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IMPULSIVE DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH INFINITE DELAY

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Abstract. This paper deals with the existence of solutions to partial functional differential equations with impulses and infinite delay, involving the Caputo fractional derivative. Our works will be conducted by using Burton-Kirk fixed point theorem.

1. Introduction

In this paper, we shall be concerned with the existence of solutions for the following impulsive partial hyperbolic differential equations:

$$({}^cD^r_{z_k}u)(x,y)=f(x,y,u_{(x,y)});\ \ \text{if } (x,y)\in J_k,\quad k=0,\dots,m, \tag{1}$$

$$u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y)),\quad \text{if } y\in[0,b],\ k=1,\dots,m, \tag{2}$$

$$u(x,y)=\phi(x,y);\ \text{if } (x,y)\in\tilde{J}, \tag{3}$$

$$u(x,0)=\varphi(x),\ x\in[0,a],\ u(0,y)=\psi(y);\ y\in[0,b], \tag{4}$$

where $J_0=[0,x_1]\times[0,b]$, $J_k:=(x_k,x_{k+1}]\times[0,b];\ k=1,\dots,m,\ z_k=(x_k,0),\ k=0,\dots,m,\ a,b>0,\ J=[0,a]\times[0,b],\ \tilde{J}=(-\infty,a]\times(-\infty,b]\backslash(0,a]\times(0,b]$, ${}^cD^r_{x_k}$ is the Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $\varphi:[0,a]\to\mathbb{R}^n$, $\psi:[0,b]\to\mathbb{R}^n$ are given continuous functions with $\varphi(x)=\phi(x,0),\ \psi(y)=\phi(0,y)$ for each $(x,y)\in J,\ \ 0=x_0<x_1<\cdots<x_m<x_{m+1}=a,\ f:J\times\mathcal{B}\to\mathbb{R}^n,\ I_k:\mathbb{R}^n\to\mathbb{R}^n,\ k=1,\dots,m,\ \phi:\tilde{J}\to\mathbb{R}^n$, are given functions. $\mathcal{B}$ is called a phase space that will be specified in the next Section. If $u:(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$, then for any $(x,y)\in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t)=u(x+s,y+t),\ \text{for } (s,t)\in[-\alpha,0]\times[-\beta,0].$$

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order in spaces of integrable functions was studies in numerous works. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. ([13, 14, 20, 26, 30]). There has been a significant development in

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ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5], Kilbas *et al.* [22], Podlubny [27], the papers of Abbas and Benchohra [1, 2], Agarwal *et al.* [6], Benchohra *et al.* [7, 8], Vityuk and Golushkov [31] and the references therein. In [3] Abbas and Benchohra considered the existence of solutions to the fractional order initial value problem

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(x,y)}),\ \text{if } (x,y)\in J, \tag{5}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y)\in \tilde{J}, \tag{6}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ (x,y)\in J. \tag{7}$$

In [4], the same authors provided sufficient conditions for the existence and uniqueness of solutions to the following fractional order implicit differential system

$$\overline{D}_{z_k}^r u(x,y) = f(x,y,u(x,y),\overline{D}_{z_k}^r u(x,y));\quad \text{if } (x,y)\in J_k,\quad k=0,\dots,m, \tag{8}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y));\quad \text{if } y\in[0,b],\ k=1,\dots,m, \tag{9}$$

$$\begin{cases} u(x,0)=\varphi(x);\ x\in[0,a],\\ u(0,y)=\psi(y);\ y\in[0,b],\\ \varphi(0)=\psi(0). \end{cases} \tag{10}$$

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [17], Hino *et al.* [21], Kolmanovskii and Myshkis [23], Lakshmikantham *et al.* [25], Smith [29], and Wu [32], and the papers [10, 16].

The theory of impulsive integer order differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [7], Lakshmikantham *et al.* [24], and Samoilenko and Perestyuk [28], and the references therein.

Motivated by the papers [3, 4], in this paper we present existence results for the problem (1)-(4). Our approach is based on Burton-Kirk fixed point theorem [9]. The present results complement and extend those devoted to problems without impulses.

2. The phase space $\mathcal{B}$

The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [16] (see [17, 21, 25]).

