GENERALIZED TWO-DIMENSIONAL FRACTIONAL FOURIER TRANSFORM AND INVERSION FORMULA

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ABSTRACT. The fractional Fourier transform is a time-frequency distribution and an extension of the classical Fourier transform. There are several known applications of the fractional Fourier transform in areas of signal processing especially in signal restoration and noise removal. This paper provides an inversion formula to the generalized two-dimensional fractional Fourier transform. Relation between two-dimensional fraction Fourier transform and two-dimensional Fourier transform were obtained. Using Inversion formula, Uniqueness theorem is proved.

1. INTRODUCTION

The idea of fractional powers of the Fourier operator appears in the mathematical literature as early as [1]. It has been rediscovered in quantum mathematics [2], optics [3] and signal processing [4]. The fractional Fourier Transform (FrFT) is a generalization of the ordinary Fourier transform [4] with an order parameter α and is identical to the ordinary Fourier transform when this order α is equal to $\pi/2$. Means that FrFT belongs to the class of time frequency representation and is a linear operator that corresponds to the rotation of the signal through a angle which is not a multiple of $\pi/2$. .

Recently, FrFT has independently discussed by lots of researches. Ozaktas etal [5] had given Applications of FrFT in optics and signal processing. Alieva T. [6] had developed Fractional transforms in optical Information processing. Bultheel A. etal [7] had given recent developments in the theory of fractional Fourier and linear canonical transforms. Djurovic [8] etal had established FrFT as a signal processing tool. Salazar F. [9] had given a new introduction to the FrFT and its applications. Kilbas [12] had presented theory and applications of fractional differential equations.

One dimensional fractional Fourier transform with parameter α of $f(x)$ difined as

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$$FrFT\{f(x)\} = F_\alpha(u) = \int_{-\infty}^{\infty} f(x)K_\alpha(x,u)dx \tag{1}$$

where the kernel

$$k_\alpha(x,u) = \sqrt{\frac{1 - icot\alpha}{2\pi}} e^{\frac{i}{2sin\alpha}[(x^2+u^2)cos\alpha - 2(xu)]} \tag{2}$$

In the present paper generalization of two-dimensional fractional Fourier transform (2DFrFT) is given. In section 2 testing function spaces is proved. Definition of distributional generalized 2DFrFT is given in section 3. Relation between 2DFrFT and 2DFT (Two-Dimensional Fourier transform) is obtained in section 4. In section 5, Inversion formula for this transform is derived. Uniqueness theorem is proved in section 6. Last section concludes the paper.

2. DISTRIBUTIONAL TWO DIMENSIONAL FRACTIONAL FOURIER TRANSFORM

2.1. **Conventional Two- dimensional Fractional Fourier transforms.** The two dimensional fractional Fourier transform with parameter α of $f(x,y)$ denoted by $FRFTf(x,y)$ performs a linear operation given by the integral transform,

$$\begin{aligned} FRFT\{f(x,y)\} &= F_\alpha\{f(x,y)\}(u,v) \\ &= F_\alpha(u,v) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)K_\alpha(x,y,u,v)dxdy, \end{aligned} \tag{3}$$

where the kernel,

$$K_\alpha(x,y,u,v) = \sqrt{\frac{1 - icot\alpha}{2\pi}} e^{\frac{i}{2sin\alpha}[(x^2+y^2+u^2+v^2)cos\alpha - 2(xu+yv)]} \tag{4}$$

2.2. **The testing function space E.** An infinitely differentiable complex values smooth function ϕ on R^n belongs to $E(R^n)$ if for each compact set $I < S_{a,b}$ where,

$$\begin{aligned} S_{a,b} &= \{x, y : x, y \in R^n, |x| \le a, |y| \le b, a > 0, b > 0\}, I \in R^n. \\ \gamma_{E,p,q}(\phi) &= \sup_{x,y \in I} | D_{x,y}^{p,q}\phi(x,y) | \\ &< \infty, \qquad where\, p, q = 1, 2, 3..... \end{aligned}$$

Thus $E(R^n)$ will denote the space of all $\phi \in E(R^n)$ with support contained in $S_{a,b}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Fourier transformable if it is a member of E^*, the dual space of E.

