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SOME PROPERTIES OF A CERTAIN SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH AN EXTENDED FRACTIONAL DIFFERINTEGRAL OPERATOR

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Abstract. Making use of an extended fractional differintegral operator (introduced recently by Patel and Mishra), we introduce a new subclass of multivalent analytic functions. Such results as subordination and superordination properties, convolution properties, inequality properties and other interesting properties of this subclass are proved.

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z : z \in C \text{ and } |z| < 1\}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_k z^k + a_{k+1}z^{k+1} + ...$, with $H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$.
Let $A_p(k)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_{n+p} z^{n+p} \left(p,\, k \in \mathrm{N} = \{1,2,3,...\}; z \in U\right), \tag{1}$$

which are analytic in the open unit disk U, and let $A_p(1) = A_p$ and $A_1(1) = A$. A function $f(z) \in A_p(k)$ is said to be in the class $S^*_{p,k}(\rho)$ of multivalent (p-valent) starlike of order $\rho(0 \leq \rho < p)$, if it satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \left(0 \leq \rho < p, z \in U\right). \tag{2}$$

Let f and F be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$(see [8,9]).

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For two functions $f(z)$ given by (1) and

$$g(z) \,=\, z^p \,+\, \sum_{n=k}^{\infty} b_{n+p} z^{n+p}, \tag{3}$$

The hadmard product (or convolution) of f and g is defined by

$$(f * g)(z) \,=\, z^p \,+\, \sum_{n=k}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z). \tag{4}$$

In [11] (see also [12] and [16]), Owa introduced the following definitions of fractional calculus (that is, fractional integrals and fractional derivatives of an arbitrary order).

Definition 1 Let the function $f(z)$ be analytic in a simply connected region (of the $z-$plane) containing the origin and let $\alpha > 0$, then the fractional integral of order α is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\alpha}} \, d\zeta \quad (\alpha > 0), \tag{5}$$

where the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2 Let the function $f(z)$ be analytic in a simply connected region (of the $z-$plane) containing the origin and let $0 \leq \alpha < 1$, then the fractional derivative of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} \, d\zeta \quad (0 \leq \alpha < 1), \tag{6}$$

where the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3 Under the hypotheses of Definition 2, the fractional derivative of order $\alpha + n$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^{\alpha} f(z) \quad (n \leq \alpha < n+1;\; n \in N_0 = N \cup \{0\}). \tag{7}$$

Very recently, Patel and Mishra [13] defined the extended fractional differintegral operator $\Omega_z^{(\alpha,p)} : A_p(k) \to A_p(k)$ for a function $f(z) \in A_p(k)$ and for a real number $\alpha\,(-\infty < \alpha < p+1)$ by

$$\Omega_z^{(\alpha,p)} f(z) = \frac{\Gamma(p-\alpha+1)}{\Gamma(p+1)} z^{\alpha} D_z^{\alpha} f(z), \tag{8}$$

where $D_z^{\alpha} f$ is, respectively the fractional integral of f of order $-\alpha$ when $-\infty < \alpha < 0$ and the fractional derivative of f of order α if $0 \leq \alpha < p+1$.
It is easily seen from (8) that for a function $f(z)$ of the form (1), we have

$$\Omega_z^{(\alpha,p)} f(z) = z^p \,+\, \sum_{n=k}^{\infty} \frac{\Gamma(n+p+1)\,\Gamma(p-\alpha+1)}{\Gamma(p+1)\,\Gamma(n+p-\alpha+1)} a_{n+p} z^{n+p} \qquad (z \in U), \tag{9}$$

and

$$\left(z\Omega_z^{(\alpha,p)}f(z)\right)' = (p-\alpha)\,\Omega_z^{(\alpha+1,p)}f(z)+\alpha\Omega_z^{(\alpha,p)}f(z)\,(-\infty<\alpha<p;\ z\in U)\,. \quad (10)$$

The fractional differential operator $\Omega_z^{(\alpha,p)}$ with $0 \le \alpha < 1$ was investigated by Srivastava and Aouf [17] and studied by Srivastava and Mishra [18]. We, further observe that $\Omega_z^{(\alpha,1)} = \Omega_z^{\alpha}$ is the operator introduced and studied by Owa and Srivastava [12].

