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INTEGRALS INVOLVING GENERALIZED MITTAG -LEFFLER FUNCTION

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Abstract. Integrals of Mittag-Leffler function due to Shukla and Prajapati [1] multiplied with Jacobi polynomials, Legendre polynomials, Hermite polynomials, Legendre functions, Bessel Maitland function, hypergeometric function and generalized hypergeometric function are given here.

1. Introduction

The special function

$$E_{\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\eta k)}, \eta \in C, \Re(\eta) > 0, z \in C \tag{1}$$

and its general form

$$E_{\eta,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu+\eta k)}, \eta \in C, \Re(\eta) > 0, \Re(\mu) > 0, z \in C \tag{2}$$

with C being the set of complex numbers are called Mittag-Leffler functions ([10], section 18.1). The former was introduced by Mittag-Leffler [6] in connection with his method of summation of some divergent series. Investigated certain properties of this function. Function defined by (2) first appeared in the work of Wiman [2]. In particular, functions (1) and (2) are entire functions of order $\rho = \frac{1}{\alpha}$ and type $\sigma = 1$; see for example ([11], page 118). By means of the series representations a generalization of (1) and (2) is introduced by Prabhakar [7] as

$$E^{\gamma}_{\eta,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\mu+\eta n)} \frac{z^n}{n!}, \eta, \mu, \gamma \in C, \Re(\eta) > 0, \Re(\mu) > 0, \tag{3}$$

where

$$(\gamma)_n = \gamma(\gamma+1)...(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{(\gamma)} \tag{4}$$

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whenever $\Gamma(\gamma)$ is defined, $(\gamma)_0 = 1$, $\gamma \neq 0$. It is an entire function of order $\rho = [\Re(\eta)]^{-1}$ and type $\rho = \left(\frac{1}{\rho}\right)\left[\{\Re(\eta)\}^{\Re(\eta)}\right]^{-\rho}$. For various properties of this function with applications, see Prabhakar [7]. Further generalization of the Mittag-Leffler function $E^{\gamma}_{\eta,\mu}(z)$ of (3) was considered earlier by Shukla and Prajapati [1] which is defined in the following way

$$E^{\gamma,q}_{\eta,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!},\ (z,\mu,\gamma \in C, \Re(\eta) > \max\{0, \Re(q)-1\}; \Re(q) > 0)\,, \tag{5}$$

which is the special case when

$$q \in (0,1) \cup N \text{and} \min\{\Re(\mu), \Re(\gamma)\} > 0. \tag{6}$$

2. Integrals with Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ ([3], p. 254) may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{array}{cc} -n, 1+\alpha+\beta+n; & \frac{1-x}{2} \\ 1+\alpha; & \end{array}\right]. \tag{7}$$

When $\alpha = \beta = 0$, the polynomial in (7) becomes the Legendre polynomial ([3], p. 157). From (7) it follows that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree precisely n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}. \tag{8}$$

In dealing with the Jacobi polynomials, it is natural to make much use of our knowledge of the ${}_2F_1$ function ([3], p. 45)

$$I_1 \equiv \int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} P_n^{(\alpha,\beta)}(x) E^{\gamma,q}_{\eta,\mu}[z(1+x)^h]dx$$

$$= \int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} P_n^{(\alpha,\beta)}(x) \sum_{k=0}^{\infty} \frac{[z(1+x)^h]^k}{\Gamma(\eta k+\mu)} \frac{(\gamma)_{kq}}{k!} dx.$$

Interchanging the order of integration and summation which is permissible under the condition, then the above expression becomes

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta+hk} P_n^{(\alpha,\beta)}(x)dx. \tag{9}$$

But we have the formula ([8], p. 52)

$$\int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{(-1)^n 2^{\alpha+\delta+1}\Gamma(\delta+1)\Gamma(n+\alpha+1)\Gamma(\delta+\beta+1)}{n!\Gamma(\delta+\beta+n+1)\Gamma(\delta+\alpha+n+2)}$$

$$\times\,{}_3F_2\left[\begin{array}{c} -\lambda, \delta+\beta+1, \delta+1 \\ \delta+\beta+n+1, \delta+\alpha+n+2 \end{array};1\right]. \tag{10}$$

Provided $\alpha > -1$ and $\beta > -1$.

