SOLUTIONS TO FRACTIONAL SYSTEM OF HEAT-AND WAVE-LIKE EQUATIONS WITH VARIATIONAL ITERATION METHOD

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ABSTRACT. This paper applies the variational iteration method to obtain analytical solutions for the system of fractional heat- and wave-like equations with variable coefficients. Numerical result proves that the proposed method is very effective and convenient.

1. INTRODUCTION

Several physical phenomena in engineering physics, chemistry, other sciences can be described very successfully by the models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional non-integer order [1, 2, 3, 4]. Fractional differential equations have gained much attention recently due to exact description of nonlinear phenomena. No analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method (VIM) was first proposed to solve fractional differential equations with great success [15]. Following the above idea, Draganescu [16], Momani and Odibat [17, 18, 19, 20, 21, 22, 23] applied VIM to more complex fractional differential equations, showing the effectiveness and accuracy of the method. In 2002 the Adomian decomposition method (ADM) was suggested to solve fractional differential equations [5]. But many researchers found it very difficult to calculate the Adomian polynomials, see [28, 29, 30, 31, 32]. In 2007, Momani and Odibat [33, 34, 35] applied the homotopy perturbation method (HPM) to fractional differential equations and showed that HPM is an alternative analytical method for fractional differential equations. Xu and Cang [45] solved the fractional heat- and wave-like equations with variable coefficients using homotopy analysis method (HAM). Another powerful analytical method is called the variational iteration method (VIM) first proposed by He [8], and also see [7, 9, 10, 11, 12, 13, 14, 15]. VIM has been successfully applied to many situations. For example, Soliman [24] used VIM to find a explicit solutions of KdV-Burger's and Lax's seventh-order KdV equations, Batiha et al. [25] applied VIM to solve heat- and wave-like equations

with singular behaviors. Furthermore, Batiha et al. [26] have expanded VIM in the form of Multistage VIM to solve a class of nonlinear system of ODEs, Wazwaz [38] applied VIM to solve linear and nonlinear Schrodinger equations. Shou et al. [36] solved heat-like and wave-like equations with variable coefficients by VIM, Sweilam [27] used VIM to solve multi-order FDEs, and in general. Very recently, Yu and Lib [39] solved the synchronization of fractional-order Rossler hyperchaotic systems using VIM. Yulita Molliq R et. al use the VIM to solve the fractional heat- and wave-like equations[46].

In this paper, we will consider the system of fractional heat- and wave-like equations of the form :

$$
\begin{cases}
\frac{\partial^\alpha u_1(x)}{\partial t^\alpha} = f_1(x,y,z)u_{1xx} + g_1(x,y,z)u_{1yy} + h_1(x,y,z)z_{1xx} \\
\frac{\partial^\alpha u_2(x)}{\partial t^\alpha} = f_2(x,y,z)u_{2xx} + g_2(x,y,z)u_{2yy} + h_2(x,y,z)z_{2xx} \\
\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\frac{\partial^\alpha u_m(x)}{\partial t^\alpha} = f_m(x,y,z)u_{mxx} + g_2(x,y,z)u_{myy} + h_2(x,y,z)z_{mxx}
\end{cases}
\tag{1}
$$

where $0 \le x \le a, 0 \le y \le b, 0 \le z \le c, 0 \le \alpha \le 2, t > 0$.
subject to the boundary conditions

$$
\begin{cases}
u_1(0,y,z,t) = f_{1_1}(y,z,t), \, u_{1x}(a,y,z,t) = f_{1_2}(y,z,t), \\
u_2(0,y,z,t) = f_{2_1}(y,z,t), \, u_{2x}(a,y,z,t) = f_{2_2}(y,z,t) \\
\qquad \vdots \\
u_m(0,y,z,t) = f_{m_1}(y,z,t), \, u_{mx}(a,y,z,t) = f_{m_2}(y,z,t)
\end{cases}
$$
,
$$
\begin{cases}
u_1(x,0,z,t) = g_{1_1}(x,z,t), \, u_{1y}(x,b,z,t) = g_{1_2}(x,z,t), \\
u_2(x,0,z,t) = g_{2_1}(x,z,t), \, u_{2y}(x,b,z,t) = g_{2_2}(x,z,t) \\
\qquad \vdots \\
u_m(x,0,z,t) = g_{m_1}(x,z,t), \, u_{my}(x,b,z,t) = g_{m_2}(x,z,t)
\end{cases}
$$
,
$$
\begin{cases}
u_1(x,y,0,t) = h_{1_1}(x,y,t), \, u_{1z}(x,y,c,t) = h_{1_2}(x,y,t), \\
u_2(x,y,0,t) = h_{2_1}(x,y,t), \, u_{2z}(x,y,c,t) = h_{2_2}(x,y,t) \\
\qquad \vdots \\
u_m(x,y,0,t) = h_{m_1}(x,y,t), \, u_{mz}(x,y,c,t) = h_{m_2}(x,y,t)
\end{cases}
$$

and the initial conditions

$$\begin{cases} u_1(x,y,z,0) = \psi_1(x,y,z), u_t(x,y,z,0) = \eta_1(y,z,z) \\ u_2(x,y,z,0) = \psi_2(x,y,z), u_t(x,y,z,0) = \eta_2(y,z,z) \\ \quad \vdots \\ u_m(x,y,z,0) = \psi_m(x,y,z), u_t(x,y,z,0) = \eta_m(y,z,z) \end{cases}.$$

