NEW PATHWAY FRACTIONAL INTEGRAL OPERATOR INVOLVING H-FUNCTIONS

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ABSTRACT. In this paper, we introduce a new fractional integration operator associated with the pathway model and pathway probability density. The ob_ject of the present paper is to study a pathway fractional integral operator associated with the pathway model and pathway probability density for certain product of special functions with general argument. Finally, importance of main result also recorded herein.

1. INTRODUCTION

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades Fractional calculus has been applied to almost every field of science, engineering and mathematics. Many applications of fractional calculus can be found in turbulence and fluid dynamics, Stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, non-linear biological systems, astrophysics. In recent years several authors have made significant contributions in the field of fractional calculus. Several definitions of the operators of the classical and generalized fractional calculus are already well known and widely used in the applications to mathematical models of fractional order. The most popular one, we are based on here, is the Riemann-Liouville fractional integral operator [15]. Recently the operators of the generalized fractional calculus have been introduced and studied, as the hyper-geometric integral operators of Saigo [14], the generalized fractional integrals involving G- and H-functions in Kiryakova [7], Eulerian integral and a main theorem based upon the fractional operator associated with $\bar{H}$-function and multivariable I-function in Chaurasia and Kumar [19] etc. Here we introduce a fractional integration operator, which may be regarded as an extension of the left-sided Riemann-Liouville fractional integral operator. We propose some results for this operator, including the images of the product of H-function, Mittag-Leffler function, Bessel function of first kind and their particular cases. First, let us recall

the definition of left sided Riemann-Liouville fractional integral operator. Let f(x) $\in$ L (a,b), $\alpha \in$ C, R(α) $>$ 0, then

$$(I_{0_+}^\alpha f)\,(x) \;=\; \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\,f(t)\,dt, \tag{1}$$

where R(.) denotes the real part of (.). For more details, we refer to: Samko-KIlbas-Marichev [15], Kiryakova [7], Kilbas-Srivastava-Trujillo [6], and other books on fractional calculus.

If f(t) is replaced by $t^\gamma\,f(t)$ in (1), the above operator turns out to be Erdélyi-Kober fractional integral; if it is replaced by $_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right)$f(t), then (1) takes the form of the Saigo hypergeometric fractional integral, see e.g. [14]:

$$\frac{\Gamma(\eta)}{x^{-\eta-\beta}}\,I_{0_+}^{\eta,\beta,\gamma}\,f(x) \;=\; \int_0^x (x-t)^{\eta-1}\,_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right)\,f(t)\,dt.$$

Many other operators of generalized fractional calculus can be obtained if on the place of f(t) one takes $\Phi(t)$ as it is done in Kiryakova [7] for a Fox's H-function $\Phi(t) \;=\; H_{m,m}^{m,0}(t)$.

The pathway fractional integration operator, as an extension of (1), is defined as follows:

$$(P_{0_+}^{(\eta,\alpha)} f)\,(x) \;=\; x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}\,f(t)\,dt, \tag{2}$$

where f(x) $\in$ L (a,b), $\eta \in$ **C**, R(η) $>$ 0, a $>$ 0 and 'pathway parameter' $\alpha < 1$.

The pathway model is introduced by Mathai [8] and studied further by Mathai and Haubold [9,10]. For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) \;=\; c\,|\,x\,|^{\gamma-1}[1-a\,(1-\alpha)|\,x\,|^\delta]^{\frac{\beta}{1-\alpha}}, \tag{3}$$

$-\infty\;<\;x\;<\;\infty, \delta\;>\;0,\;\beta\;\geq\;0,\;1-a\,(1-\alpha)\,|\,x\,|^\delta\;>\;0,\;\gamma\;>\;0,$where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c \;=\; \frac{1}{2}\,\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)}, \quad for\;\alpha\;<\;1, \tag{4}$$

$$=\;\frac{1}{2}\,\frac{\delta\,[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta}\right)},\quad for\,\frac{1}{\alpha-1}-\frac{\gamma}{\delta}\;>\;0,\;\alpha\;>\;1, \tag{5}$$

$$=\;\frac{1}{2}\,\frac{\delta(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}\quad for\;\;\alpha\;\to\;1. \tag{6}$$