Now using (9) and (10), we get

$$= \frac{(-1)^n 2^{\alpha+\delta+1}\Gamma(n+\alpha+1)}{n!} \sum_{k=0}^{\infty} \frac{\Gamma(\delta+hk+1)\Gamma(\delta+hk+\beta+1)}{\Gamma(\delta+hk+\beta+n+1)\Gamma(\delta+hk+\alpha+n+2)}$$

$$\times E_{\eta,\mu}^{\gamma,q}(2^h z) {}_3F_2\left[\begin{matrix} -\lambda, \delta+hk+\beta+1, \delta+hk+1 \\ \delta+hk+\beta+n+1, \delta+hk+\alpha+n+2 \end{matrix} ;1\right]. \tag{11}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\lambda) > -1, \alpha > -1$ and $\beta > -1$.

$$I_2 \equiv \int_{-1}^{+1} (1-x)^{\delta}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\sigma)}(x) E_{\eta,\mu}^{\gamma,q}[z(1-x)^h]dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{-1}^{+1} (1-x)^{\delta+hk}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\sigma)}(x)dx. \tag{12}$$

Now using (7) in the above expression we get,

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{(1+\rho)_m}{m!} \sum_{k=0}^{\infty} \frac{(-m)_k(1+\rho+\sigma+m)_k}{(1+\rho)_k} \frac{1}{2^k k!}$$

$$\times \int_{-1}^{+1} (1-x)^{\delta+hk+k}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)dx. \tag{13}$$

Again using (7) in (13) we obtain

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m!n!\Gamma(1+\alpha)}$$

$$\times \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\alpha+\beta+n)_k}{2^{2k}(k!)^2\Gamma(1+\rho+k)\Gamma(1+\alpha+k)}$$

$$\times \int_{-1}^{+1} (1-x)^{\delta+hk+2k}(1+x)^{\beta}dx. \tag{14}$$

But by the formula ([3], p. 261)

$$\int_{-1}^{+1} (1-x)^{n+\alpha}(1+x)^{n+\beta}dx = 2^{2n+\alpha+\beta+1} B(1+\alpha+n, 1+\beta+n). \tag{15}$$

Hence (14) becomes,

$$= \frac{2^{\beta+\delta+1}\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m!n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\alpha+\beta+n)_k}{(k!)^2\Gamma(1+\rho+k)\Gamma(1+\alpha+k)}$$

$$\times E_{\eta,\mu}^{\gamma,q}(2^h z) B(1+\delta+hk+2k, 1+\beta). \tag{16}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\beta) > -1$, h and δ are positive numbers.

$$I_3 \equiv \int_{-1}^{+1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) E_{\eta,\mu}^{\gamma,q}[z(1-x)^h(1+x)^t]dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{-1}^{+1} (1-x)^{\rho+hk}(1+x)^{\sigma+tk} P_n^{(\alpha,\beta)}(x)dx. \tag{17}$$

Now using (7) in (17) we get,

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{2^k k!(1+\alpha)_k} \times \int_{-1}^{+1} (1-x)^{n+\rho+hk+k-n}(1+x)^{n+\sigma+tk-n} dx. \tag{18}$$

Using (15) in (18), we get

$$= 2^{\rho+\sigma+1} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{k!(1+\alpha)_k} \times E_{\eta,\mu}^{\gamma,q}(2^{h+t}z) B(1+\rho+hk+k, 1+\sigma+tk). \tag{19}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\alpha) > -1$ and $\Re(\beta) > -1$.

$$I_4 \equiv \int_{-1}^{+1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) E_{\eta,\mu}^{\gamma,q}[z(1+x)^{-h}] dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{-1}^{+1} (1-x)^{\rho}(1+x)^{\sigma-hk} P_n^{(\alpha,\beta)}(x) dx. \tag{20}$$

Now using (7) in (20) we get,

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{2^k k!(1+\alpha)_k} \times \int_{-1}^{+1} (1-x)^{n+\rho+k-n}(1+x)^{n+\sigma-hk-n} dx. \tag{21}$$