For any $(x,y)\in J$ denote $E_{(x,y)} := [0,x]\times\{0\}\cup\{0\}\times[0,y]$, furthermore in case $x=a,\ y=b$ we write simply E. Consider the space $(\mathcal{B},\|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0]\times(-\infty,0]$ into $\mathbb{R}^n$ satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $z:(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ and $z_{(x,y)}\in\mathcal{B}$, for all $(x,y)\in E$, then there are constants $H,K,M>0$ such that for any $(x,y)\in J$ the following conditions hold:

(i) $z_{(x,y)}$ is in $\mathcal{B}$;

(ii) $\|z(x,y)\|\le H\|z_{(x,y)}\|_{\mathcal{B}}$,

(iii) $\|z_{(x,y)}\|_B\le K\sup_{(s,t)\in[0,x]\times[0,y]}\|z(s,t)\|+M\sup_{(s,t)\in E_{(x,y)}}\|z_{(s,t)}\|_{\mathcal{B}}$,

(A_2) The space $\mathcal{B}$ is complete.

For examples of phase spaces we refer, for instance to ([3, 11, 12]).

3. Preliminaries

In this section, we introduce notations and definitions which are used throughout this paper. By $L^1(J,\mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u:J\to\mathbb{R}^n$ with the norm

$$\|u\|_{L^1}=\int_0^a\int_0^b\|u(x,y)\|dydx.$$

Let $C(J,\mathbb{R}^n)$ be the space of continuous functions $u:J\to\mathbb{R}^n$ with the norm

$$\|u\|_\infty=\sup_{(x,y)\in J}\|u(x,y)\|.$$

Definition 3.1. *([22]): Let $r_1,\ r_2>0$ and $r=(r_1,r_2)$. For $u\in L^1(J,\mathbb{R}^n)$, the expression*

$$(I^r_{z_k}u)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}u(s,\tau)d\tau ds,$$

where $\Gamma(.)$ is the gamma function, is called the left-sided mixed Riemann-Liouville integral of order r.

Definition 3.2. *([22]): For $u\in L^1(J,\mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression*

$$({}^cD^r_{z_k}u)(x,y)=(I^{1-r}_{z_k}\frac{\partial^2}{\partial t\partial x}u)(x,y).$$

We need the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 3.3. *([18]) Let $\upsilon:J\to[0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c>0$ and $0<r_1,r_2<1$ such that*

$$\upsilon(x,y)\le\omega(x,y)+c\int_0^x\int_0^y\frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

then there exists a constant $\delta=\delta(r_1,r_2)$ such that

$$\upsilon(x,y)\le\omega(x,y)+\delta c\int_0^x\int_0^y\frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

for every $(x,y)\in J$.

Theorem 3.4. *(Burton-Kirk)([9]) Let X be a Banach space, and $A,B:X\to X$ two operators satisfying:*

(i) A is completely continuous, and

(ii) B is a contraction.

Then either

(a) *the operator equation* $u = A(u) + B(u)$ *has a solution, or*
(b) *the set* $\mathcal{E} = \{u \in X : u = \lambda A(u) + \lambda B(\frac{u}{\lambda})\}$ *is unbounded for* $\lambda \in (0,1)$.

4. Auxiliary Results

To define the solutions of problem (1)-(4), we shall consider the space

$$\Omega = \big\{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n : u_{(x,y)} \in \mathcal{B} \text{ for } (x,y) \in E \text{ and there exist } u(x_k^-, .),\ u(x_k^+, .) \text{ exist with } u(x_k^-, .) = u(x_k, .);\ k = 1, \dots, m, \text{ and } u \in C(J_k, \mathbb{R}^n); k = 0, \dots, m\big\},$$

where $J_k = (x_k, x_{k+1}] \times (0, b]$. Let us define what we mean by a solution of problem (1)-(4). Set

$$J' := J \setminus \{(x_1, y), \dots, (x_m, y), y \in [0, b]\}.$$

For $u \in \Omega$, we denote by $\tilde{u}_k$, for $k = 0, 1, \dots, m$, the function $\tilde{u}_k \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$ given by $\tilde{u}_k(x, y) = u(x, y)$ for $(x, y) \in (x_k, x_{k+1}] \times [0, b]$ and $\tilde{u}_k(x_k, y) = \lim_{k \to x_k^+} u(x, y)$. Moreover, for a set $D \subset \Omega$, we represent by $\tilde{D}_k$, for $k = 0, 1, \dots, m$ the set $\tilde{D}_k = \{\tilde{u}_k : u \in D\}$.