For $\alpha < 1$, it is a finite range density with $1-a\,(1-\alpha)\,|$x$|\delta > 0$ and (3) remains in the extended generalized type - 1 beta family. The pathway density in (3) for $\alpha <$ 1, includes the extended type - 1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 1$, writing $1-\alpha=-(\alpha-1)$, we have

$$f(x) \;=\; c \mid x \mid^{\gamma-1} [1 + a\,(\alpha-1) \mid x \mid^{\delta}]^{-\frac{\beta}{\alpha-1}}, \tag{7}$$

$-\infty \;<\; x \;<\; \infty, \delta \;>\; 0,\ \beta \;\geq\; 0,\ \alpha \;>\; 1$, which is extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the student - t density, the Cauchy density and many more.

Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (3) and (7) take the exponential form, since

$$\lim_{\alpha \to 1} c \mid x \mid^{\gamma-1}[1 - a(1-\alpha \mid x \mid^{\delta}]^{\frac{\beta}{1-\alpha}} \;=\; \lim_{x \to 1} c \mid x \mid^{\gamma-1}[1 + a\,(\alpha-1) \mid x \mid^{\delta}]^{-\frac{\beta}{\alpha-1}}$$

$$=\; c \mid x \mid^{\gamma-1} e^{-a\,\eta \mid x \mid^{\delta}}. \tag{8}$$

This includes the generalized Gamma-, the Weibull-, the Chi-square, the Laplace- and the Maxwell-Boltzmann and other related densities. Therefore, the operator introduced in this paper can be related and applicable to a wide variety of statistical densities.

For more details on the pathway model, the reader is referred to the recent papers of Mathai and Haubold [9,10]. It is seen that the pathway fractional integral operator (2), based on the pathway model of Mathai and Haubold, and using the pathway parameter α, can lead to other interesting examples of fractional calculus operators, related to some probability density functions and applications in statistics.

2. Pathway integral operator of product of two H-function

Charles Fox [2] made a detailed study of a Mellin-Barnes integral, which is now known in the literature as Fox's H-function. This function is defined and represented by means of the following Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[z \left|^{(a_1,\alpha_1),\dots,(a_p,\alpha_p)}_{(b_1,\beta_1),\dots,(b_q,\beta_q)} \right.\right] \;=\; \frac{1}{2\pi i} \int_{L} \phi(s)\, z^{-s}\, ds, \quad i \;=\; +\; \sqrt{-1} \tag{9}$$

where

$$\phi(s) \;=\; \frac{\left\{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s)\right\} \left\{\prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)\right\}}{\left\{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)\right\} \left\{\prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s)\right\}}, \tag{10}$$

with α_j, j = 1,2,...,p and βj, j = 1,2,...,q are real positive numbers, a_j, j = 1,2,...,p and b_j, j = 1,2,...,q are complex numbers, L is a suitable contour separating the poles of $\Gamma(b_j + \beta_j s)$, $j \;=\; 1,2,...,m\ from\ those\ of\ \Gamma(1 - a_j - \alpha_j s)$, $j \;=\; 1,2,...,n.$ For the convergence conditions, existence of various contours L and other properties, see Mathai and Saxena [11], Kilbas and Saigo [4], Kilbas-Srivastava-Trujillo [6], etc. The importance of Fox's H-function lies in the fact that almost all the elementary and special functions in the literature follow as its special cases. These special functions appear in various problems arising in theoretical and applied branches of mathematics, statistics, physics, engineering and other areas.