By making use of the differintegral operator $\Omega_z^{(\alpha,p)}$ and the above mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class $A_p(k)$ of p-valent analytic functions.

Definition 4 A function $f(z) \in A_p(k)$ is said to be in the class $S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ if it satisfies the following subordination condition:

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+Az}{1+Bz}, \quad (11)$$

$(-\infty<\alpha<p;\ -1\le B<A\le 1;\ A\ne B;\ A\in \mathrm{R};\ p,k\in \mathrm{N};\ \lambda\in \mathrm{C}$ and $Re(\mu)>0)$.

It may be noted that for suitable choice of μ, A, B, p, λ and α the class $S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ extends several classes of analytic and p-valent functions studied by several authors such as Aouf and Seoudy [3], Yang [19], Zhou and Owa [20] and Liu [4].

To prove our results, we need the following definitions and lemmas.

Definition 5 ([8]). Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\bar{U}/E(f)$ where

$$E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \ne 0$ for $\zeta \in \partial U/E(f)$.

Lemma 1 ([9]). Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0) = 1$. Suppose also that the function $g(z)$ given by

$$g(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + ... \quad (12)$$

is analytic in U. If

$$g(z) + \frac{zg'(z)}{\gamma} \prec h(z) \quad (\Re(\gamma) > 0;\ \gamma \ne 0;\ z \in U)\,, \quad (13)$$

then

$$g(z) \prec q(z) = \frac{\gamma}{k} z^{-\frac{\gamma}{k}} \int h(t)\, t^{\frac{\gamma}{k}}\, dt \prec h(t),$$

and $q(z)$ is the best dominant of (13).

Lemma 2 ([15]). Let $q(z)$ be a convex univalent function in U and let $\alpha \in \mathrm{C}, \eta \in \mathrm{C}^* = \mathrm{C}\backslash\{0\}$ with

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ is analytic in U and

$$\sigma g(z) + \eta z g'(z) \prec \sigma q'(z) + \eta z q'(z)\,,$$

then $g(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 ([9]). Let $q(z)$ be convex univalent function in U and let $k \in$ C. Further assume $Re(k) > 0$. If $g(z) \in H[q(0), 1] \bigcap Q$, and $g(z) + kzg'(z)$ is univalent in U, then

$$q(z) + kzq'(z) \prec g(z) + kzg'(z)\,,$$

implies $g(z) \prec q(z)$ and $q(z)$ is the best subordinate.

Lemma 4 ([16]). Let the function F be analytic and convex in U. If $f, g \in A$ and $f, g \prec F$, then $\lambda f + (1-\lambda)g \prec F \quad (0 \leq \lambda \leq 1)$.

Lemma 5 ([14]). Let $f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$, be analytic in U and $g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ be analytic and convex in U. If $f(z) \prec g(z)$, then

$$|a_k| < |b_1| \qquad (k \in \mathrm{N})\,.$$

Lemma 6 ([6]). Let $0 \neq \delta \in \mathrm{R},\ \frac{p}{\delta} > 0,\ 0 \leq \rho < 1,\ g(z) \in H[1,k]$ and

$$g(z) \prec +Lz \quad \left(L = \frac{\nu M}{k\delta + \nu}\right),$$

where

$$M = M_k(\delta, \nu, \rho) = \frac{(1-\rho)\,|\delta|\left(1 + \frac{k\delta}{\nu}\right)}{|1 - \delta + \rho\delta| + \sqrt{1 + \left(1 + \frac{k\delta}{\nu}\right)^2}}.$$

If $h(z) \in H[1,k]$ satisfies the following subordination condition;

$$g(z)\left[1 - \delta + \delta(1-\rho)h(z) + \rho\right] \prec 1 + Mz,$$

then

$$\Re(h(z)) > 0 \quad (z \in U).$$

In the present paper, we aim to prove some subordination and superordination properties, convolution properties associated with the fractional differintegral operator $\Omega_z^{(\alpha,p)}$. Sandwich-type result involving this operator is also derived . A similar problem for analytic functions was studied by Aouf and Seoud [3] and Muhamad [10].

