

ON A FRACTIONAL DIFFERENTIAL INCLUSION WITH STRIP BOUNDARY CONDITIONS

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ABSTRACT. We study the existence of solutions for fractional differential inclusions of order $q \in (1, 2]$ with strip boundary conditions. We establish Filippov type existence results in the case of nonconvex set-valued maps.

1. INTRODUCTION

This paper is concerned with the following boundary value problem

$$\begin{aligned} D_c^q x(t) &\in F(t, x(t)) \quad a.e. \text{ } ([0, 1]), \\ x(0) &= \sigma \int_{\alpha}^{\beta} x(s) ds, \quad x(1) = \eta \int_{\gamma}^{\delta} x(s) ds, \end{aligned} \quad (1.1)$$

where $q \in (1, 2]$, D_c^q is the Caputo fractional derivative, $F : I \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map, $\sigma, \eta \in \mathbf{R}$ and $0 < \alpha < \beta < \gamma < \delta < 1$.

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([15], [17] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([12]). Recently several qualitative results for fractional differential inclusions were obtained in [2], [7]-[10] etc..

The present paper is motivated by a recent paper of Ahmad and Ntouyas ([1]) where existence results for problem (1.1) are established for convex as well as nonconvex set-valued maps. The existence results in [1] are based on a nonlinear alternative of Leray-Schauder type and Covitz-Nadler contraction principle for set-valued maps. For the motivation, discussion on strip boundary conditions, examples and a consistent bibliography on these problems we refer to [1] and the references therein.

The aim of our paper is to consider the situation when $F(\cdot, \cdot)$ has nonconvex values and to present two new existence results for problem (1.1) which are Filippov type existence results for this problem.

In our first approach we obtain an existence result by the application of the set-valued contraction principle in the space of derivatives of solutions instead of

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the space of solutions as in [1]. We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([11]) in the space of derivatives of the trajectories belongs to Tallos ([14], [18]) and it was already used for similar results obtained for other classes of differential inclusions ([5]-[7], [9]).

In our second approach we show that Filippov's ideas ([13]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1). Recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem ([13]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. PRELIMINARIES

In this short section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space and consider a set valued map T on X with nonempty values in X . T is said to be a λ -contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in X,$$

where $d_H(\cdot, \cdot)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

The set-valued contraction principle ([11]) states that if X is complete, and $T : X \rightarrow \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then $T(\cdot)$ has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T . Obviously, $Fix(T)$ is closed.

Lemma 2.1. ([16]) *Let X be a complete metric space and suppose that T_1, T_2 are λ -contractions with closed values in X . Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z \in X} d(T_1(z), T_2(z)).$$

Let $I = [0, 1]$, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to \mathbf{R} with the norm $\|x(\cdot)\|_C = \sup_{t \in I} |x(t)|$ and $L^1(I, \mathbf{R})$ is the Banach space of integrable functions $u(\cdot) : I \rightarrow \mathbf{R}$ endowed with the norm $\|u(\cdot)\|_1 = \int_0^1 |u(t)| dt$.

Definition 2.2. ([15]) a) *The fractional integral of order $q > 0$ of a Lebesgue integrable function $f(\cdot) : (0, \infty) \rightarrow \mathbf{R}$ is defined by*

$$I^q f(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

b) *The Caputo fractional derivative of order $q > 0$ of a function $f(\cdot) : [0, \infty) \rightarrow \mathbf{R}$ is defined by*

$$D_c^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{-q+n-1} f^{(n)}(s) ds,$$

where $n = [q] + 1$. It is assumed implicitly that $f(\cdot)$ is n times differentiable whose n -th derivative is absolutely continuous.

We recall (e.g., [15]) that if $q > 0$ and $f(\cdot) \in C(I, \mathbf{R})$ or $f(\cdot) \in L^\infty(I, \mathbf{R})$ then $(D_c^q I^q f)(t) \equiv f(t)$.