Lemma 4.1. [19] *A set* $D \subset \Omega$ *is relatively compact if and only if, each set* $\tilde{D}_k, k = 0, 1, \dots, m$*, is relatively compact in* $C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$.

Definition 4.2. A function $u \in \Omega$ is said to be a solution of (1)-(4) if u satisfies $({}^cD_0^r u)(x, y) = f(x, y, u(x, y))$ on J' and conditions (2), (3) and (4) are satisfied.

Let $h \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, and

$$\mu_k(x, y) = u(x, 0) + u(x_k^+, y) - u(x_k^+, 0), \quad k = 0, \dots, m.$$

For the existence of solutions for the problem $(1)-(4)$, we need the following lemma:

Lemma 4.3. *A function* $u \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$; $k = 0, \dots, m$ *is a solution of the differential equation*

$$({}^cD_{z_k}^r u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b],$$

if and only if $u(x, y)$ *satisfies*

$$u(x, y) = \mu_k(x, y) + (I_{z_k}^r h)(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{11}$$

Proof: Let $u(x, y)$ be a solution of $({}^cD_{z_k}^r u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b]$. Then, taking into account the definition of the derivative $({}^cD_{z_k}^r u)(x, y)$, we have

$$I_{z_k}^{1-r}(D_{xy}^2 u)(x, y) = h(x, y).$$

Hence, we obtain

$$I_{z_k}^r(I_{z_k}^{1-r} D_{xy}^2 u)(x, y) = (I_{z_k}^r h)(x, y),$$

then

$$I_{z_k}^1 D_{xy}^2 u(x, y) = (I_{z_k}^r h)(x, y).$$

Since

$$I_{z_k}^1(D_{xy}^2 u)(x, y) = u(x, y) - u(x, 0) - u(x_k^+, y) + u(x_k^+, 0),$$

we have

$$u(x, y) = \mu_k(x, y) + (I_{z_k}^r h)(x, y).$$

Now let $u(x,y)$ satisfies (11). It is clear that $u(x,y)$ satisfies

$$({}^cD_0^r u)(x,y) = h(x,y), \text{ on } [x_k, x_{k+1}] \times [0,b].$$

Lemma 4.4. [5] *Let $0 < r_1, r_2 \le 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \phi(x,y) & \text{if } (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum\limits_{0<x_k<x} \Big(I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0)) \Big) & \text{if } (x,y) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds & k = 1,\dots,m, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \end{cases} \tag{12}$$

if and only if u is a solution of the fractional initial value problem

$$^cD_{z_k}^r u(x,y) = h(x,y), \quad (x,y) \in J_k, \ k = 0,\dots,m, \tag{13}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \quad k = 1,\dots,m. \tag{14}$$

5. Main Result

Our main result in this section is based upon the fixed point theorem due to Burton and Kirk. Let us introduce the following hypotheses which are assumed hereafter.

- $(H1)$ The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$, and $f : J \times \mathcal{B} \to \mathbb{R}^n$ are continuous.
- $(H2)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that

$$\|f(t,x,u)\| \le p(t,x) + q(t,x)\|u\|_{\mathcal{B}}, \text{ for } (t,x) \in J \text{ and each } u \in \mathcal{B}.$$

- $(H3)$ There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \le l\|u - v\| \text{ for each } u, v \in \mathbb{R}^n.$$

Theorem 5.1. *Assume that hypotheses $(H1)$-$(H3)$ hold. If*

$$2ml < 1, \tag{15}$$

then the IVP (1)-(4) has at least one solution on J.