The series representation is given in [11] is as follows:

$$H_{P,Q}^{M,N}[z] \;=\; H_{P,Q}^{M,N}\left[z \left|^{(e_P,E_P)}_{(f_Q,F_Q)}\right.\right] = \sum_{h=1}^{N} \sum_{v=0}^{\infty} \frac{(-1)^v\, \chi(\xi)}{v\,!\, E_h} \left(\frac{1}{z}\right)^{\xi}, \tag{11}$$

where $\xi = \frac{e_h - 1 - v}{E_h}$, and $h \;=\; 1, 2, ..., N$

and

$$\chi(\xi) = \frac{\left\{\prod_{j=1}^{M}\Gamma(f_j + F_j\xi)\right\}\left\{\prod_{j=1,j\neq h}^{N}\Gamma(1 - e_j - E_j\xi)\right\}}{\left\{\prod_{j=M+1}^{Q}\Gamma(1 - f_j - F_j\xi)\right\}\left\{\prod_{j=N+1}^{P}\Gamma(e_j + \xi E_j)\right\}}.$$

For convergence conditions and other details of the above function see [11].

Theorem 1. Let $\eta,\ \rho \in C, R(\beta) > 0,\ R(\delta) > 0,\ R\left(1 + \frac{h}{1-\alpha}\right) > 0,\ R(\rho) > 0$ *and* $\alpha < 1$,

b $\in$ R, c $\in$ R. Then for the pathway fractional integral $P_{0+}^{(\eta,\alpha)}$ the following formula hold for the image of product of two H-function:

$$\left(P_{0+}^{(\eta,\alpha)}\ t^{P-1}\ H_{P,Q}^{M,N}\left[c\,t^\delta\ \Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]\ H_{p,q}^{m,n}\left[b\,t^\beta\ \Big|_{(b_q,\beta_q)}^{(a_p,\alpha_p)}\right]\right)$$

$$= \frac{x^{\eta+\rho}\Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^\rho}H_{P,Q}^{M,N}\left[\frac{c\,x^\delta}{[a(1-\alpha)]^\delta}\ \Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]$$

$$\times H_{p+1,q+1}^{m,\ n+1}\left[\frac{b\,x^\beta}{[a(1-\alpha)]^\beta}\ \Big|_{(b_1,\beta_1),...,(b_q,\beta_q),\left(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta\right)}^{(1-\rho+\delta\xi,\beta),(a_1,\alpha_1),...,(a_p,\alpha_p)}\right]. \qquad (12)$$

Proof. Using the definitions (2), (9) and (11), we have

$$\left(P_{0+}^{(\eta,\alpha)}\ t^{P-1}\ H_{P,Q}^{M,N}\left[c\,t^\delta\ \Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]\ H_{p,q}^{m,n}\left[b\,t^\beta\ \Big|_{(b_q,\beta_q)}^{(a_p,\alpha_p)}\right]\right)$$

$$= x^\eta\left\{\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}t^{\rho-1}\sum_{h=1}^{N}\sum_{v=0}^{\infty}\frac{(-1)^v\,\chi(\xi)}{v\,!\,E_h}\left(\frac{1}{c\,t^\delta}\right)^\xi\right.$$

$$\left.\cdot\frac{1}{2\pi i}\int_L\phi(s)\,(bt^\beta)^{-s}\,ds\right\}\,dt.$$

Interchanging the order of integration, using a known result [12] and evaluating the integral by the beta function formula, it yields

$$\left(P_{0+}^{(\eta,\alpha)}\ t^{P-1}\ H_{P,Q}^{M,N}\left[c\,t^\delta\ \Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]\ H_{p,q}^{m,n}\left[b\,t^\beta\ \Big|_{(b_q,\beta_q)}^{(a_p,\alpha_p)}\right]\right)$$

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^\rho}\sum_{h=1}^{N}\sum_{v=0}^{\infty}\frac{(-1)^v\chi(\xi)}{v\,!\ E_h}\left[\frac{c\,x^\delta}{[a(1-\alpha)]^\delta}\right]^{-\xi}$$

$$\cdot\frac{1}{2\pi i}\int_L\phi(s)\,\frac{\Gamma(\rho - \delta\xi - \beta s)}{\Gamma\left(1 + \rho + \frac{\eta}{1-\alpha} - \delta\xi - \beta s\right)}\left(\frac{bx^\beta}{[a(1-\alpha)]^\beta}\right)\,ds.$$

$$= \frac{x^{\eta+\rho}\Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^\rho}H_{P,Q}^{M,N}\left[\frac{c\,x^\delta}{[a(1-\alpha)]^\delta}\ \Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]$$

$$\cdot H_{p+1,q+1}^{m,\ n+1}\left[\frac{b\,x^\beta}{[a(1-\alpha)]^\beta}\ \Big|_{(b_1,\beta_1),...,(b_q,\beta_q),\left(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta\right)}^{(1-\rho+\delta\xi,\beta),(a_1,\alpha_1),...,(a_p,\alpha_p)}\right].$$

This completes the proof of Theorem 1.