2. MAIN RESULT

Theorem 1 Let $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$. Then

$$\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \prec q(z) = \frac{(p-\alpha)\mu}{\lambda k} \int_0^1 \frac{1 + Azu}{1 + Bzu}\, u^{\frac{(p-\alpha)\mu}{\lambda k} - 1} du \prec \frac{1 + Az}{1 + Bz}\,, \qquad (14)$$

and $q(z)$ is the best dominant.

Proof. Define the function $g(z)$ by

$$g(z) = \left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \ (z \in U). \qquad (15)$$

Then $g(z)$ is of the form (12) and analytic in U. Differentiating (14) with respect to z and using (10), we get

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}=g(z)+\frac{\lambda zg'(z)}{(p-\alpha)\mu}$$
$$\prec\frac{1+Az}{1+Bz} \tag{16}$$

Applying Lemma 1 to (16) with $\gamma=\frac{(p-\alpha)\mu}{\lambda}$, we get

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec q(z)=\frac{(p-\alpha)\mu}{\lambda k}\int_0^z\frac{1+At}{1+Bt}\,t^{\frac{(p-\alpha)\mu}{\lambda k}-1}dt$$
$$=\frac{(p-\alpha)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\prec\frac{1+Az}{1+Bz}, \tag{17}$$

and $q(z)$ is the best dominant.

Theorem 2 Let $q(z)$ be univalent function in U and let $\lambda\in\mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0,\,-\Re\left(\frac{(p-\alpha)\mu}{\lambda}\right)\right\}. \tag{18}$$

If $f\in A_p$ satisfies the following subordination:

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec q(z)+\frac{\lambda zq'(z)}{(p-\alpha)\mu}, \tag{19}$$

then

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (15). we know that (16) holds true. Combining (16) and (19), we find that

$$g(z)+\frac{\lambda zg'(z)}{(p-\alpha)\mu}\prec q(z)+\frac{\lambda zq'(z)}{(p-\alpha)\mu}. \tag{20}$$

By using Lemma 2 and (20), we easily get the assertion of Theorem 2.
Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 2, we get the following result.

Corollary 1 Let $\lambda\in\mathbb{C}^*$ and $-1\le B<A\le 1$. Suppose also that

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0,\,-\Re\left(\frac{(p-\alpha)\mu}{\lambda}\right)\right\}.$$

If $f\in A_p$ satisfies the following subordination:

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}$$
$$\prec\frac{1+Az}{1+Bz}+\frac{\lambda}{(p-\alpha)\mu}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 3 Let $q(z)$ be convex univalent function in U and let $\lambda \in$ C with $\Re(\lambda) > 0$. Also let

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu \in H[q(0),\,1]\bigcap Q,$$

and

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu$$

be univalent in U. If

$$q(z) + \frac{\lambda z q'(z)}{(p-\alpha)\mu} \prec (1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu,$$

then

$$q(z) \prec \left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu,$$

and $q(z)$ is the best subordinate.

Proof. Let the function $g(z)$ be defined by (15). Then

$$q(z) + \frac{\lambda z q'(z)}{(p-\alpha)\mu} \prec (1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu$$

$$= g(z) + \frac{\lambda z g'(z)}{(p-\alpha)\mu}.$$

By using Lemma 3 we easily get the assertion of theorem 3.
Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3, we get the following result.

Corollary 2 Let $q(z)$ be convex univalent function in U and $-1 \le B < A \le 1$, $\lambda \in$ C with $\Re(\lambda) > 0$. Also let

$$0 \ne \left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu \in H[q(0),\,1]\bigcap Q,$$

and

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu$$

be univalent in U. If

$$\frac{1+Az}{1+Bz} + \frac{\lambda}{(p-\alpha)\mu}\frac{(A-B)z}{(1+Bz)^2} \prec (1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu,$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^\mu,$$

and $\frac{1+Az}{1+Bz}$ is the best subordinate.
Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

Corollary 3 Let $q_1(z)$ be convex function in U and let $q_2(z)$ be univalent function in U, let $\lambda \in \mathrm{C}$ with $\Re(\lambda) > 0$. let $q_2(z)$ satisfy (18). If

$$0 \neq \left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \in H[q(0),\, 1] \bigcap Q,$$

and

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} + \lambda \left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu}$$

is univalent in U, and also

$$q_1(z) + \frac{\lambda z q_1'(z)}{(p-\alpha)\mu} \prec (1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} + \lambda \left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu}$$

$$\prec q_2(z) + \frac{\lambda z q_2'(z)}{(p-\alpha)\mu},$$

then

$$q_1(z) \prec \left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are respectively, the best subordinate and dominant.