2. FRACTIONAL CALCULUS

In this section we give some basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 2.1: A real function $f(x), x > 0$, is said to be in space $C^\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in (0, \infty)$, and it is said to be in the space C_μ^m if and only if $f^m \in C^m, m \in N$.

Definition 2.2: The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a functional $f^m \in C_\mu, \mu > -1$, is defined as:

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \ x > 0,$$

$$J_a^0 f(x) = f(x).$$

Properties of the operator J^α can be found in [3],we mention the following:
For $f^m \in C_\mu, \alpha, \beta > 0, \mu - 1, \gamma > -1$

(1) $J_a^\alpha f(x)$ exist for almost every $x \in [a, b]$
(2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x),$
(3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x),$
(4) $J_a^\beta (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma},$

Definition 2.3: The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D_a^\alpha f(x) = J_a^{m-\alpha} D_a^m f(x) = \frac{1}{\Gamma(m-a)} \int_0^x (x-\xi)^{m-\alpha-1} f^{(m)}(\xi) d\xi, \qquad (2)$$

For $m - 1 < \alpha \le m, m \in N, x > 0, f \in C_{-1}^m$
Also, we need here two basic properties of the Caputos fractional derivative[3].

Lemma 2.4: *if* $m - 1 < \alpha \le m, m \in N, x > 0, f \in C_\mu^m, \mu > -1$ *then*

$$D_a^\alpha J_a^\alpha f(x) = f(x),$$

and $$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, \qquad x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

In this paper, we consider multi-dimensional time fractional system of heat- and wave-like equations , where the unknown function $u = u(x, t)$ is assumed to be a casual function of time, i.e, vanishing for $t > 0$, and the fractional derivative is taken in Caputo sense to be:

Definition 2.5: For m to be the smallest integer that exceeds α, the Caputo fractional derivative of order$\alpha > 0$ is defined as:

$$D_a^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-a)} \int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m (x-\xi)}{\partial \xi^m} d\xi, & for \quad m-1 < \alpha \leq m \\ \frac{\partial^m (x-t)}{\partial t^m} & for \quad \alpha = m \in N. \end{cases} \quad (3)$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references [1, 2, 3, 4].

3. Variational iteration method

In this section, to illustrate the basic concepts of VIM, we consider the following non-homogeneous system of differential Equations:

$$\begin{cases} L_1 u_1(x,y,z,t) + N_1(x,y,z,t) = g_1(x,y,z,t) \\ L_2 u_2(x,y,z,t) + N_2(x,y,z,t) = g_2(x,y,z,t) \\ \qquad\qquad \vdots \\ L_m u_m(x,y,z,t) + N_m(x,y,z,t) = g_m(x,y,z,t) \end{cases} \quad (4)$$

In the above system of equations $L_1, L_2, \cdots, L_m$ are linear differential operators with respect to t and $N_1, N_2, \cdots, N_m$ are nonlinear operators and $g_1(x,y,z,t)$, $g_2(x,y,z,t), \cdots, g_m(x,y,z,t)$ are some given functions.

According to the variational iteration method, we can construct a correct functional as follows :

$$\begin{cases} u_{1_{n+1}}(x,y,z,t) = u_{1_n}(x,y,z,t)+ \\ \qquad + \int_0^t \lambda_1(\xi) \left\{ L_1 u_{1_n}(x,y,z,\xi) + N_1(x,y,z,\xi) - g_1(x,y,z,\xi) \right\} d\xi \\ \\ u_{2_{n+1}}(x,y,z,t) = u_{2_n}(x,y,z,t)+ \\ \qquad + \int_0^t \lambda_2(\xi) \left\{ L_2 u_{2_n}(x,y,z,\xi) + N_2(x,y,z,\xi) - g_2(x,y,z,\xi) \right\} d\xi \\ \\ \qquad\qquad \vdots \\ \\ u_{m_{n+1}}(x,y,z,t) = u_{m_n}(x,y,z,t)+ \\ \qquad + \int_0^t \lambda_m(\xi) \left\{ L_m u_{m_n}(x,y,z,\xi) + N_m(x,y,z,\xi) - g_m(x,y,z,\xi) \right\} d\xi \end{cases}$$

$$(5)$$

Where $\lambda_1(\xi), \lambda_2(\xi), \cdots, \lambda_m(\xi)$ are general Lagrange multipliers, which can be identified optimally via variational theory. The second terms on the right-hand side in (5) are called the correction and the subscript n denotes the nth order approximation. Under a suitable restricted variational assumptions (i.e $\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_m,$)

then the Lagrange multipliers are identified.