3. Pathway integral operator of product of an H-function and Mittag-Leffler functions

In 1903, the Swedish mathematician Gosta Mittag-Leffler introduced the function $E_\beta(z)$ [3] defined as,

$$E_\beta(z) \ = \ \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \tag{13}$$

where z is a complex variable and $\Gamma(.)$ is a gamma function $\beta \geq 0$. The Mittag-Leffler function is the direct generalization of the exponential function to which it reduces for $\beta = 1$. For $0 < \beta < 1$, it interpolates between the pure exponential and a hypergeometric function $\frac{1}{1-z}$. Mittag-Leffler function naturally occurs as the solution of the fractional order differential equations and (or) fractional order integral equations.
Wiman [16] studied the generalization of $E_\beta(z)$, that is given by

$$E_{\beta,\rho}(z) \ = \ \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \rho)}, \quad \beta, \rho \in \ \mathbf{C}, R(\rho) > 0, R(\beta) > 0, \tag{14}$$

which is known as Wiman's function.
Prabhakar [13] investigated the function $E_{\beta,p}^\gamma(z)$ as

$$E_{\beta,\rho}^\gamma(z) \ = \ \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta k + \rho)} \frac{z^k}{k\,!}, \quad \rho, \beta, \gamma \ \in \mathbf{C}, R(\beta) > 0, R(\gamma) > 0, R(\rho) > 0 \tag{15}$$

The Mittag-Leffler type functions belong to H-functions family, since they can be represented in terms of the H-function:

$$E_\beta(z) \ = \ H_{1,2}^{1,1} \left[-z \ \Big|_{(0,1),\ (0,\beta)}^{(0,1)} \right], \quad \beta \ \in \ \mathbf{C}, R(\beta) > 0, \tag{16}$$

$$E_{\beta,\rho}(z) \ = \ H_{1,2}^{1,1} \left[-z \ \Big|_{(0,1),(1-\rho,\beta)}^{(0,1)} \right], \quad \beta, \ \rho, \ \in \ \mathbf{C}, R(\beta) > 0, \tag{17}$$

$$E_{\beta,\rho}^\gamma(z) \ = \ \frac{1}{\Gamma\gamma} H_{1,2}^{1,1} \left[-z \ \Big|_{(0,1),\ (1-\rho,\beta)}^{(1-\gamma,1)} \right], \quad \rho, \ \beta, \ \gamma \in \mathbf{C}, R(\beta) > 0. \tag{18}$$

$$= \ \frac{1}{2\pi i\,\Gamma\gamma} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\,\Gamma(\gamma - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s}\, ds. \tag{19}$$

The relation connecting the Wright's function $_p\psi_q(z)$ and the H-function is given for the first time in the monograph of Mathai and Saxena [11, page 11, equation (1.7.81.)] as:

$$_p\psi_q \left[{}_{(b_1,B_1),...,(b_q,B_q)}^{(a_1,A_1),...,(a_p,A_p)} \Big| \ z \right] = \ H_{p,\,q+1}^{1,p} \left[-z \ \Big|_{(0,1),\ (1-b_1,B_1),...,(1-b_q,B_q)}^{(1-a_1,A_1),...,\ (1-a_p,A_p)} \right], \tag{20}$$

where $_p\psi_q(z)$ is the Wright's generalized hypergeometric function will be required in the proof.
Theorem 2. Let $\eta, \gamma, \rho \ \in C, \ R(\eta) > 0, \ R(\gamma) > 0, \ R(\rho) > 0, \ R\left(1 + \frac{\eta}{1-\alpha}\right) > max\,[0, -R(\rho)], \ b \ \in R, \ c \ \in R, \beta > 0, \ \delta > 0 \ and\, \alpha < 1$. Then the image