Theorem 4 If $\lambda,\ \mu > 0$ and $f(z) \in S_{p,k}^{0,\mu}(\alpha; 1-2\rho, -1)(0 \leq \rho < 1)$, then $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; 1-2\rho, -1)$ for $|z| < R$, where

$$R = \left(\sqrt{\left(\frac{\lambda k}{(p-\alpha)\mu}\right)^2 + 1} - \frac{\lambda k}{(p-\alpha)\mu}\right)^{\frac{1}{k}}. \tag{21}$$

The bound R is the best possible.
Proof. We begin by writing

$$\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} = \rho + (1-\rho) g(z) \qquad (z \in U;\, 0 \leq \rho < 1). \tag{22}$$

Then, clearly, the function $g(z)$ is of the form (12), is analytic and has a positive real part in U. Differentiating (22) with respect to z and using the identity (10), we get

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} + \lambda \left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} - \rho\right\}$$

$$= g(z) + \frac{\lambda z g'(z)}{(p-\alpha)\mu}. \tag{23}$$

By making use of the following well-known estimate (see [7]):

$$\frac{|zg'(z)|}{\Re(g(z))} \leq \frac{2kr^k}{1-r^{2k}} \qquad (|z| < r < 1)$$

In (23), we obtain that

$$\Re\left(\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}-\rho\right\}\right)$$
$$\geq \Re\{g(z)\}\left(1-\frac{2kr^k\lambda}{(p-\alpha)\mu(1-r^{2k})}\right). \tag{24}$$

It is seen that the right-hand side of (24) is positive, provided that $r < R$, where R is given by (21).

In order to show that the bound R is the best possible, we consider the function $f(z) \in A_p(k)$ defined by

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}=\rho+(1-\rho)\frac{1+z^k}{1-z^k} \qquad (z\in U;\ 0\leq\rho<1).$$

Noting that

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}-\rho\right\}$$
$$=\frac{1+z^k}{1-z^k}+\frac{2k\lambda z^k}{(p-\alpha)\mu(1+z^k)^2}=0. \tag{25}$$

for $|z| < R$,we conclude that the bound is the best possible. Theorem 4 is thus proved.

Theorem 5 Let$f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with$\Re(\lambda) > 0$. Then

$$f(z)=\left(z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\frac{1}{\mu}}\right)*\left(z^p+\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\sum_{n=k}^{\infty}\frac{\Gamma(n+p-\alpha+1)}{\Gamma(n+p+1)}z^{n+p}\right), \tag{26}$$

where $w(z)$ is an analytic function with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.

Proof. Let$f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$. It follows from (14) that

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}=\frac{1+Aw(z)}{1+Bw(z)}, \tag{27}$$

where $w(z)$ is an analytic function with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$. By virtue of (27), we easily find that

$$\Omega_z^{(\alpha,p)}f(z)=z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\frac{1}{\mu}}. \tag{28}$$

Combining (9) and (28), we have

$$\left(z^p+\frac{\Gamma(p-\alpha+1)}{\Gamma(p+1)}\sum_{n=k}^{\infty}\frac{\Gamma(n+p+1)}{\Gamma(n+p-\alpha+1)}z^{n+p}\right)*f(z)=\left(z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\frac{1}{\mu}}\right) \tag{29}$$

The assertion (26) of Theorem 5 can now easily be derived from (29).