By using (15) in (21), we get

$$= 2^{\rho+\sigma+1} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{k!(1+\alpha)_k} \times E_{\eta,\mu}^{\gamma,q}(2^{-h}z) B(1+\rho+k, 1+\sigma-hk). \tag{22}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\alpha) > -1$ and $\Re(\beta) > -1$.

$$I_5 \equiv \int_{-1}^{+1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) E_{\eta,\mu}^{\gamma,q}[z(1-x)^{h}(1+x)^{-t}] dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{-1}^{+1} (1-x)^{\rho+hk}(1+x)^{\sigma-tk} P_n^{(\alpha,\beta)}(x) dx \tag{23}$$

using (7) in (23), we have

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{2^k (1+\alpha)_k k!} \times \int_{-1}^{+1} (1-x)^{n+\rho+hk+k-n}(1+x)^{\sigma-tk-n+n} dx. \tag{24}$$

Finally using (15) in (24), we get

$$= 2^{\rho+\sigma+1}\frac{(1+\alpha)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\alpha+\beta+n)_k}{k!(1+\alpha)_k}$$

$$\times E_{\eta,\mu}^{\gamma,q}(2^{h-t}z)B(1+\rho+hk+k, 1+\sigma-tk). \tag{25}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\alpha) > -1$ and $\Re(\beta) > -1$.

3. Special Cases

(i) If we replace δ by $\lambda - 1$ and put $\alpha = \beta = \rho = \sigma = 0$ then the integral I_2 transforms into the following integral involving Legendre polynomials [3]

$$I_6 \equiv \int_{-1}^{+1}(1-x)^{\lambda-1}P_n(x)E_{\eta,\mu}^{\gamma,q}[z(1-x)^h]dx$$

$$= \sum_{k=0}^{\infty}\frac{2^{\lambda}(-m)_k(m+1)_k}{(k!)^2}\frac{(-n)_k(n+1)_k}{(k!)^2}$$

$$\times E_{\eta,\mu}^{\gamma,q}(2^h z)B(\lambda+hk+2k, 1). \tag{26}$$

(ii) If $\alpha = \beta = 0$, ρ is replaced by $\rho - 1$ and σ by $\sigma - 1$, then I_3 transforms into the following integral involving Legendre polynomials [3]

$$I_7 \equiv \int_{-1}^{+1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_n(x)E_{\eta,\mu}^{\gamma,q}[z(1-x)^h(1+x)^t]dx$$

$$= \sum_{k=0}^{\infty}\frac{2^{\rho+\sigma-1}(-n)_k(n+1)_k}{(k!)^2}$$

$$\times E_{\eta,\mu}^{\gamma,q}(2^{h+t}z)B(\rho+hk+k, \sigma+tk). \tag{27}$$

(iii) Replacing ρ by $\rho - 1$ and σ by $\sigma - 1$ and putting $\alpha = \beta = 0$, the integral I_5 takes the form of the following integral involving Legendre polynomials [3]

$$I_8 \equiv \int_{-1}^{+1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_n(x)E_{\eta,\mu}^{\gamma,q}[z(1-x)^h(1+x)^{-t}]dx$$

$$= \sum_{k=0}^{\infty}\frac{2^{\sigma+\rho-1}(-n)_k(n+1)_k}{(k!)^2}$$

$$\times E_{\eta,\mu}^{\gamma,q}(2^{h-t}z)B(\rho+hk+k, \sigma-tk). \tag{28}$$

4. Integral with Bessel Maitland function

The special case of the Wright function ([11], vol. 3, section 18.1) and ([4], [5]) in the form

$$\phi(B,b;z) \equiv {}_0\Psi_1\left[\begin{matrix} \\ (b,B)\end{matrix}\ \Big|\ z\right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Bk+b)}\frac{z^k}{k!} \tag{29}$$

with complex $z, b \in C$ and real $B \in R$. When $B = \delta$, $b = \nu + 1$ and z is replaced by $-z$, the function $\phi(\delta, \nu+1; -z)$ is defined by $J_\nu^\delta(z)$:

$$J_\nu^\delta(z) \equiv \phi(\delta,\nu+1;-z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k+\nu+1)}\frac{(-z)^k}{k!} \tag{30}$$

and such a function is known as the Bessel Maitland function, or the Wright generalized Bessel function, see ([9], p. 352).