of product of an arbitrary Fox H-function and Mittag-Leffler function under the pathway integral operator

$$\left(P_{0+}^{(\eta,\alpha)} \; t^{P-1} \; H_{P,Q}^{M,N} \left[c \, t^{\delta} \; \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right] \; E_{\beta,\rho}^{\gamma}(b \, t^{\beta}) \right)$$

$$= \frac{1}{\Gamma\gamma} \frac{x^{\eta+\rho} \, \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} H_{P,Q}^{M,N} \left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}} \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right]$$

$$\cdot_2\psi_2 \left[_{(\rho,\beta),\, \left(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi,\beta\right)}^{(\rho-\delta\xi,\beta),(\gamma,1)} \; \Big| \; \frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}} \right]. \tag{21}$$

Proof. This result can be derived from Theorem 1, by putting m = 1, n = 1, p = 1, q = 2, $b_1 = 0$, $\beta_1 = 1$, $b_2 = 1 - \rho$, $\beta_2 = \beta$, $a_1 = 1 - \gamma$, $\alpha_1 = 1$, $b = -b$, then (12) reduces to

$$\left(P_{0+}^{(\eta,\alpha)} \; t^{\rho-1} \; H_{P,Q}^{M,N} \left[c \, t^{\delta} \; \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right] \; H_{1,2}^{1,1} \left[-b \, t^{\beta} \; \Big|_{(0,1),(1-\rho,\beta)}^{(1-\gamma,\alpha)} \right] \right)$$

$$= \frac{x^{\eta+\rho} \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} H_{P,Q}^{M,N} \left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}} \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right]$$

$$\cdot H_{2,3}^{1,2} \left[-\frac{b \, x^{\beta}}{[a(1-\alpha)]^{\beta}} \; \Big|_{(0,1),(1-\rho,\beta),\, \left(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta\right)}^{(1-\rho+\delta\xi,\beta),\, (1-\gamma,1)} \right].$$

Or

$$\left(P_{0+}^{(\eta,\alpha)} \; t^{\rho-1} \; H_{P,Q}^{M,N} \left[c \, t^{\delta} \; \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right] \; E_{\beta,\rho}^{\gamma}(b \, t^{\beta}) \right)$$

$$= \frac{1}{\Gamma\gamma} \frac{x^{\eta+\rho} \, \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} H_{P,Q}^{M,N} \left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}} \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right]$$

$$\cdot H_{2,3}^{1,2} \left[-\frac{b \, x^{\beta}}{[a(1-\alpha)]^{\beta}} \; \Big|_{(0,1),(1-\rho,\beta),\, \left(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta\right)}^{(1-\rho+\delta\xi,\beta),\, (1-\gamma,1)} \right].$$

$$= \frac{1}{\Gamma\gamma} \frac{x^{\eta+\rho} \, \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} H_{P,Q}^{M,N} \left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}} \Big|_{(f_Q,F_Q)}^{(e_P,E_P)} \right]$$

$$\cdot_2\psi_2 \left[_{(\rho,\beta),\, \left(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi,\beta\right)}^{(\rho-\delta\xi,\beta),(\gamma,1)} \; \Big| \; \frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}} \right].$$

This completes the proof of Theorem 2.

4. Pathway integral operator of product of an H-function and Bessel functions

The Bessel function of the first kind Jν(x) is defined for complex x $\in$ **C**, x $\neq$ 0 and $\nu \in$ C, R(ν) $> -$ 1 by

$$J_\nu(x) \;=\; \left(\frac{x}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{\Gamma(\nu+k+1)\; k\;!}, \tag{22}$$

and its H-function representation is given by

$$J_\nu(x) \;=\; \left(\frac{x}{2}\right)^\nu \; H_{0,2}^{1,0}\left[\frac{x^2}{4}\;\bigg|_{(0,1),(-\nu,1)}\right], \qquad \nu \in \mathbf{C}, R(\nu) > 0. \tag{23}$$

The asymptotic behavior and other properties of this function can be seen from the papers of Wright [17,18] and the handbook Erdlyi et al. [1]. We find here its pathway integral image.