The iteration formula (5) will give several approximations of $u_1(t), u_2(t), \cdots, u_m(t)$ and the exact solutions are obtained at the limit of the resulting successive approximations .i.e.

$$
\begin{aligned}
u_1(t) &= \lim_{n \to \infty} u_{1_n}(t) \\
u_2(t) &= \lim_{n \to \infty} u_{2_n}(t) \\
&\vdots \\
u_m(t) &= \lim_{n \to \infty} u_{m_n}(t).
\end{aligned} \tag{6}
$$

4. Fractional variational iteration method

Consider the following general fractional differential equation

$$
\frac{D^\alpha u}{Dt^\alpha} + f = 0 \tag{7}
$$

In the case $0 < \alpha < 1$, we rewrite (7) in the form

$$
\frac{du}{dt} + \frac{D^\alpha u}{Dt^\alpha} - \frac{du}{dt} + f = 0 \tag{8}
$$

and the variational iteration algorithms are given as follows

$$
\begin{aligned}
u_{n+1}(t) &= u_n(t) - \int_0^t \left(\frac{D^\alpha u_n}{Dt^\alpha} + f_n \right) ds \\[2mm]
u_{n+1}(t) &= u_0(t) - \int_0^t \left(\frac{D^\alpha u_n}{Dt^\alpha} - \frac{du_n}{dt} + f_n \right) ds \\[2mm]
u_{n+1}(t) &= u_0(t) - \int_0^t \left\{ \left(\frac{D^\alpha u_n}{Dt^\alpha} - \frac{du_n}{dt} + f_n \right) \right. \\
&\qquad \left. - \left(\frac{D^\alpha u_{n-1}}{Dt^\alpha} - \frac{du_{n-1}}{dt} + f_{n-1} \right) \right\} ds
\end{aligned} \tag{9}
$$

In the case $1 < \alpha < 2$, the above iteration formulas are also valid.We can rewrite (7) in the form

$$
\frac{d^2 u}{dt^2} + \frac{D^\alpha u}{Dt^\alpha} - \frac{d^2 u}{dt^2} + f = 0 \tag{10}
$$

and the variational iteration algorithms in the case are given as follows

$$
\begin{aligned}
u_{n+1}(t) &= u_n(t) - \int_0^t (s - t) \left(\frac{D^\alpha u_n}{Dt^\alpha} + f_n \right) ds \\[2mm]
u_{n+1}(t) &= u_0(t) - \int_0^t (s - t) \left(\frac{D^\alpha u_n}{Dt^\alpha} - \frac{d^2 u_n}{dt^2} + f_n \right) ds \\[2mm]
u_{n+1}(t) &= u_0(t) - \int_0^t (s - t) \left\{ \left(\frac{D^\alpha u_n}{Dt^\alpha} - \frac{d^2 u_n}{dt^2} + f_n \right) \right. \\
&\qquad \left. - \left(\frac{D^\alpha u_{n-1}}{Dt^\alpha} - \frac{d^2 u_{n-1}}{dt^2} + f_{n-1} \right) \right\} ds
\end{aligned} \tag{11}
$$

5. Solution to system of fractional heat-like equations

In this section, we consider the following system of fractional wave-like equations of the form:

$$\begin{cases} \frac{\partial^\alpha u_1(x)}{\partial t^\alpha} = f_1(x,y,z)u_{1xx} + g_1(x,y,z)u_{1yy} + h_1(x,y,z)z_{1xx} \\ \frac{\partial^\alpha u_2(x)}{\partial t^\alpha} = f_2(x,y,z)u_{2xx} + g_2(x,y,z)u_{2yy} + h_2(x,y,z)z_{2xx} \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \frac{\partial^\alpha u_m(x)}{\partial t^\alpha} = f_m(x,y,z)u_{mxx} + g_2(x,y,z)u_{myy} + h_2(x,y,z)z_{mxx} \end{cases} \tag{12}$$

where $0 \le x \le a, 0 \le y \le b, 0 \le z \le c, 0 < \alpha \le 1, t > 0$.
The variational iteration method for the system of fractional wave-like equations (12) can be displayed as follows:

$$\begin{cases} u_{1_{n+1}}(x,y,z,t) = u_{1_n}(x,y,z,t) + \int_0^t \lambda_1(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{1_n}(x,y,z,\xi) - \right. \\ \quad -f_2(x,y,z)\frac{\partial^2}{\partial x^2}{}_1(x,y,z,\xi) - g_1(x,y,z)\frac{\partial^2}{\partial y^2}{}_1(x,y,z,\xi) \\ \quad \left. -h_1(x,y,z)\frac{\partial^2}{\partial z^2}{}_1(x,y,z,\xi) - q_1(x,y,z,\xi) \right) \\ u_{2_{n+1}}(x,y,z,t) = u_{2_n}(x,y,z,t) + \int_0^t \lambda_2(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{2_n}(x,y,z,\xi) - \right. \\ \quad -f_2(x,y,z)\frac{\partial^2}{\partial x^2}{}_2(x,y,z,\xi) - g_2(x,y,z)\frac{\partial^2}{\partial y^2}{}_2(x,y,z,\xi) \\ \quad \left. -h_2(x,y,z)\frac{\partial^2}{\partial z^2}{}_2(x,y,z,\xi) - q_2(x,y,z,\xi) \right) \\ \qquad\qquad\qquad \vdots \\ u_{m_{n+1}}(x,y,z,t) = u_{m_n}(x,y,z,t) + \int_0^t \lambda_m(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{m_n}(x,y,z,\xi) - \right. \\ \quad -f_m(x,y,z)\frac{\partial^2}{\partial x^2}{}_m(x,y,z,\xi) - g_m(x,y,z)\frac{\partial^2}{\partial y^2}{}_m(x,y,z,\xi) \\ \quad \left. -h_m(x,y,z)\frac{\partial^2}{\partial z^2}{}_m(x,y,z,\xi) - q_m(x,y,z,\xi) \right) \end{cases} \tag{13}$$

Making the above correction functional stationary, and noting that $\delta \tilde{u}_i = 0$

$$\delta u_{1_{n+1}}(x,y,z,t) = \delta u_{1_n}(x,y,z,t) + \delta \int_0^t \lambda_1(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{1_n}(x,y,z,\xi) - q_1(x,y,z,\xi) \right) d\xi$$
$$\delta u_{2_{n+1}}(x,y,z,t) = \delta u_{2_n}(x,y,z,t) + \delta \int_0^t \lambda_2(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{2_n}(x,y,z,\xi) - q_2(x,y,z,\xi) \right) d\xi$$

$$\vdots$$

$$\delta u_{m_{n+1}}(x,y,z,t) = \delta u_{m_n}(x,y,z,t) + \delta \int_0^t \lambda_m(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{m_n}(x,y,z,\xi) - q_m(x,y,z,\xi) \right) d\xi$$

We follow the stationary conditions for above system for $s = 1$ as:

$$\begin{cases} 1 + \lambda_1(t)|_{\xi=t} = 0 \\ \lambda_1(\xi) = 0 \end{cases}, \begin{cases} 1 + \lambda_2(t)|_{\xi=t} = 0 \\ \lambda_2(\xi)| = 0 \end{cases}, \dots, \begin{cases} 1 + \lambda_m(t)|_{\xi=t} = 0 \\ \lambda_m(\xi) = 0 \end{cases}$$

The general Lagrange multipliers, therefore, can be identified:

$$\lambda_1(\xi) = -1, \lambda_2(\xi) = -1, \dots, \lambda_m(\xi) = -1 \tag{14}$$

As a result, we obtain the following iteration formula:

$$
\begin{cases}
u_{1_{n+1}}(x,y,z,t) = u_{1_n}(x,y,z,t) - \int_0^t \left(\frac{\partial^s}{\partial \xi^s} u_{1_n}(x,y,z,\xi) - \right. \\
\quad - f_2(x,y,z)\frac{\partial^2}{\partial x^2}_1(x,y,z,\xi) - g_1(x,y,z)\frac{\partial^2}{\partial y^2}_1(x,y,z,\xi) \\
\quad \left. - h_1(x,y,z)\frac{\partial^2}{\partial z^2}_1(x,y,z,\xi) - q_1(x,y,z,\xi) \right) \\[2mm]
u_{2_{n+1}}(x,y,z,t) = u_{2_n}(x,y,z,t) - \int_0^t \left(\frac{\partial^s}{\partial \xi^s} u_{2_n}(x,y,z,\xi) - \right. \\
\quad - f_2(x,y,z)\frac{\partial^2}{\partial x^2}_2(x,y,z,\xi) - g_2(x,y,z)\frac{\partial^2}{\partial y^2}_2(x,y,z,\xi) \\
\quad \left. - h_2(x,y,z)\frac{\partial^2}{\partial z^2}_2(x,y,z,\xi) - q_2(x,y,z,\xi) \right) \\[2mm]
\qquad\qquad\qquad \vdots \\[2mm]
u_{m_{n+1}}(x,y,z,t) = u_{m_n}(x,y,z,t) - \int_0^t \left(\frac{\partial^s}{\partial \xi^s} u_{m_n}(x,y,z,\xi) - \right. \\
\quad - f_m(x,y,z)\frac{\partial^2}{\partial x^2}_m(x,y,z,\xi) - g_m(x,y,z)\frac{\partial^2}{\partial y^2}_m(x,y,z,\xi) \\
\quad \left. - h_m(x,y,z)\frac{\partial^2}{\partial z^2}_m(x,y,z,\xi) - q_m(x,y,z,\xi) \right).
\end{cases}
\tag{15}
$$