$$I_9 \equiv \int_0^\infty x^\rho J_\nu^\tau(x) E_{\eta,\mu}^{\gamma,q}(zx^\alpha)dx = \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)}\frac{z^k}{k!}\int_0^\infty x^{\rho+\alpha k} J_\nu^\tau(x)dx. \tag{31}$$

Now we know that the formula ([8], p. 55)

$$\int_0^\infty x^\rho J_\nu^\tau(x)dx = \frac{\Gamma(\rho+1)}{\Gamma(1+\nu-\tau-\tau\rho)}, \tag{32}$$

provided $\Re(\rho) > -1, 0 < \tau < 1$.
Now using (32) in (31), we get

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)}\frac{z^k}{k!}\frac{\Gamma(\rho+\alpha k+1)}{\Gamma(1+\nu-\tau-\tau(\rho+\alpha k))}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\rho+\alpha k+1)}{\Gamma(1+\nu-\tau-\tau(\rho+\alpha k))} E_{\eta,\mu}^{\gamma,q}(z). \tag{33}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\alpha - \tau\alpha > -1$, and $\alpha > 0$
(iii) $0 < \tau < 1$ and $\Re(\rho+1) > 0$.

5. Integrals With Legendre Functions

The Legendre functions are solution of Legendre's differential equation ([10], sec 3.1, vol. 1)

$$(1-z^2)\frac{d^2\omega}{dz^2} - 2z\frac{d\omega}{dz} + [\nu(\nu+1) - \mu^2(1-z^2)^{-1}]\omega = 0, \tag{34}$$

where z, ν, μ unrestricted.
Under the substitution $\omega = (z^2-1)^{\frac{1}{2}\mu}\nu$, (34) becomes

$$(1-z^2)\frac{d^2\nu}{dz^2} - 2(\mu+1)z\frac{d\nu}{dz} + (\nu-\mu)(\nu+\mu+1)\nu = 0 \tag{35}$$

and, with $\xi = \frac{1}{2} - \frac{1}{2}z$ as the independent variable, this differential equation becomes

$$\xi(1-\xi)\frac{d^2\nu}{d\xi^2} + (\mu+1)(1-2\xi)\frac{d\nu}{d\xi} + (\nu-\mu)(\nu+\mu+1)\nu = 0. \tag{36}$$

This is the Gauss hypergeometric type equation with $a = \mu - \nu$, $b = \nu + \mu + 1$ and $c = \mu + 1$.
Hence it follows that the function

$$\omega = P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} F\left[\ -\nu, \nu+1; 1-\mu;\ \frac{1}{2} - \frac{1}{2}z\right], \mid 1-z \mid < 2 \tag{37}$$

is a solution of (34).
The function $P_\nu^\mu(z)$ is know as the Legendre function of first kind ([10], vol. 1). It is one valued and regular in z-plane supposed cut along the real axis form 1 to $-\infty$.

$$\begin{aligned} I_{10} &\equiv \int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} P_\nu^\delta(x) E_{\eta,\mu}^{\gamma,q}(zx^\alpha)dx \\ &= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_0^1 x^{\sigma-1+\alpha k}(1-x^2)^{\frac{\delta}{2}} P_\nu^\delta(x)dx. \end{aligned} \tag{38}$$

Now integral in (38) can be solved by using the formula ([10], vol. 1, section 3.12)

$$\begin{aligned} &\int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} P_\nu^\delta(x)dx \\ &= \frac{(-1)^\delta \pi^{1/2} 2^{-\sigma-\delta}\Gamma(\sigma)\Gamma(1+\delta+\nu)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}+\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}+\frac{\delta}{2}+\frac{\nu}{2}\right)\Gamma\left(1-\delta+\nu\right)}. \end{aligned} \tag{39}$$