Lemma 2.3. ([1]) *For a given function $f(\cdot) \in C(I, \mathbf{R})$ the unique solution of the boundary value problem*

$$\begin{aligned} D_c^q x(t) &= f(t), \\ x(0) &= \sigma \int_\alpha^\beta x(s) ds, \quad x(1) = \eta \int_\gamma^\delta x(s) ds \end{aligned}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds + \frac{\sigma}{\Delta} [-(\frac{\eta}{2}(\delta^2 - \gamma^2) - 1) + \\ &t(\eta(\delta - \gamma) - 1) \int_\alpha^\beta (\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m) dm) ds + \frac{1}{\Delta} [\frac{\sigma}{2}(\beta^2 - \alpha^2) - \\ &(\sigma(\beta - \alpha) - 1)t] [\eta \int_\gamma^\delta (\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m) dm) ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s) ds], \end{aligned} \tag{2.1}$$

where

$$\Delta = [\frac{\eta}{2}(\delta^2 - \gamma^2) - 1][\sigma(\beta - \alpha) - 1] - [\frac{\sigma}{2}(\beta^2 - \alpha^2)][\eta(\delta - \gamma) - 1] \neq 0.$$

Remark 2.4. Denote $A(t, s) = \frac{(t-s)^{q-1}}{\Gamma(q)} \chi_{[0,t]}(s)$, $B(t, s) = \frac{\sigma}{\Delta \Gamma(q)} [-(\frac{\eta}{2}(\delta^2 - \gamma^2) - 1) + t(\eta(\delta - \gamma) - 1)] \frac{1}{q} [(\beta - s)^q \chi_{[0,\beta]}(s) - (\alpha - s)^q \chi_{[0,\alpha]}(s)]$, $C(t, s) = \frac{1}{\Delta \Gamma(q)} [\frac{\sigma}{2}(\beta^2 - \alpha^2) - (\sigma(\beta - \alpha) - 1)t] \frac{\eta}{q} [(\delta - s)^q \chi_{[0,\delta]}(s) - (\gamma - s)^q \chi_{[0,\gamma]}(s)]$, $D(t, s) = -\frac{1}{\Delta \Gamma(q)} [\frac{\sigma}{2}(\beta^2 - \alpha^2) - (\sigma(\beta - \alpha) - 1)t] (1-s)^{q-1}$ and $G(t, s) = A(t, s) + B(t, s) + C(t, s) + D(t, s)$, where $\chi_S(\cdot)$ is the characteristic function of the set S . Then the solution $x(\cdot)$ in Lemma 2.3 may be written as

$$x(t) = \int_0^1 G(t, s) f(s) ds, \tag{2.2}$$

Moreover, for any $t, s \in I$ we have

$$\begin{aligned} |G(t, s)| &\leq \frac{1}{\Gamma(q)} + \frac{\sigma}{|\Delta| \Gamma(q)} [|\frac{\eta}{2}(\delta^2 - \gamma^2) - 1| + |\eta(\delta - \gamma) - 1|] \frac{\beta^q + \alpha^q}{q} + \\ &\frac{1}{|\Delta| \Gamma(q)} [|\frac{\sigma}{2}(\beta^2 - \alpha^2)| + |\sigma(\beta - \alpha) - 1|] [\frac{\eta}{q}(\delta^q + \gamma^q) + 1] =: M. \end{aligned}$$

Definition 2.5. A function $x(\cdot) \in C(I, \mathbf{R})$ with its Caputo derivative of order q existing on $[0, 1]$ is a solution of problem (1.1) if there exists a function $f(\cdot) \in L^1(I, \mathbf{R})$ such that $f(t) \in F(t, x(t))$ a.e. (I) and (2.1) is satisfied.

3. THE MAIN RESULTS

We study first problem (1.1) with fixed point techniques. In order to do this we introduce the following hypothesis.