6. Solution to system of fractional wave-like equations

In this section, we consider the following system of fractional wave-like equations of the form:

$$
\begin{cases}
\frac{\partial^\alpha u_1(x)}{\partial t^\alpha} = f_1(x,y,z)u_{1xx} + g_1(x,y,z)u_{1yy} + h_1(x,y,z)z_{1xx} \\
\frac{\partial^\alpha u_2(x)}{\partial t^\alpha} = f_2(x,y,z)u_{2xx} + g_2(x,y,z)u_{2yy} + h_2(x,y,z)z_{2xx} \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\frac{\partial^\alpha u_m(x)}{\partial t^\alpha} = f_m(x,y,z)u_{mxx} + g_2(x,y,z)u_{myy} + h_2(x,y,z)z_{mxx}
\end{cases}
\tag{16}
$$

where $0 \le x \le a, 0 \le y \le b, 0 \le z \le c, 1 < \alpha \le 2, t > 0$.
The variational iteration method for the system of fractional wave-like equations

(16) can be displayed as follows:

$$
\begin{cases}
\begin{aligned}
u_{1_{n+1}}(x,y,z,t) &= u_{1_n}(x,y,z,t) + \int_0^t \lambda_1(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{1_n}(x,y,z,\xi) - \right. \\
&\quad - f_2(x,y,z) \frac{\partial^2}{\partial x^2}{}_1(x,y,z,\xi) - g_1(x,y,z) \frac{\partial^2}{\partial y^2}{}_1(x,y,z,\xi) \\
&\quad \left. - h_1(x,y,z) \frac{\partial^2}{\partial z^2}{}_1(x,y,z,\xi) - q_1(x,y,z,\xi) \right) \\[2mm]
u_{2_{n+1}}(x,y,z,t) &= u_{2_n}(x,y,z,t) + \int_0^t \lambda_2(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{2_n}(x,y,z,\xi) - \right. \\
&\quad - f_2(x,y,z) \frac{\partial^2}{\partial x^2}{}_2(x,y,z,\xi) - g_2(x,y,z) \frac{\partial^2}{\partial y^2}{}_2(x,y,z,\xi) \\
&\quad \left. - h_2(x,y,z) \frac{\partial^2}{\partial z^2}{}_2(x,y,z,\xi) - q_2(x,y,z,\xi) \right) \\[1mm]
&\qquad\qquad\qquad \vdots \\[1mm]
u_{m_{n+1}}(x,y,z,t) &= u_{m_n}(x,y,z,t) + \int_0^t \lambda_m(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{m_n}(x,y,z,\xi) - \right. \\
&\quad - f_m(x,y,z) \frac{\partial^2}{\partial x^2}{}_m(x,y,z,\xi) - g_m(x,y,z) \frac{\partial^2}{\partial y^2}{}_m(x,y,z,\xi) \\
&\quad \left. - h_m(x,y,z) \frac{\partial^2}{\partial z^2}{}_m(x,y,z,\xi) - q_m(x,y,z,\xi) \right)
\end{aligned}
\end{cases}
\tag{17}
$$

Making the above correction functional stationary, and noting that $\delta \tilde{u}_i = 0$

$$
\delta u_{1_{n+1}}(x,y,z,t) = \delta u_{1_n}(x,y,z,t) + \delta \int_0^t \lambda_1(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{1_n}(x,y,z,\xi) - q_1(x,y,z,\xi) \right) d\xi
$$
$$
\delta u_{2_{n+1}}(x,y,z,t) = \delta u_{2_n}(x,y,z,t) + \delta \int_0^t \lambda_2(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{2_n}(x,y,z,\xi) - q_2(x,y,z,\xi) \right) d\xi
$$

$$\vdots$$

$$
\delta u_{m_{n+1}}(x,y,z,t) = \delta u_{m_n}(x,y,z,t) + \delta \int_0^t \lambda_m(\xi) \left(\frac{\partial^s}{\partial \xi^s} u_{m_n}(x,y,z,\xi) - q_m(x,y,z,\xi) \right) d\xi
$$

We follow the stationary conditions for above system for $s = 1$ as:

$$
\begin{cases}
1 + \lambda_1'(t)|_{\xi=t} = 0 \\
\lambda_1(\xi)|_{\xi=t} = 0 \\
\lambda_1''(\xi) = 0
\end{cases}
,
\begin{cases}
1 + \lambda_2'(t)|_{\xi=t} = 0 \\
\lambda_2(\xi)|_{\xi=t} = 0 \\
\lambda_2''(\xi) = 0
\end{cases}
, \ldots ,
\begin{cases}
1 + \lambda_m'(t)|_{\xi=t} = 0 \\
\lambda_m(\xi)|_{\xi=t} = 0 \\
\lambda_m''(\xi) = 0
\end{cases}
$$