Provided, $\Re(\sigma) > 0, \delta = 1, 2, 3, ...$.
Now (38) becomes,

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{(-1)^\delta \pi^{1/2} 2^{-\sigma-\alpha k-\delta}\Gamma(\sigma+\alpha k)\Gamma(1+\delta+\nu)}{\Gamma\left(\frac{1}{2}+\frac{\sigma+\alpha k}{2}+\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma+\alpha k}{2}+\frac{\delta}{2}+\frac{\nu}{2}\right)\Gamma\left(1-\delta+\nu\right)} \\ &= \frac{(-1)^\delta \pi^{1/2} 2^{-\sigma-\delta}\Gamma(1+\delta+\nu)}{\Gamma\left(1-\delta+\nu\right)} \sum_{k=0}^{\infty} \frac{\Gamma(\sigma+\alpha k)}{\Gamma\left(\frac{1}{2}+\frac{\sigma+\alpha k}{2}+\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma+\alpha k}{2}+\frac{\delta}{2}+\frac{\nu}{2}\right)} \\ &\qquad \times E_{\eta,\mu}^{\gamma;q}\left(\frac{z}{2^\alpha}\right). \end{aligned} \tag{40}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\sigma > 0$ and δ is non negative integer.

$$\begin{aligned} I_{11} &\equiv \int_0^1 x^{\sigma-1}(1-x^2)^{-\frac{\delta}{2}} P_\nu^\delta(x) E_{\eta,\mu}^{\gamma;q}(zx^\alpha)dx \\ &= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_0^1 x^{\sigma-1+\alpha k}(1-x^2)^{-\frac{\delta}{2}} P_\nu^\delta(x)dx. \end{aligned} \tag{41}$$

Now we have the formula ([10], vol. 1, section 3.12)

$$\int_0^1 x^{\sigma-1}(1-x^2)^{-\frac{\delta}{2}} P_\nu^\delta(x)dx$$

$$= \frac{\pi^{1/2} 2^{-\sigma+\delta} \Gamma(\sigma)}{\Gamma\left(\frac{1}{2} + \frac{\sigma}{2} - \frac{\delta}{2} - \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma}{2} - \frac{\delta}{2} - \frac{\nu}{2}\right)}. \tag{42}$$

Provided, $\Re(\sigma) > 0$, $\delta = 1, 2, 3, \ldots$.
Finally using (42) in (41), we get

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k + \mu)} \frac{z^k}{k!} \frac{\pi^{1/2} 2^{-\sigma-\alpha k+\delta} \Gamma(\sigma + \alpha k)}{\Gamma\left(\frac{1}{2} + \frac{\sigma+\alpha k}{2} - \frac{\delta}{2} - \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma+\alpha k}{2} - \frac{\delta}{2} - \frac{\nu}{2}\right)}$$

$$= \pi^{1/2} 2^{\delta-\sigma} \sum_{k=0}^{\infty} \frac{\Gamma(\sigma + \alpha k)}{\Gamma\left(\frac{1}{2} + \frac{\sigma+\alpha k}{2} - \frac{\delta}{2} - \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma+\alpha k}{2} - \frac{\delta}{2} - \frac{\nu}{2}\right)} \times E_{\eta,\mu}^{\gamma,q}\left(\frac{z}{2^{\alpha}}\right). \tag{43}$$

Provided,
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0, 1) \cup N$
(ii) $\Re(\sigma) > 0$ and $\Re(\delta) > 1$.

6. Integrals with Hermite Polynomials

Hermite polynomials $H_n(x)$ ([3], p. 187) may be defined by means of the relation

$$exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} \tag{44}$$

valid for all finite x and t. Since

$$exp(2xt - t^2) = exp(2xt) exp(-t^2)$$

$$= \left(\sum_{n=0}^{\infty} \frac{(2x)^n t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k} t^n}{k!(n-2k)!}.$$

It follows from (44) that

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!}. \tag{45}$$

Examination of equation (45) show that $H_n(x)$ is a polynomial of degree precisely n in x and that

$$H_n(x) = 2^n x^n + \pi_{n-2}(x), \tag{46}$$

in which $\pi_{n-2}(x)$ is a polynomial of degree (n-2) in x.