3. DISTRIBUTIONAL TWO-DIMENSIONAL FRACTIONAL FOURIER TRANSFORM

The two-dimensional distributional fractional Fourier transform of $f(x,y) \in E(R^n)$ can be defined by

$$FRFT\{F(X,Y)\} = F_\alpha(u,v) = \langle f(x,y),\, k_\alpha(x,y,u,v) \rangle, \tag{5}$$

where

$$K_\alpha(x,y,u,v) = C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}, C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}, C_{2\alpha} = \frac{1}{2sin\alpha}$$
$$(6)$$

R.H.S. of eq. (5) has a meaning as the application of $f \in E^*$ to $k_\alpha(x,y,u,v) \in E$
It can be extended to the complex space as an entire function given by

$$FRFT\{F(x,y)\} = F_\alpha(g,h) = \langle f(x,y), k_\alpha(x,y,g,h)\rangle \qquad (7)$$

where

$$k_\alpha(x,y,g,h) = C_{1\alpha}e^{iC_{2\alpha}[(x^2+g^2+y^2+h^2)cos\alpha-2(xg+yh)]}$$

The right hand side of (7) is meaningful because for each $g, h \in c^n$, $k_\alpha(x,y,u,v) \in E$, as a function of x, y

4. CONVERSION OF TWO-DIMENSIONAL FRACTIONAL FOURIER TRANSFORM TO TWO-DIMENSIONAL FOURIER TRANSFORM

4.1. Show that $F_\alpha(u,v) = C_{1\alpha}e^{iC_{2\alpha}[(u^2+v^2)cos\alpha-2t]F\{\tilde{f}(x,y)\}}$**.** The two dimensional fractional Fourier transform is given as

$$
\begin{aligned}
F_\alpha(u,v) &= FRFT\{f(x,y)\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)C_{1\alpha}e^{C_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}dxdy \\
&= \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i}{2sin\alpha}[(u^2+v^2)cos\alpha]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{\frac{i}{2sin\alpha}[(x^2+y^2)cos\alpha]}e^{\frac{-i}{2sin\alpha}(xu+yv)}dxdy, \\
&= C_{1\alpha}e^{iC_{2\alpha}[(u^2+v^2)cos\alpha-2i]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x,y)e^{-i[xu+yv]}dxdy,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{f}(x,y) &= f(x,y)e^{\frac{i}{2sin\alpha}[(x^2+y^2)cos\alpha]} \\
F_\alpha(u,v) &= C_{1\alpha}e^{iC_{2\alpha}[(u^2+v^2)cos\alpha-2i]F\{\tilde{f}(x,y)\}}
\end{aligned}
$$

Thus if $f(x,y)$ is any signal then $F_\alpha(u,v) = C_{1\alpha}e^{iC_{2\alpha}[(u^2+v^2)cos\alpha-2i]F\{\tilde{f}(x,y)\}}$

5. INVERSION FORMULA

It is possible to recover the function $f(t,x)$ by means of inversion formula of conventional Two-dimensional Fourier transform.

Theorem 1 If $FRFT\{f(x,y)\} = F_\alpha(u,v), 0<\alpha \le \frac{\pi}{2}$ then prove that

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_\alpha(u,v)\tilde{k}_\alpha(x,y,u,v)dudv$$

where $\tilde{k}_\alpha(x,y,u,v) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{1-icot\alpha}}\frac{1}{sin^2\alpha}e^{-C_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}$

Proof Recall the two-dimensional fractional Fourier transform as

$$FRFT\{f(x,y)\} = F_\alpha(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)k_\alpha(x,y,u,v)dxdy \qquad (8)$$

where the kernel

$$
\begin{aligned}
k_\alpha(x,y,u,v) &= \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i}{2sin\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]} \\
&= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}
\end{aligned}
$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}, C_{2\alpha} = \frac{1}{2sin\alpha}$