The general Lagrange multipliers, therefore, can be identified:

$$
\lambda_1(\xi) = \xi - t, \lambda_2(\xi) = \xi - t, \ldots, \lambda_m(\xi) = \xi - t
$$

As a result, we obtain the following iteration formula:

$$
\begin{cases}
u_{1_{n+1}}(x,y,z,t) = u_{1_n}(x,y,z,t) + \int_0^t \xi - t \left(\frac{\partial^s}{\partial \xi^s} u_{1_n}(x,y,z,\xi) - \right. \\
\quad - f_2(x,y,z) \frac{\partial^2}{\partial x^2} {}_1(x,y,z,\xi) - g_1(x,y,z) \frac{\partial^2}{\partial y^2} {}_1(x,y,z,\xi) \\
\quad \left. - h_1(x,y,z) \frac{\partial^2}{\partial z^2} {}_1(x,y,z,\xi) - q_1(x,y,z,\xi) \right) d\xi \\
u_{2_{n+1}}(x,y,z,t) = u_{2_n}(x,y,z,t) + \int_0^t \xi - t \left(\frac{\partial^s}{\partial \xi^s} u_{2_n}(x,y,z,\xi) - \right. \\
\quad - f_2(x,y,z) \frac{\partial^2}{\partial x^2} {}_2(x,y,z,\xi) - g_2(x,y,z) \frac{\partial^2}{\partial y^2} {}_2(x,y,z,\xi) \\
\quad \left. - h_2(x,y,z) \frac{\partial^2}{\partial z^2} {}_2(x,y,z,\xi) - q_2(x,y,z,\xi) \right) d\xi \\
\qquad\qquad\qquad \vdots \\
u_{m_{n+1}}(x,y,z,t) = u_{m_n}(x,y,z,t) + \int_0^t \xi - t \left(\frac{\partial^s}{\partial \xi^s} u_{m_n}(x,y,z,\xi) - \right. \\
\quad - f_m(x,y,z) \frac{\partial^2}{\partial x^2} {}_m(x,y,z,\xi) - g_m(x,y,z) \frac{\partial^2}{\partial y^2} {}_m(x,y,z,\xi) \\
\quad \left. - h_m(x,y,z) \frac{\partial^2}{\partial z^2} {}_m(x,y,z,\xi) - q_m(x,y,z,\xi) \right) d\xi.
\end{cases}
\tag{18}
$$

7. Numerical Examples

In this section we illustrate the variational iteration method (VIM) to system of fractional heat-and wave-like equations.

Example 7.1: We consider the system of fractional heat-like equation:

$$
\begin{cases}
D_t^\alpha u_1 = u_{1xx} + u_{1yy} \\
D_t^\alpha u_2 = x^4 y^4 z^4 + \frac{1}{36} \left[x^2 u_{2xx} + y^2 u_{2yy} + z^2 z_{2xx} \right],
\end{cases}
\tag{19}
$$

$$ x,y > 0, z < 1, t > 0, 0 < \alpha \le 1, $$

subject to the boundary conditions

$$
\begin{cases}
u_1(0,y,t) = 0, \, u_1(2\pi,y,t) = 0 \\
u_2(0,y,z,t) = 0, \, u_2(1,y,z,t) = y^4 z^4 (e^t - 1),
\end{cases}
$$

$$
\begin{cases}
u_1(x,0,t) = 0, \, u_1(x,2\pi,t) = 0 \\
u_2(x,0,z,t) = 0, \, u_2(x,1,z,t) = x^4 z^4 (e^t - 1),
\end{cases}
$$

$$
\begin{cases}
u_2(x,y,0,t) = 0, \, u_2(x,y,1,t) = x^4 y^4 (e^t - 1),
\end{cases}
$$

and the initial condition

$$
\begin{cases}
u_1(x,y,0) = \sin x \sin y \\
u_2(x,y,z,0) = 0,
\end{cases}
$$

The exact solution, $(\alpha = 1)$ was found to be [6],

$$\begin{cases} u_1(x,t) = e^{-2t}\sin x \cos y \\ \\ u_2(x,y,z,t) = x^4 y^4 z^4 (e^t - 1), \end{cases}$$

To solve system (19) by means of VIM, we use the formula of iteration (15) to find the iteration for the system (19) given by

$$\begin{cases} u_{1_{n+1}}(x,t) = u_{1_n}(x,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial t} u_{1_n}(x,y,\xi) - \right. \\ \qquad \left. -\frac{1}{2}\left[y^2 \frac{\partial^2 u_{1_n}(x,y,\xi)}{\partial x^2} + x^2 \frac{\partial^2 u_{1_n}(x,y,\xi)}{\partial y^2} \right] \right) d\xi \\ \\ u_{2_{n+1}}(x,y,t) = u_{2_n}(x,y,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial t} u_{2_n}(x,y,\xi) - x^4 y^4 z^4 - \frac{1}{36}\left[x^2 \right. \right. \\ \qquad \left. \left. \frac{\partial^2 u_{1_n}(x,y,\xi)}{\partial x^2} + y^2 \frac{\partial^2 u_{1_n}(x,y,\xi)}{\partial y^2} + z^2 \frac{\partial^2 u_{1_n}(x,y,\xi)}{\partial z^2} \right] \right), \end{cases} \qquad (20)$$