Theorem 6 Let $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$. Then

$$\frac{1}{z^p}\left[\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\left(z^p+\frac{\Gamma(p-\alpha+1)}{\Gamma(p+1)}\sum_{n=k}^{\infty}\frac{\Gamma(n+p+1)}{\Gamma(n+p-\alpha+1)}z^{n+p}\right)*f(z)\right.$$
$$\left.-z^p\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0\,(z\in U;\ 0<\theta<2\pi). \tag{30}$$

Proof. Let $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$. We know that (14) holds true, which implies that

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\ (z\in U;\ 0<\theta<2\pi). \tag{31}$$

It is easy to see that the condition (31) can be written as follows:

$$\frac{1}{z^p}\left[\Omega_z^{(\alpha,p)}f(z)\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}-z^p\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0\,(z\in U;\ 0<\theta<2\pi). \tag{32}$$

Combining (9) and (23), we easily get the convolution property (30) asserted by Theorem 6.

Theorem 7 Let $\lambda_2 \geq \lambda_1 \geq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. then

$$S_{p,k}^{\lambda_2,\mu}(\alpha; A_2, B_2) \subset S_{p,k}^{\lambda_1,\mu}(\alpha; A_1, B_1). \tag{33}$$

Proof. Let $f(z) \in S_{p,k}^{\lambda_2,\mu}(\alpha; A_2, B_2)$. Then

$$(1-\lambda_2)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda_2\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$(1-\lambda_2)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda_2\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}, \tag{34}$$

that is $f(z) \in S_{p,k}^{\lambda_2,\mu}(\alpha; A_1, B_1)$. Thus the assertion (33) holds for $\lambda_2 = \lambda_1 \geq 0$. If $\lambda_2 \geq \lambda_1 \geq 0$, by Theorem 1 and (34), we know that $f(z) \in S_{p,k}^{\lambda_0,\mu}(\alpha; A_1, B_1)$, that is

$$\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_1z}{1+B_1z}, \tag{35}$$

At the same time, we have

$$(1-\lambda_1)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda_1\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}=\left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}$$
$$+\frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda_2\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\right]. \tag{36}$$

Moreover, $0 \leq \frac{\lambda_1}{\lambda_2} < 1$, and the function $\frac{1+A_1z}{1+B_1z}(-1 \leq B_1 < A_1 \leq 1;\ z \in U)$ is analytic and convex in U. Combining (34)-(36) using Lemma 4, we find that

$$(1-\lambda_1)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}+\lambda_1\left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_1z}{1+B_1z},$$

that is $f(z) \in S_{p,k}^{\lambda_1,\mu}(\alpha; A_1, B_1)$, which implies that the assertion (33) of Theorem 7 holds.

Theorem 8 Let $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$ and $-1 \leq B < A \leq 1$. Then

$$\frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du$$

$$< \Re\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu} < \frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du. \tag{37}$$

The extremal function of (37) is defined by

$$\Omega_z^{(\alpha,p)}F(z) = z^p\left(\frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1+Azu}{1+zBu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}. \tag{38}$$

Proof. Let$f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$. From Theorem 1 we know that (14) holds true, which implies that

$$\Re\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu} < \sup_{z\in U}\Re\left\{\frac{(p-\alpha)\mu}{\lambda k} \in t_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right\}$$

$$\leq \frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du$$

$$\leq \frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du, \tag{39}$$

and

$$\Re\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu} > \inf_{z\in U}\Re\left\{\frac{(p-\alpha)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right\}$$

$$\geq \frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \inf_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du$$

$$\geq \frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du, \tag{40}$$

Combining (37) and (40), we get (37). By noting that the function $\Omega_z^{(\alpha,p)}F(z)$ defined by (38) belongs to the class $S_{p,k}^{\lambda,\mu}(\alpha; A, B)$, we obtain that equality (37) is sharp. The proof of Theorem 8 is evidently completed.
In view of Theorem 8, we easily derive the following distortion theorems for the class $S_{p,k}^{\lambda,\mu}(\alpha; A, B)$**.**

Corollary 4 Let $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$ and $-1 \leq B < A \leq 1$. Then for $|z| = r < 1$, we have

$$r^p\left(\frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1-Aur}{1-Bur}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}$$

$$< \left|\Omega_z^{(\alpha,p)}f(z)\right| < r^p\left(\frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1+Aur}{1+Bur}\,u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}. \tag{41}$$