By equation (8)

$$
\begin{aligned}
e^{-iC_{2\alpha}(u^2+v^2)cos\alpha}F_\alpha(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)C_{1\alpha}e^{iC_{2\alpha}[(u^2+v^2)cos\alpha}e^{-2iC_{2\alpha}(xu+yv)]}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-2ic_{2\alpha}[xu+yv]}dxdy
\end{aligned}
$$

where
$$
\begin{aligned}
g(x,y) &= f(x,y)C_{1\alpha}e^{-ic_{2\alpha}[(u^2+v^2)cos\alpha]} \qquad (9) \\
&= F\{g(x,y)\}(2c_{2\alpha}u, 2c_{2\alpha}v) \\
&= F\{g(x,y)\}(\eta,\delta) \qquad (10)
\end{aligned}
$$

where $\eta = 2c_{2\alpha}u, \delta = 2c_{2\alpha}v$

$$e^{-ic_{2\alpha}[(u^2+v^2)cos\alpha]}F_\alpha\left(\frac{\eta}{c_{2\alpha}},\frac{\delta}{c_{2\alpha}}\right) = F\{g(x,y)\}(\eta,\delta)$$

using Two-dimensional inverse Fourier transform

$$g(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\eta,\delta)e^{i[x\eta+y\delta]}d\eta d\delta \qquad (11)$$

where $G(\eta,\delta) = e^{-ic_{2\alpha}[(u^2+v^2)cos\alpha]}F_\alpha\left(\frac{\eta}{2c_{2\alpha}},\frac{\delta}{2c_{2\alpha}}\right)$

$$f(x,y)c_{1\alpha}e^{ic_{2\alpha}[(x^2+y^2)cos\alpha]} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\eta,\delta)e^{i[x\eta+y\delta]}d\eta d\delta$$

$$
\begin{aligned}
f(x,y) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (c_{1\alpha})^{-1}e^{-ic_{2\alpha}[(x^2+y^2)cos\alpha]}G(\eta,\delta)e^{i[x\eta+y\delta]}d\eta d\delta \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\sqrt{\frac{1-icot\alpha}{2\pi}}\right)^{-1}e^{-ic_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}F_\alpha(u,v)4(C_{2\alpha})^2dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_\alpha(u,v)\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{1-icot\alpha}}\frac{1}{sin^2\alpha}e^{-iC_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_\alpha(u,v)\tilde{k}_\alpha(x,y,u,v)dudv,
\end{aligned}
$$

where $\tilde{k}_\alpha(x,y,u,v) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{1-icot\alpha}}\frac{1}{sin^2\alpha}e^{-C_{2\alpha}[(x^2+y^2+u^2+v^2)cos\alpha-2(xu+yv)]}$

6. Uniqueness Theorem

If $F_\alpha(g,h) = FRFTf(x,y)$ and $G_\alpha(g,h) = FRFTg(x,y)$ for $0<\alpha \leq \dfrac{\pi}{2}$ and $supf \subset S_{a,b}, supg \subset S_{a,b}$, where $S_{a,b} = \{x,y\colon x,y \in R^n, |x| \leq a, |y| \leq b, a>0, b>0\}$. If $F_\alpha(g,h) = G_\alpha(g,h)$ then $f = g$ in the sense of equality in $E^*(R^n)$.

Proof By inversion theorem

$$
\begin{aligned}
f - g &= \lim_{N \to \infty} \int_{-N}^{N} \int_{-N}^{N} \tilde{k}_\alpha(x,y,u,v)[F_\alpha(g,h) - G_\alpha(g,h)]dgdh \\
&= 0
\end{aligned}
$$

Then $f = g$ in $E^*(R^n)$

7. Conclusion

The present work generalized 2DFRFT. Relation between 2DFRFT and 2DFT are presented. Inversion formula and uniqueness theorem is proved.

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