To get the iteration, we start with an initial approximation

$$\begin{cases} u_{10}(x,y,t) = u_1(x,y,t) = \sin x \cos y \\ \\ u_{20}(x,y,t) = x^4 y^4 z^4 \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha \end{cases}$$

we obtain the following successive approximations as follows:

$$\begin{cases} u_{1,0}(x,y,t) = \sin x \cos y \\ \\ u_{2,0}(x,y,z,t) = x^4 y^4 z^4 \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha \end{cases}$$

$$\begin{cases} u_{1,1}(x,y,t) = (1 - 2t)\sin x \cos y \\ \\ u_{2,1}(x,y,z,t) = u_{2,0}(x,y,z,t) + \frac{x^4 y^4 z^4 t^{\alpha+1}}{\Gamma(\alpha+2)} \end{cases}$$

$$\begin{cases} u_{1,2}(x,y,t) = \left(1 - 4t + 2t^2 + \frac{2t^{(\alpha-2)}}{\Gamma(3-\alpha)} \right)\sin x \cos y \\ \\ u_{2,2}(x,y,z,t) = u_{2,1}(x,y,z,t) + x^4 y^4 z^4 t \end{cases}$$

$$\begin{cases} u_{1,3}(x,y,t) = \left(1 - 6t + 6t^2 + \frac{1}{\Gamma(4-\alpha)}[6\alpha t^{(4-\alpha)} - 8t^{(3-\alpha)} + 18t^{(2-\alpha)}] \right. \\ \qquad \left. \frac{2t^{(3-5\alpha)}}{\Gamma(4-2\alpha)} \right)\sin x \cos y \\ u_{2,3}(x,y,z,t) = u_{2,2}(x,y,z,t) + \frac{1}{2}x^4 y^4 z^4 t \left[\frac{10x^4 y^4 z^4 t\alpha^2}{(\alpha^2-1)(\alpha^2-4)} \right. \\ \qquad - \frac{2x^4 y^4 z^4 t\alpha^4}{(\alpha^2-1)(\alpha^2-4)} - \frac{2\sin(\pi(\alpha+2))t^{(2-\alpha)}x^4 y^4 z^4}{\Gamma(\alpha+3)\pi\alpha(\alpha^2-1)(\alpha^2-4)} \\ \qquad - \frac{8x^4 y^4 z^4 t}{(\alpha^2-1)(\alpha^2-4)} + \frac{5x^4 y^4 z^4 t^2\alpha^2}{(\alpha^2-1)(\alpha^2-4)} - \frac{4x^4 y^4 z^4 t^2}{(\alpha^2-1)(\alpha^2-4)} \\ \qquad - \frac{10t^{(\alpha+2)}\alpha^2}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} + \frac{x^4 y^4 z^4 s^2\alpha^4\Gamma(\alpha+3)}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} \\ \qquad - \frac{20t^{(\alpha+1)}\alpha^2}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} + \frac{2\alpha^5 t^{(\alpha+1)}}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} \\ \qquad + \frac{4t^{(\alpha+1)}\alpha^4}{\pi\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} + \frac{8t^{(\alpha+1)}\alpha}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} \\ \qquad - \frac{10\alpha^2 t^{(\alpha+2)}}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} + \frac{8t^{(\alpha+2)}}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} \\ \qquad + \frac{2\alpha^4 t^{(\alpha+2)}}{\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} + \frac{16t^{(\alpha+1)}}{\alpha\Gamma(\alpha+3)(\alpha^2-1)(\alpha^2-4)} \right] \end{cases}$$

$$\vdots$$

Example 7.2 We consider the system of fractional wave-like equation:

$$
\begin{cases}
D_{tt}^{\alpha} u_1 = \frac{1}{12}(x^2 u_{1xx} + y^2 u_{1yy}) \\[2mm]
D_{tt}^{\alpha} u_2 = x^2 y^2 z^2 + \frac{1}{2}\left[x^2 u_{2xx} + y^2 u_{2yy} + z^2 z_{2xx}\right],
\end{cases}
\tag{21}
$$

$$x, y > 0, z < 1, t > 0, 1 < \alpha \le 2,$$

subject to the boundary conditions

$$
\begin{cases}
u_1(0, y, t) = 0,\, u_1(1, y, t) = 4\cosh t \\[2mm]
u_2(0, y, z, t) = y^2(e^t - 1) + z^2(e^{-t} - 1),\, u_2(1, y, z, t) = (1 + y^2)(e^t - 1) + z^2(e^{-t} - 1)
\end{cases}
$$