Proof. We shall reduce the existence of solutions of (1)-(4) to a fixed point problem. Consider the operator $N : \Omega \longrightarrow \Omega$ defined by

$$N(u)(x,y) = \begin{cases} \phi(x,y) & \text{if } (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum\limits_{0<x_k<x} \big(I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0)) \big) & \text{if } (x,y) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds & k = 1,\dots,m, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds. \end{cases}$$

Consider the operators $A, B : \Omega \to \Omega$ defined by,

$$A(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} & k=1,\dots,m \\ \times f(s,t,u(s,t))dtds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times f(s,t,u(s,t))dtds, & (x,y) \in J, \end{cases}$$

and

$$B(u)(x,y) = \begin{cases} 0, & (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(x_k^-,0))), & (x,y) \in J. \end{cases}$$

Let $v(.,.) : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n$ be a function defined by,

$$v(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}. \\ \mu(x,y), & (x,y) \in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in E$.
For each $w \in (J,\mathbb{R}^n)$ with $w(x,y) = 0$ for each $(x,y) \in E$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} 0, & (x,y) \in \tilde{J}, \\ w(x,y) & (x,y) \in J. \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(x,y) \quad = \quad \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})dtds,$$

we can decompose $u(.,.)$ as $u(x,y) = \overline{w}(x,y) + v(x,y);\ (x,y) \in (x_k, x_{k+1}] \times [0,b]$, which implies $u_{(x,y)} = \overline{w}_{(x,y)} + v_{(x,y)}$, for every $(x,y) \in J \times [0,b]$ and the function $w(.,.)$ satisfies

$$\begin{aligned} w(x,y) = & \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(_k^-,0))) \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(t-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds. \end{aligned}$$

Set

$$C_0 = \{w \in \Omega :\ w(x,y) = 0 \ \text{ for } (x,y) \in E\},$$

and let $\|.\|_{C_0}$ be the norm in C_0 defined by

$$\|w\|_{C_0} = \sup_{(x,y)\in E} \|w_{(x,y)}\|_{\mathcal{B}} + \sup_{(x,y)\in J} \|w(x,y)\| = \sup_{(x,y)\in J} \|w(x,y)\|,\ w \in C_0.$$

C_0 is a Banach space with norm $\|.\|_{C_0}$. Let the operators $A, B: \ C_0 \to C_0$ defined by

$$(Aw)(x,y) = \begin{cases} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1}, \quad k = 1,\dots,m \\ \times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ \times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds, \qquad (x,y) \in J, \end{cases}$$

and

$$(Bw)(x,y) = \mu(x,y) + \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(x_k^-,0))), (x,y) \in J.$$

Then the problem of finding the solution of the IVP (1)–(4) is reduced to finding the solutions of the operator equation $A(w) + B(w) = w$. We shall show that the operators A and B satisfy the conditions of Theorem 3.4. The proof will be given by a couple of steps.

Step 1: A is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0, then for each $(x,y) \in J$

$$\begin{aligned}
&\|A(w_n)(x,y) - A(w)(x,y)\| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|f(s,t,\overline{w}_{n(s,t)} + v_{n(s,t)}) - f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\| dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|f(s,t,\overline{w}_{n(s,t)} + v_{n(s,t)}) - f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\| dtds. \\
&\le \frac{\|f(.,.,\overline{w}_{n(.,.)} + v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)} + v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&+ \frac{\|f(.,.,\overline{w}_{n(.,.)} + v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)} + v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds.
\end{aligned}$$

Since f is continuous function, we have

$$\|A(w_n) - A(w)\|_{C_0} \le \frac{2a^{r_1}b^{r_2}\|f(.,.,w_{n(.,.)}) - f(.,.,w_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n \to \infty$$

Thus A is continuous.

Step 2: A maps bounded sets into bounded sets in C_0.
Indeed, it is enough show that for any η^*, there exists a positive constant l such that, for each $w \in B_{\eta^*} = \{w \in C_0 : \|w\|_{C_0} \le \eta^*\}$ we have $\|A(w)\|_{C_0} \le l$. By $(H2)$

we have for each $(x,y)\in(x_k,x_{k+1}]\times[0,b]$,

$$\begin{aligned}\|A(w)(x,y)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)}+v_{(s,t)})\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)}+v_{(s,t)})\|dtds\\ &\leq \frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}dtds\\ &+\frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds.\end{aligned}$$

Thus

$$\|A(w)\|_B \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty+\|q\|_\infty\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := l,$$

where

$$\begin{aligned}\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,t)}\|_{\mathcal{B}}+\|v_{(s,t)}\|_{\mathcal{B}}\\ &\leq K\eta^*+K\|\phi(0,0)\|+M\|\phi\|_{\mathcal{B}} := \eta.\end{aligned}$$

Hence $\|A(w)\|_{C_0}\leq l$.