Hypothesis 3.1. (i) $F(\cdot, \cdot) : I \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x \in \mathbf{R}$, $F(\cdot, x)$ is measurable.

(ii) There exists $L(\cdot) \in L^1(I, \mathbf{R}_+)$ such that for almost all $t \in I$, $F(t, \cdot)$ is $L(t)$ -Lipschitz in the sense that

$$d_H(F(t, x), F(t, y)) \leq L(t)|x - y| \quad \forall x, y \in \mathbf{R}.$$

(iii) $d(0, F(t, 0)) \leq L(t)$ a.e. (I)

Denote $L_0 := \int_0^1 L(s) ds$.

Theorem 3.2. *Assume that Hypothesis 3.1 is satisfied and $ML_0 < 1$. Let $y(\cdot) \in C(I, \mathbf{R})$ be such that $y(0) = \sigma \int_{\alpha}^{\beta} y(s)ds$, $y(1) = \eta \int_{\gamma}^{\delta} y(s)ds$ and there exists $p(\cdot) \in L^1(I, \mathbf{R}_+)$ with $d(D_c^q y(t), F(t, y(t))) \leq p(t)$ a.e. (I) .*

Then for every $\varepsilon > 0$ there exists $x(\cdot) \in C(I, \mathbf{R})$ a solution of problem (1.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{M}{1 - ML_0} \int_0^1 p(t)dt + \varepsilon.$$

Proof. For $u(\cdot) \in L^1(I, \mathbf{R})$ define the following set-valued maps

$$M_u(t) = F(t, \int_0^1 G(t, s)u(s)ds), \quad t \in I,$$

$$T(u) = \{\phi(\cdot) \in L^1(I, \mathbf{R}); \quad \phi(t) \in M_u(t) \quad \text{a.e. } (I)\}.$$

It follows from Lemma 2.2 that $x(\cdot)$ is a solution of problem (1.1) if and only if $D_c^q x(\cdot)$ is a fixed point of $T(\cdot)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I, \mathbf{R})$. The fact that the set valued map $M_u(\cdot)$ is measurable is well known. For example the map $t \rightarrow \int_0^1 G(t, s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [4]) we infer that $M_u(\cdot)$ admits a measurable selection ϕ . One has

$$\begin{aligned} |\phi(t)| &\leq d(0, F(t, 0)) + d_H(F(t, 0), F(t, \int_0^1 G(t, s)u(s)ds)) \leq \\ &\leq L(t)(1 + M \int_0^1 |u(s)|ds), \end{aligned}$$

which shows that $\phi \in L^1(I, \mathbf{R})$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $\|\phi_n - \phi\|_1 \rightarrow 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \rightarrow \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(\cdot)$ is a contraction on $L^1(I, \mathbf{R})$.

Let $u, v \in L^1(I, \mathbf{R})$ be given and $\phi \in T(u)$. Consider the following set-valued map

$$H(t) = M_v(t) \cap \{x \in \mathbf{R}; \quad |\phi(t) - x| \leq L(t) \left| \int_0^1 G(t, s)(u(s) - v(s))ds \right|\}.$$

From Proposition III.4 in [4], $H(\cdot)$ is measurable and from Hypothesis 3.1 ii) $H(\cdot)$ has nonempty closed values. Therefore, there exists $\psi(\cdot)$ a measurable selection of $H(\cdot)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$\begin{aligned} \|\phi - \psi\|_1 &= \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 L(t) \left(\int_0^1 |G(t, s)| \cdot |u(s) - v(s)|ds \right) dt \\ &= \int_0^1 \left(\int_0^1 L(t)|G(t, s)|dt \right) |u(s) - v(s)|ds \leq ML_0 \|u - v\|_1. \end{aligned}$$

We deduce that

$$d(\phi, T(v)) \leq ML_0 \|u - v\|_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq ML_0 \|u - v\|_1,$$

thus $T(\cdot)$ is a contraction on $L^1(I, \mathbf{R})$.