$$I_{12} \equiv \int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x) E_{\eta,\mu}^{\gamma,q}(z x^{-2h}) dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k + \mu)} \frac{z^k}{k!} \int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x) x^{-2hk} dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k + \mu)} \frac{z^k}{k!} \int_{-\infty}^{\infty} x^{2\rho-2hk} e^{-x^2} H_{2\nu}(x) dx. \tag{47}$$

Now from the formula ([8], p. 59),

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x)dx = \frac{\pi^{1/2} 2^{2(\nu-\rho)} \Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)}. \tag{48}$$

Hence, (47) becomes,

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \frac{\pi^{1/2} 2^{2(\nu-\rho+hk)} \Gamma(2\rho-2hk+1)}{\Gamma(\rho-hk-\nu+1)}$$

$$= \sqrt{\pi} 2^{2(\nu-\rho)} \sum_{k=0}^{\infty} \frac{\Gamma(2\rho-2hk+1)}{\Gamma(\rho-hk-\nu+1)} E_{\eta,\mu}^{\gamma,q}(2^{2h} z). \tag{49}$$

Provided,
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $h > 0$ and $\rho = 0, 1, 2, ...$.

$$I_{13} \equiv \int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x) E_{\eta,\mu}^{\gamma,q}(zx^{2h})dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{-\infty}^{\infty} x^{2\rho+2hk} e^{-x^2} H_{2\nu}(x)dx. \tag{50}$$

Using the formula mentioned in (48). Then the above expression (50) becomes

$$= \sqrt{\pi} 2^{2(\nu-\rho)} \sum_{k=0}^{\infty} \frac{\Gamma(2\rho+2hk+1)}{\Gamma(\rho+hk-\nu+1)} E_{\eta,\mu}^{\gamma,q}(2^{-2h} z). \tag{51}$$

Provided,
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $h > 0$ and $\rho = 0, 1, 2, ...$.

7. Integral with Hypergeometric Function

In the study of second-order linear differential equations with three regular singular points, there arise the function

$$F(a, b, ; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{52}$$

for c neither zero nor a negative integer in (52) the notation

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1), n \geq 1, \tag{53}$$
$$(\alpha)_0 = 1, \alpha \neq 0$$

is called the factorial function and the function in (52) is called the hypergeometric function ([3], p. 45).

$$I_{14} \equiv \int_{1}^{\infty} x^{-\rho}(x-1)^{\sigma-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho; \\ \sigma; \end{matrix} (1-x) \right] E_{\eta,\mu}^{\gamma,q}(zx)dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{1}^{\infty} x^{-\rho+k}(x-1)^{\sigma-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho; \\ \sigma; \end{matrix} (1-x) \right] dx.$$

Putting $x = t+1$ and $dx = dt$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)} \frac{z^k}{k!} \int_{0}^{\infty} (t+1)^{k-\rho} t^{\sigma-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho; \\ \sigma; \end{matrix} -t \right] dt.$$

$$= E_{\eta,\mu}^{\gamma,q}(z)\sum_{k=0}^{\infty}\frac{(\nu+\sigma-\rho)_k(\lambda+\sigma-\rho)_k}{(\sigma)_k}\frac{(-1)^k}{k!}$$
$$\times\int_0^{\infty} t^{k+\sigma-1}(t+1)^{k-\rho}dt.$$
$$= E_{\eta,\mu}^{\gamma,q}(z)_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho;\\ \sigma;\end{array} -1\right]B(k+\sigma,\rho-2k-\sigma). \quad (54)$$

Provided, $\Re(\eta)>0,\Re(\mu)>0,\Re(\gamma)>0$ and $q\in(0,1)\cup N$.