Step 3: A maps bounded sets into equicontinuous sets in C_0.
Let $(x_1,y_1),(x_2,y_2)\in(0,a]\times(0,b]$, $x_1<x_2$, $y_1<y_2$, B_{η^*} be a bounded set as in

Step 2. Let $w \in B_{\eta^*}$, then

$$
\begin{aligned}
&\|A(w)(x_2,y_2) - A(w)(x_1,y_1)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1-1}[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}] \\
&\times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \\
&\times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtdx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k-s)^{r_1-1}[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}]dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k-s)^{r_1-1}[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}]dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} [2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2} \\
&+ x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}
$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $A : C_0 \to C_0$ is continuous and completely continuous.

Step 4: B is a contraction.
Let $w, w^* \in C_0$, then for each $(x,y) \in J$, we have

$$\begin{aligned}
&\|B(w)(x,y) - B(w^*)(x,y)\| \\
&\leq \sum_{k=1}^{m}(\|I_k(w(x_k^-,y)) - I_k(w^*(x_k^-,y))\| + \|I_k(w(x_k^-,0)) - I_k(w^*(x_k^-,0))\|) \\
&\leq \sum_{k=1}^{m} l(\|w-w^*\|_{C_0} + \|w-w^*\|_{C_0}) \\
&\leq 2ml\|w-w^*\|_{C_0}.
\end{aligned}$$

Thus

$$\|B(w) - B(w^*)\|_{C_0} \leq 2ml\|w-w^*\|_{C_0}.$$

Hence by (15), B is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E} = \{w \in C_0 : w = \lambda B\left(\frac{w}{\lambda}\right) + \lambda A(w) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $w \in \mathcal{E}$, then $w = \lambda B\left(\frac{w}{\lambda}\right) + \lambda A(w)$. Thus, for each $(x,y) \in J$ we have

$$\begin{aligned}
w(x,y) &= \lambda \sum_{k=1}^{m}(\|I_k \frac{(u(x_k^-,y))}{\lambda})\| + \|I_k \frac{(u(x_k^-,0))}{\lambda}\|) \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dt ds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dt ds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(x, y) \in J$, we have

$$\begin{aligned}
\|w(x,y)\| &\leq \sum_{k=1}^{m} \lambda(\|I_k \frac{u(x_k^-, y)}{\lambda}\| - \|I_k(0)\| + \|I_k \frac{u(x_k^-, 0)}{\lambda}\| - \|I_k(0)\|) \\
&+ 2\lambda \sum_{k=1}^{m} \|I_k(0)\| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&\leq l \sum_{k=1}^{m} (\|u(t_k^-, x)\| + \|u(t_k^-, 0)\|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds,
\end{aligned}$$

where

$$I^* = \sum_{k=1}^{m} \|I_k(0)\|,$$

and

$$\begin{aligned}
\|\overline{w}_{(s,t)} + v_{(s,t)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,t)}\|_{\mathcal{B}} + \|v_{(s,t)}\|_{\mathcal{B}} \\
&\leq K \sup\{w(\tilde{s}, \tilde{t}) : (\tilde{s}, \tilde{t}) \in [0, s] \times [0, t]\} \\
&\quad + M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\|. \qquad (16)
\end{aligned}$$

If we name $\gamma(s, t)$ the right hand side of (16), then we have

$$\|\overline{w}_{(s,t)} + v_{(s,t)}\|_{\mathcal{B}} \leq \gamma(x, y),$$

and therefore, for $\gamma(x, y) \in J$ we obtain

$$\begin{aligned}
\|w(t,x)\| &\leq l \sum_{k=1}^{m} (\|u(t_k^-, x)\| + \|u(t_k^-, 0)\|) + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \Big(\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \gamma(s,\tau) d\tau ds
\end{aligned}$$

$$+\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big). \tag{17}$$

Using the above inequality and the definition of γ for each $(x,y)\in J$ we have

$$\begin{aligned}\gamma(t,x) &\leq M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\ &+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\left(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\right. \\ &+ \left.\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\right) + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}.\end{aligned}$$

