}^{\alpha}vg(b)| \leq$$
$$M[J_{q_1}^{\alpha}v(b)J_{q_2}^{\alpha}wg^2(b) + J_{q_2}^{\alpha}w(b)J_{q_1}^{\alpha}vg^2(b) - 2J_{q_1}^{\alpha}vg(b)J_{q_2}^{\alpha}wg(b)]. \quad (27)$$

Theorem 5 Let f and g be two lipschitzian functions on $[0,+\infty[$ with the constants L_1 and L_2 and let v,w be two positive functions on $[0,+\infty[$. Then, the inequality

$$|J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}vfg(b)+J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}wfg(b)-J_{q_1}^{\alpha}vf(b)J_{q_2}^{\beta}wg(b)-J_{q_2}^{\beta}wf(b)J_{q_1}^{\alpha}vg(b)|$$
$$\leq L_1L_2(J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}(\tau^2w)(b)+J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}(\tau^2v)(b)-2J_{q_1}^{\alpha}(\tau v)(b)J_{q_2}^{\beta}(\tau w)(b))$$

is valid.

Proof. For all $\tau,\rho\in[0,+\infty[$, we have

$$|f(\tau)-f(\rho)|\leq L_1|\tau-\rho|,\quad |g(\tau)-g(\rho)|\leq L_2|\tau-\rho|. \tag{28}$$

Hence

$$|H(\tau,\rho)|\leq L_1L_2(\tau-\rho)^2. \tag{29}$$

Setting

$$R(\tau,\rho):=L_1L_2(\tau-\rho)^2, \tag{30}$$

then, multiplying (30) by $\frac{(b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}v(\tau)w(\rho)$ and integrating with respect to τ and ρ on $[0,+\infty[^2$, we get

$$|\frac{1}{\Gamma_{q_1}(\alpha)\Gamma_{q_2}(\beta)}\int_0^b\int_0^b(b-q_1\tau)^{(\alpha-1)}(b-q_2\rho)^{(\beta-1)}v(\tau)w(\rho)R(\tau,\rho)d_{q_1}\tau d_{q_2}\rho|$$
$$=L_1L_2\left(J_{q_1}^{\alpha}v(b)J_{q_2}^{\beta}(\tau^2w)(b)+J_{q_2}^{\beta}w(b)J_{q_1}^{\alpha}(\tau^2v)(b)-2J_{q_1}^{\alpha}(\tau v)(b)J_{q_2}^{\beta}(\tau w)(b)\right).$$

The result is thus proved.

Theorem 6 Let f and g be two lipschitzian functions on $[0,+\infty[$ with the constants L_1 and L_2 and let v,w be two positive functions on $[0,+\infty[$. The inequality

$$|J_{q_2}^{\alpha}w(b)J_{q_1}^{\alpha}vfg(b)+J_{q_1}^{\alpha}v(b)J_{q_2}^{\alpha}wfg(b)-J_{q_2}^{\alpha}wf(b)J_{q_1}^{\alpha}vg(b)-J_{q_1}^{\alpha}vf(b)J_{q_2}^{\alpha}wg(b)|$$
$$\leq L_1L_2(J_{q_2}^{\alpha}w(b)J_{q_1}^{\alpha}(\tau^2v)(b)+J_{q_1}^{\alpha}v(b)J_{q_2}^{\alpha}(\tau^2w)(b)-J_{q_2}^{\alpha}(\tau w)(b)J_{q_1}^{\alpha}(\tau v)(b)). \tag{31}$$

is valid.

Proof. same approach, we take $\alpha=\beta$ in Theorem 5.

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