8. Integrals with Generalized Hypergeometric Function

A generalized hypergeometric function ([3], p. 73) may be defined by

$${}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,...,\alpha_p;\\ \beta_1,\beta_2,...,\beta_q;\end{array} z\right] = 1+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}. \quad (55)$$

In which no denominator parameter β_j is allowed to be zero or a negative integer. If any numerator parameter α_i in (55) is zero or a negative integer, the series terminates.

$$I_{15}\equiv\int_0^t x^{\rho-1}(t-x)^{\sigma-1}{}_pF_q\left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right]E_{\eta,\mu}^{\gamma,q}\left[zx^{u}(t-x)^{v}\right]dx$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)}\frac{z^k}{k!}t^{\sigma+vk-1}\int_0^t x^{\rho+uk-1}\left(1-\frac{x}{t}\right)^{\sigma+vk-1}{}_pF_q\left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right]dx.$$

Now, putting $x=st$ and $dx=tds$, then we get

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{kq}}{\Gamma(\eta k+\mu)}\frac{(t^{u+v}z)^k}{k!}t^{\rho+\sigma-1}\int_0^1 s^{\rho+uk-1}(1-s)^{\sigma+vk-1}$$
$$\times{}_pF_q\left[(g_p);(h_q);at^{\alpha+\beta}s^{\alpha}(1-s)^{\beta}\right]ds$$
$$=t^{\rho+\sigma-1}\sum_{k=0}^{\infty}f(k)t^{(\alpha+\beta)k}E_{\eta,\mu}^{\gamma,q}\left(zt^{u+v}\right)$$
$$\times B\left(\rho+uk+\alpha k,\sigma+vk+\beta k\right), \quad (56)$$

where

$$f(k)=\frac{(g_1)_k...(g_p)_k}{(h_1)_k...(h_q)_k}\frac{a^k}{k!}. \quad (57)$$

Provided
(i) $\Re(\eta)>0,\Re(\mu)>0,\Re(\gamma)>0$ and $q\in(0,1)\cup N$
(ii) $\Re(\alpha)\geq 0$, $\Re(v)\geq 0$ (both are not zero simultaneously)
(iii) α and β are non-negative integer such that $\alpha+\beta\geq 1$.

$$I_{16}\equiv\int_0^t x^{\rho-1}(t-x)^{\sigma-1}{}_pF_q\left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right]E_{\eta,\mu}^{\gamma,q}\left[zx^{-u}(t-x)^{-v}\right]dx$$

$$=t^{\rho+\sigma-1}\sum_{k=0}^{\infty}f(k)t^{(\alpha+\beta)k}E_{\eta,\mu}^{\gamma,q}\left(zt^{-u-v}\right)$$
$$\times B\left(\rho+\alpha k-uk,\sigma+\beta k-vk\right), \quad (58)$$

where $f(k)$ is defined in (57).
Provided
(i) $\Re(\eta)>0,\Re(\mu)>0,\Re(\gamma)>0$ and $q\in(0,1)\cup N$

(ii) $\Re(\alpha) \geq 0$, $\Re(v) \geq 0$ (both are not zero simultaneously)
(iii) α and β are non-negative integer such that $\alpha + \beta \geq 1$.

$$I_{17} \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1} {}_pF_q \left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right] E_{\eta,\mu}^{\gamma,q}\left[zx^{u}(t-x)^{-v}\right] dx$$

$$= t^{\rho+\sigma-1} \sum_{k=0}^{\infty} f(k) t^{(\alpha+\beta)k} E_{\eta,\mu}^{\gamma,q}\left(zt^{u-v}\right) \times B\left(\rho+\alpha k+uk, \sigma+\beta k-vk\right). \tag{59}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\alpha) \geq 0$, $\Re(v) \geq 0$ (both are not zero simultaneously)
(iii) α and β are non-negative integer such that $\alpha + \beta \geq 1$.

$$I_{18} \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1} {}_pF_q \left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right] E_{\eta,\mu}^{\gamma,q}\left[zx^{-u}(t-x)^{v}\right] dx$$

$$= t^{\rho+\sigma-1} \sum_{k=0}^{\infty} f(k) t^{(\alpha+\beta)k} E_{\eta,\mu}^{\gamma,q}\left(zt^{-u+v}\right) \times B\left(\rho+\alpha k-uk, \sigma+\beta k+vk\right). \tag{60}$$

Provided
(i) $\Re(\eta) > 0, \Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$
(ii) $\Re(\alpha) \geq 0$, $\Re(v) \geq 0$ (both are not zero simultaneously)
(iii) α and β are non-negative integer such that $\alpha + \beta \geq 1$.

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