$$
\begin{cases}
u_1(x, 0, t) = 0,\, u_1(x, 1, t) = 4\sinh t \\[2mm]
u_2(x, 0, z, t) = x^2(e^t - 1) + z^2(e^{-t} - 1),\, u_2(x, 1, z, t) = (1 + x^2)(e^t - 1) + z^2(e^{-t} - 1)
\end{cases}
$$

$$
\left\{ u_2(x, y, 0, t) = x^2(e^t - 1) + y^2(e^{-t} - 1),\, u_2(x, y, 1, t) = (1 + x^2)(e^t - 1) + y^2(e^{-t} - 1) \right.
$$

and the initial condition

$$
\begin{cases}
u_1(x, y, 0) = x^4,\, u_{1t}(x, y, 0) = y^4 \\[2mm]
u_2(x, y, z, 0) = 0,\, u_{2t}(x, y, z, 0) = x^2 + y^2 + z^2
\end{cases}
$$

The exact solution system (21) for $(\alpha = 2)$ is

$$
\begin{cases}
u_1(x, y, t) = x^4 \cosh t\, y^4 \sinh t \\[2mm]
u_2(x, y, z, t) = -(x^2 + y^2 + z^2) + (x^2 + y^2)e^{-t} + z^2 e^{-t},
\end{cases}
\tag{22}
$$

To solve system (21) by means of VIM, we use the formula of iteration (18) to find the iteration for the system (21) given by

$$
\begin{cases}
u_{1_{n+1}}(x, y, t) = u_{1_n}(x, y, t) + \int_0^t (\xi - t)\left(\frac{\partial^\alpha}{\partial \xi^\alpha} u_{1_n}(x, y, \xi) - \right. \\[2mm]
\qquad \left. -\frac{1}{12}\left[y^2 \frac{\partial^2 u_{1_n}(x, y, \xi)}{\partial x^2} + x^2 \frac{\partial^2 u_{1_n}(x, y, \xi)}{\partial y^2}\right]\right) d\xi \\[4mm]
u_{2_{n+1}}(x, y, z, t) = u_{2_n}(x, y, z, t) + \int_0^t (\xi - t)\left(\frac{\partial^\alpha}{\partial t} u_{2_n}(x, y, z, \xi) - x^2 - y^2 - z^2 \right. \\[2mm]
\qquad \left. -\frac{1}{2}\left[x^2 \frac{\partial^2 u_{1_n}(x, y, z, \xi)}{\partial x^2} + y^2 \frac{\partial^2 u_{1_n}(x, y, z, \xi)}{\partial y^2} + z^2 \frac{\partial^2 u_{1_n}(x, y, z, \xi)}{\partial z^2}\right]\right)
\end{cases}
\tag{23}
$$

To get the iteration, we start with an initial approximation

$$\begin{cases} u_{1,0}(x,y,t) = x^4 + y^4 t \\[2ex] u_{2,0}(x,y,t) = (x^2 + y^2)\left(t + \frac{t^\alpha}{\Gamma(\alpha+1)}\right) + z^2\left(-t + \frac{t^\alpha}{\Gamma(\alpha+1)}\right) \end{cases},$$

we obtain the following successive approximations as follows

$$\begin{cases} u_{1,1}(x,y,t) = u_{10}(x,y,t) + \frac{1}{6}y^4 t^3 + \frac{1}{2}x^4 t^2 \\[2ex] u_{2,1}(x,y,z,t) = u_{20}(x,y,z,t) + (x^2 + y^2 - z^2)\frac{t^3}{6} + (x^2 + y^2 + z^2)\frac{t^{(2+\alpha)}}{\Gamma(3+\alpha)} \end{cases},$$

$$\begin{cases} u_{1,2}(x,y,t) = x^4 + y^4 t + \frac{1}{3}y^4 t^3 + \frac{1}{2}x^4 t^2 + \frac{t^4 y^4}{24} + \frac{t^5 y^4}{120} + \frac{x^4 t^{(5-\alpha)}}{\Gamma(3-\alpha)(4-\alpha)} \\[3ex] \qquad\qquad + \frac{y^4 t^{(5-\alpha)}}{\Gamma(5-\alpha)(5-\alpha)} - \frac{x^4 t^{(4-\alpha)}}{\Gamma(3-\alpha)(4-\alpha)} - \frac{y^4 t^{(5-\alpha)}}{\Gamma(4-\alpha)(4-\alpha)} \\[3ex] u_{2,2}(x,y,z,t) = \ldots \end{cases}$$

8. Conclusions

In this paper, the variation iteration method (VIM) has been successfully employed to obtain the approximate analytical solutions of fractional system of heat-and wave-like Equations. The results show that He's variational iteration method is very effective and convenient for fractional problem.

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