We consider next the following set-valued maps

$$F_1(t, x) = F(t, x) + p(t)[-1, 1], \quad (t, x) \in I \times \mathbf{R},$$

$$M_u^1(t) = F_1(t, \int_0^1 G(t, s)u(s)ds),$$

$$T_1(u) = \{\psi(\cdot) \in L^1(I, \mathbf{R}); \quad \psi(t) \in M_u^1(t) \quad a.e. (I)\}, \quad u(\cdot) \in L^1(I, \mathbf{R}).$$

Obviously, $F_1(\cdot, \cdot)$ satisfies Hypothesis 3.1.

Repeating the previous step of the proof we obtain that T_1 is also a ML_0 -contraction on $L^1(I, \mathbf{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq \int_0^1 p(t)dt. \tag{3.1}$$

Let $\phi \in T(u)$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbf{R}; \quad |\phi(t) - z| \leq p(t)\}.$$

With the same arguments used for the set valued map $H(\cdot)$, we deduce that $H_1(\cdot)$ is measurable with nonempty closed values. Hence let $\psi(\cdot)$ be a measurable selection of $H_1(\cdot)$. It follows that $\psi \in T_1(u)$ and one has

$$\|\phi - \psi\|_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 p(t)dt.$$

As above we obtain (3.1).

We apply Lemma 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1 - ML_0} \int_0^1 p(t)dt.$$

Since $v(\cdot) = D_c^q y(\cdot) \in Fix(T_1)$ it follows that for any $\varepsilon > 0$ there exists $u(\cdot) \in Fix(T)$ such that

$$\|v - u\|_1 \leq \frac{1}{1 - ML_0} \int_0^1 p(t)dt + \frac{\varepsilon}{M}.$$

We define $x(t) = \int_0^1 G(t, s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq \int_0^1 |G(t, s)| \cdot |u(s) - v(s)|ds \leq \frac{M}{1 - ML_0} \int_0^1 p(t)dt + \varepsilon$$

which completes the proof.

The assumption in Theorem 3.2 is satisfied, in particular, for $y(\cdot) = 0$ and therefore, via Hypothesis 3.1 (iii), with $p(\cdot) = L(\cdot)$. We obtain the following consequence of Theorem 3.2.

Corollary 3.3. *Assume that Hypothesis 3.1 is satisfied and $ML_0 < 1$. Then for every $\varepsilon > 0$ there exists $x(\cdot)$ a solution of problem (1.1) satisfying for all $t \in I$*

$$|x(t)| \leq \frac{ML_0}{1 - ML_0} + \varepsilon. \tag{3.2}$$

We present next the main result of this paper.

Theorem 3.4. *Assume that Hypothesis 3.1 (i), (ii) is satisfied and $ML_0 < 1$. Let $y(\cdot) \in C(I, \mathbf{R})$ be such that $y(0) = \sigma \int_{\alpha}^{\beta} y(s)ds$, $y(1) = \eta \int_{\gamma}^{\delta} y(s)ds$ and there exists $p(\cdot) \in L^1(I, \mathbf{R}_+)$ with $d(D_c^q y(t), F(t, y(t))) \leq p(t)$ a.e. (I).*

Then there exists $x(\cdot) \in C(I, \mathbf{R})$ a solution of problem (1.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{M}{1 - ML_0} \int_0^1 p(t)dt. \tag{3.3}$$

Proof. The set-valued map $t \rightarrow F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{D_c^q y(t) + p(t)[-1, 1]\} \neq \emptyset \quad \text{a.e. (I)}.$$