Theorem 3. Let $\eta, \gamma, \nu \in C$, $R(\eta) > 0$, $R\left(1+\frac{\eta}{1-\alpha}\right) > 0$, $R(\gamma + \nu) > 0$, $c \in R$, $\delta > 0$ and $\alpha < 1$. Let $P_{0+}^{(\eta,\alpha)}$ be the pathway fractional integral. Then there holds the image

$$\left(P_{0+}^{(\eta,\alpha)}\left(\frac{t}{2}\right)^{\gamma-1} H_{P,Q}^{M,N}\left[c\left(\frac{t}{2}\right)^\delta\bigg|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] J_\nu(t)\right)$$

$$= \frac{x^{\eta+\nu+\gamma}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma+\nu}2^{\nu+\gamma+\eta}} H_{P,Q}^{M,N}\left[\frac{cx^\delta}{[2a(1-\alpha)]^\delta}\bigg|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]$$

$$\times\;\; {}_1\psi_2\left[{}_{(\nu+1,1),\,(1+\nu,\gamma-\delta\xi+\frac{\eta}{1-\alpha},2)}^{(\gamma+\nu-\delta\xi,2)}\bigg|-\frac{x^2}{4a^2(1-\alpha)^2}\right]. \tag{24}$$

Proof. This result can be obtained from theorem 1, by putting m = 1, n = 0, p = 0, q = 2, $b_1 = 0$, $\beta_1 = 1$, $b_2 = -\nu$, $\beta_2 = 1$, $\rho = \gamma + \nu$, $b = 1$, $\beta = 2$ *and replacing t by* $\frac{t}{2}$, then (12) reduces to

$$\left(P_{0+}^{(\eta,\alpha)}\left(\frac{t}{2}\right)^{\gamma+\nu-1} H_{P,Q}^{M,N}\left[c\left(\frac{t}{2}\right)^\delta\bigg|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]\right) H_{0,2}^{1,0}\left[\frac{t^2}{4}\bigg|_{(0,1),\,(-\nu,1)}\right]$$

$$= \frac{x^{\eta+\gamma+\nu}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma+\nu}\,2^{\eta+\gamma+\nu}} H_{P,Q}^{M,N}\left[\frac{c\,x^\delta}{[2a(1-\alpha)]^\delta}\bigg|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]$$

$$\times H_{1,3}^{1,1}\left[\frac{x^2}{4a^2(1-\alpha)^2}\bigg|_{(0,1),(-\nu,1),\left(-\nu-\gamma+\delta\xi-\frac{\eta}{1-\alpha},2\right)}^{(1-\gamma-\nu+\delta\xi,2)}\right].$$

Or

$$\left(P_{0+}^{(\eta,\alpha)}\left(\frac{t}{2}\right)^{\gamma-1} H_{P,Q}^{M,N}\left[c\left(\frac{t}{2}\right)^\delta\bigg|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] J_\nu(t)\right)$$

$$= \frac{x^{\eta+\nu+\gamma}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma+\nu}2^{\eta+\nu+\gamma}} H_{P,Q}^{M,N}\left[\frac{cx^\delta}{[2a(1-\alpha)]^\delta}\bigg|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]$$

$$\times \; {}_1\psi_2 \left[\begin{matrix} (\gamma+\nu-\delta\xi,2) \\ (\nu+1,1),\,(1+\nu+\gamma-\delta\xi+\frac{\eta}{1-\alpha},2) \end{matrix} \,\middle|\, -\frac{x^2}{4a^2(1-\alpha)^2} \right].$$

Here $_p\psi_q(x)$ denotes the generalized Wright hypergeometric function.

This completes the proof of Theorem 3.

The importance of our results lies in their manifold generality. In view of the generality of H-functions, on specializing the various parameters, we can obtained from our main result, several result containing remarkably wide variety of useful functions and their various special cases. Thus the main result presented in this article would at once yield a very large number of results containing a large variety of simpler special functions occurring scientific and technological fields.

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