If $(t,x)\in J$, then Lemma 3.3 implies that there exists $\tilde{k}=\tilde{k}(r_2,r_2)$ such that

$$\begin{aligned}\gamma(t,x) &\leq \left(M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|)\right. \\ &+ \left. 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \\ &\times \left(1+\tilde{k}\frac{\|p\|_\infty}{\Gamma(r_2)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\right) \\ &\leq \left(M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|)\right. \\ &+ \left. 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \\ &\times \left(1+\tilde{k}\frac{\|p\|_\infty a^{r_1}b^{r_2}}{\Gamma(r_2+1)\Gamma(r_2+1)}\right) := \tilde{R}.\end{aligned}$$

Since for every $(t,x)\in J$, $\|w_{(t,x))}\|_\infty \leq \gamma(x,y)$.
This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.4 we deduce that $A+B$ has a fixed point which is a solution of problem (1)-(4).

6. Example

Consider the following impulsive partial hyperbolic functional differential equations

$$({}^cD^r_{z_k}u)(x,y) = \frac{e^{-x-y}}{9+e^{x+y}}\frac{2+|u(x,y)|}{(1+|u(x,y)|)},\ (x,y)\in J = \left[0,\frac{1}{2}\right]\times[0,1]\cup\left(\frac{1}{2},1\right]\times[0,1], \tag{18}$$

$$u\left(\frac{1}{2}^+,y\right) = u\left(\frac{1}{2}^-,y\right) + \frac{\left|u\left(\frac{1}{2}^-,y\right)\right|}{\frac{1}{4}+\left|u\left(\frac{1}{2}^-,y\right)\right|},\ \text{if } y\in[0,1], \tag{19}$$

$$u(x,y) = x+y^2,\ \text{if } (x,y)\in[-1,1]\times[-2,1]\setminus(0,1]\times(0,1], \tag{20}$$

$$u(x,0) = x,\ u(0,y) = y^2,\ x\in[0,1],\ y\in[0,1], \tag{21}$$

where $z_0=(0,0), z_1=\left(\frac{1}{2},0\right)$. Let $\gamma\in\mathbb{R}$, and C_γ be the set of all piece-wise continuous functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty}e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm

$$\|\phi\|_{C_\gamma}=\sup_{(s,t)\in(-\infty,0]\times(-\infty,0]}e^{\gamma(s+t)}\|\phi(s,t)\|.$$

Set

$$f(x,y,\varphi)=\frac{e^{-x-y}(2+|\varphi|)}{(9+e^{x+y})(1+|\varphi|)},\ (x,y)\in[0,1]\times[0,1],\ \varphi\in C,$$

and

$$I_1(u)=\frac{|u|}{\frac{1}{4}+|u|},\ u\in\mathbb{R}.$$

It is clear that the functions f and I_1 are continuous, and for $(x,y)\in[0,1]\times[0,1]$ and $\varphi\in C$, we have

$$|f(x,y,\varphi)|\le\frac{e^{-x-y}}{9+e^{x+y}}(2+|\varphi|).$$

Hence (H2) is satisfied with

$$p(x,y)=\frac{2e^{-x-y}}{9+e^{x+y}}\quad\text{and}\quad q(x,y)=\frac{e^{-x-y}}{9+e^{x+y}}.$$

Also, for $u_1,u_2\in\mathbb{R}$, we have

$$\begin{aligned}|I_1(u_1)-I_1(u_2)| &= \left|\frac{|u_1|}{\frac{1}{4}+|u_1|}-\frac{|u_2|}{\frac{1}{4}+|u_2|}\right|\\ &= \frac{1}{4}\left|\frac{|u_1|-|u_2|}{(\frac{1}{4}+|u_1|)(\frac{1}{4}+|u_2|)}\right|\\ &\le \frac{1}{4}|u_1-u_2|.\end{aligned}$$

Thus (H3) is satisfied with $l=\frac{1}{4}$. Finally condition (15) is satisfied. Theorem 5.1 implies that problem (18)-(21) has at least one solution defined on $(-\infty,1]\times(-\infty,1]$.

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