It follows (e.g., Theorem 1.14.1 in [3]) that there exists a measurable selection $f_1(t) \in F(t, y(t))$ a.e. (I) such that

$$|f_1(t) - D_c^q y(t)| \leq p(t) \quad \text{a.e. (I)} \tag{3.4}$$

Define $x_1(t) = \int_0^1 G(t, s)f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq M \int_0^1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(\cdot) \in C(I, \mathbf{R})$, $f_n(\cdot) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 G(t, s)f_n(s)ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad \text{a.e. (I), } n \geq 1, \tag{3.6}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad \text{a.e. (I), } n \geq 1. \tag{3.7}$$

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\leq \int_0^1 |G(t, t_1)| \cdot |f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq \\ &M \int_0^1 L(t_1)|x_n(t_1) - x_{n-1}(t_1)| dt_1 \leq M \int_0^1 L(t_1) \int_0^1 |G(t_1, t_2)| \cdot \\ &|f_n(t_2) - f_{n-1}(t_2)| dt_2 \leq M^2 \int_0^1 L(t_1) \int_0^1 L(t_2)|x_{n-1}(t_2) - x_{n-2}(t_2)| dt_2 dt_1 \\ &\leq (M)^n \int_0^1 L(t_1) \int_0^1 L(t_2) \dots \int_0^1 L(t_n)|x_1(t_n) - y(t_n)| dt_n \dots dt_1 \leq \\ &\leq (ML_0)^n M \int_0^1 p(t)dt. \end{aligned}$$

Therefore $\{x_n(\cdot)\}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence converging uniformly to some $x(\cdot) \in C(I, \mathbf{R})$. Therefore, by (3.7), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in \mathbf{R} . Let $f(\cdot)$ be the pointwise limit of $f_n(\cdot)$.

Moreover, one has

$$\begin{aligned} |x_n(t) - y(t)| &\leq |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq \\ &M \int_0^1 p(t)dt + \sum_{i=1}^{n-1} (M \int_0^1 p(t)dt)(ML_0)^i = \frac{M \int_0^1 p(t)dt}{1 - ML_0}. \end{aligned} \tag{3.8}$$

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all $t \in I$

$$\begin{aligned} |f_n(t) - D_c^q y(t)| &\leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + \\ &+ |f_1(t) - D_c^q y(t)| \leq L(t) \frac{M \int_0^1 p(t) dt}{1 - ML_0} + p(t). \end{aligned}$$

Hence the sequence $f_n(\cdot)$ is integrably bounded and therefore $f(\cdot) \in L^1(I, \mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that $x(\cdot)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(\cdot)$.

It remains to construct the sequences $x_n(\cdot), f_n(\cdot)$ with the properties in (3.5)-(3.7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(\cdot) \in C(I, \mathbf{R})$ and $f_n(\cdot) \in L^1(I, \mathbf{R})$, $n = 1, 2, \dots, N$ satisfying (3.5), (3.7) for $n = 1, 2, \dots, N$ and (3.6) for $n = 1, 2, \dots, N - 1$. The set-valued map $t \rightarrow F(t, x_N(t))$ is measurable. Moreover, the map $t \rightarrow L(t)|x_N(t) - x_{N-1}(t)|$ is measurable. By the Lipschitzianity of $F(t, \cdot)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1, 1]\} \neq \emptyset.$$

Theorem 1.14.1 in [3] yields that there exist a measurable selection $f_{N+1}(\cdot)$ of $F(\cdot, x_N(\cdot))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad a.e. (I).$$

We define $x_{N+1}(\cdot)$ as in (3.5) with $n = N + 1$. Thus $f_{N+1}(\cdot)$ satisfies (3.6) and (3.7) and the proof is complete.

Remark 3.5. Obviously, Theorem 3.4 extends Theorem 3.2. We do not suppose that $d(0, F(t, 0)) \leq L(t) \quad a.e. (I)$ and the estimate in (3.3) is better than the one in Theorem 3.2.

Even if Theorem 3.4 improves Theorem 3.2, we chosen to present both results; on one hand because the methods used in their proofs are different and on the other hand to show that there exists situations when the fixed point approaches are less powerful.

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