The extremal function of (41) is defined by (38).
By noting that

$$(\Re(\nu))^{\frac{1}{2}} \leq \Re\left(\nu^{\frac{1}{2}}\right) \leq \left|\nu^{\frac{1}{2}}\right| \quad (\nu \in \mathrm{C};\ \Re(\nu) \geq 0).$$

From Theorem 8, we easily get the following results.
Corollary 5 Let $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, B)$ with $\Re(\lambda) > 0$ and $-1 \leq B < A \leq 1$. Then

$$\left(\frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\, u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}}$$

$$< \Re\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\frac{\mu}{2}} < \left(\frac{(p-\alpha)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\, u^{\frac{(p-\alpha)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}}.$$

Theorem 9 Let $f(z)$ defined by (1) be in the class $S_{p,k}^{\lambda,\mu}(\alpha; A, B)$, Then

$$|a_{n+p}| \leq \frac{\Gamma(p+1)\Gamma(k+p-\alpha+1)}{\Gamma(k+p+1)\Gamma(p-\alpha)}\left|\frac{A-B}{\lambda k+\mu(p-\alpha)}\right|. \tag{42}$$

The inequality (42) is sharp, with the extremal function defined by (38).
Proof. Combining (1) and (11), we obtain

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu}$$

$$= 1 + [\lambda k + \mu(p-\alpha)]\frac{\Gamma(k+p+1)\Gamma(p-\alpha)}{\Gamma(p+1)\Gamma(k+p-\alpha+1)}a_{p+k}z^k + ...$$

$$\prec \frac{1+Az}{1+Bz} = 1 + (A-B)z + ... \tag{43}$$

An application of Lemma 5 to (43) yields

$$\frac{\Gamma(k+p+1)\Gamma(p-\alpha)}{\Gamma(p+1)\Gamma(k+p-\alpha+1)}\left|[\lambda k + \mu(p-\alpha)]\, a_{n+p}\right| \leq |A-B|. \tag{44}$$

Thus, from (44), we easily arrive at (42) asserted by Theorem 9.
Theorem 10 Let $0 \neq \lambda \in \mathrm{R}$, $\mu \in \mathrm{R}$, $\alpha > 1$, $\frac{\mu}{\lambda} > 0$ and $0 \leq \rho < 1$. If $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, 0)$ with

$$A = \frac{(1-\rho)|\lambda|\left(1+\frac{k\lambda}{(P-\alpha)\mu}\right)}{|1-\lambda+\rho\lambda| + \sqrt{1+\left(1+\frac{k\lambda}{(P-\alpha)\mu}\right)^2}},$$

then

$$\Omega_z^{(\alpha,p)} f(z) \in S_{p,k}^{*}(p\rho - (p-\alpha)(1-\rho)).$$

Proof. Suppose that $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha; A, 0)$. By (11), we have

$$(1-\lambda)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \prec 1 + Az. \tag{45}$$

Let the function $g(z)$ be defined by (15). We then find from (14) and (45) that

$$g(z) \prec \frac{(p-\alpha)\mu}{\lambda k} z^{-\frac{(p-\alpha)\mu}{\lambda k}} \int_0^z (1+At)\, t^{\frac{(p-\alpha)\mu}{\lambda k}-1} dt$$
$$= 1 + \frac{(p-\alpha)\mu}{\lambda k + (p-\alpha)\mu} z.$$

We now suppose that

$$\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)} = (1-\rho)h(z) + \rho \quad (\alpha > 1;\ 0 \leq \rho < 1;\ z \in U). \tag{46}$$

Then $h \in H[1,k]$. It follows from (45) and (46) that

$$g(z)\left\{(1-\lambda) + \lambda[(1-\rho)h(z) + \rho]\right\} \prec 1 + Az \qquad (z \in U). \tag{47}$$

An application of Lemma 6 to (47) yields

$$\Re(h(z)) > 0 \qquad (z \in U). \tag{48}$$

Combining (46) and (48), we find that

$$\Re\left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right) = (1-\rho)\Re(h(z)) + \rho > \rho \qquad (\alpha > 1;\ 0 \leq \rho < 1;\ z \in U). \tag{49}$$

The assertion of Theorem 10 can now easily be derived from (10) and (49).

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