

## REMARKS ON FRACTIONAL KINETIC DIFFERINTEGRAL EQUATIONS AND M-SERIES

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**ABSTRACT.** This paper is devoted to investigate certain generalized fractional kinetic differintegral equations using Laplace transform technique. Fractional kinetic differintegral equations involving M-series are also studied and results are obtained in the form suitable for numerical computation. Several special cases containing generalized Mittag-Leffler function are discussed. An alternative method is suggested for solving certain fractional differential equations.

### 1. INTRODUCTION

In last few decades fractional kinetic equations have been extensively used in describing and solving various problems of applied sciences. In view of the usefulness and importance of the kinetic equation in certain physical problems governing reaction-diffusion in complex systems and anomalous diffusion fractional kinetic equations are studied by Gloeckle and Nonnenmacher [4], Saichev and Zaslavsky [10], Saxena et al. [12]-[14]. Recently, in a series of papers Saxena et al. [15]-[18], Haubold et al. [5] have investigated the solution of certain fractional differintegral equations related to reaction diffusion equations. Chouhan and Saraswat [2] have studied the solution of a generalized fractional kinetic equation involving the generalized fractional integral operator.

In the present paper we are investigating certain generalized fractional kinetic differintegral equations. Several special cases involving Mittag-Leffler function and M-series are also presented. In section 4 we have consider more generalized fractional kinetic equation and its solution involving M-series and generalized fractional integral operator operator containing a generalized Mittag-Leffler function in its kernel.

### 2. PRELIMINARIES AND DEFINITIONS

**Definition 1** Wiman [21] studied the following Mittag-Leffler function of two parameters

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$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} t^n \quad (t, \alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0) \quad (1)$$

Prabhakar [9] introduced the more generalized function  $E_{\alpha,\beta}^{\gamma}(t)$  in the form

$$E_{\alpha,\beta}^{\gamma}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{t^n}{n!} \quad (t, \alpha, \beta, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0) \quad (2)$$

**Definition 2** The Riemann – Liouville operators:

The right sided Riemann – Liouville fractional integral operator  $D_{a+}^{-\nu}$  and the right sided Riemann – Liouville fractional derivative operator  $D_{a+}^{\nu}$  are defined by Samko et al., [11] for  $\operatorname{Re}(\nu) > 0$ ,

$$(I_{a+}^{\nu} f)(t) = (D_{a+}^{-\nu} f)(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-x)^{\nu-1} f(x) dx \quad (3)$$

and,

$$(D_{a+}^{\nu} f)(t) = \left( \frac{d}{dx} \right)^n (I_{a+}^{n-\nu} f)(t) \quad (n = [\operatorname{Re}(\nu)] + 1) \quad (4)$$

where  $[x]$  denotes the greatest integer in the real number  $x$ .

Hilfer [6] generalized the Riemann – Liouville fractional derivative operator  $D_{a+}^{\nu}$  in (4) by introducing a right sided fractional derivative operator of order and type with respect to  $x$  as follows:

$$(D_{a+}^{\nu,\mu} f)(t) = I_{a+}^{\mu(1-\nu)} \frac{d}{dt} (I_{a+}^{(1-\mu)(1-\nu)} f)(t). \quad (5)$$

**Definition 3** Prabhakar [9] studied some properties of the integral operator

$$\left( \mathbf{E}_{\alpha,\beta,\omega;a+}^{\gamma} \varphi \right) (t) = \int_a^t (t-x)^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega(t-x)^{\alpha}) \varphi(x) dx \quad (6)$$

with  $\alpha, \beta, \gamma, \omega \in \mathbb{C}$ ,  $\operatorname{Re}(\alpha) > 0$ ,  $\operatorname{Re}(\beta) > 0$  containing the function (2) in its kernel. The fractional integral operator (6) was further investigated by Kilbas et al. [7].

**Definition 4** The Parseval theorem [20] for Laplace transform is defined as

$$\mathcal{L} \left[ \int_0^t f(t-x) g(x) dx \right] (s) = \mathcal{L}[f(t)](s) \mathcal{L}[g(t)](s) \quad (7)$$

Also Prabhakar [9] introduced the Laplace transform formula for the generalized Mitta-Leffler function  $E_{\alpha,\beta}^{\gamma}(t)$  as

$$\mathcal{L} \left[ t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega t^{\alpha}) \right] (s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\omega)^{\gamma}} \left( \alpha, \beta, \omega \in \mathbb{C}, \operatorname{Re}(\beta) > 0, \operatorname{Re}(s) > 0, \left| \frac{\omega}{s^{\alpha}} \right| < 1 \right) \quad (8)$$

Using equations (7) and (8) it can be easily shown that

$$\mathcal{L} \left[ \left( \mathbf{E}_{\alpha,\beta,\omega;0+}^{\gamma} \varphi \right) (t) \right] = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\omega)^{\gamma}} \bar{\varphi}(s) \quad (9)$$

where  $\bar{\varphi}(s) = \mathcal{L}[\varphi(t)]$ .

The Laplace transform of Riemann – Liouville fractional integral operator  $(D_{0+}^v f)(t)$  and the Hilfer operator  $(D_{a+}^{\nu,\mu} f)(t)$  are given as [6]:

$$\mathcal{L} [(D_{0+}^v f)(t)](s) = s^v \mathcal{L}[f(t)](s) - \sum_{k=0}^{n-1} s^k D_{0+}^{v-k-1} f(0+), (n - 1 < v < n) \quad (10)$$

and,

$$\mathcal{L} [(D_{a+}^{\nu,\mu} f)(t)](s) = s^v \mathcal{L}[f(t)](s) - s^{\mu(v-1)} \left( I_{0+}^{(1-\mu)(1-v)} f \right) (0+), (0 < v < 1) \quad (11)$$

**Definition 5** Sharma and Jain [19] introduced the generalized M-series as the function defined by means of the power series

$$\begin{aligned} {}_p^{\alpha} M_q^{\beta} (a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) &= {}_p^{\alpha} M_q^{\beta}(z) = {}_p^{\alpha} M_q^{\beta} ((a_j)_1^p; (b_j)_1^q; z) \\ &= \sum_{m=0}^{\infty} \frac{(a_1)_m \dots (a_p)_m}{(b_1)_m \dots (b_q)_m} \frac{z^m}{\Gamma(\alpha m + \beta)} (z, \alpha, \beta \in C, R(\alpha) > 0) \end{aligned} \quad (12)$$

The M-series yields the following relationship with various classical special functions:

- The Mittag-Leffler function (1) can be obtain from (12) for  $p = q = 0$ , we have

$$E_{\alpha,\beta}(z) = {}_0^{\alpha} M_0^{\beta} (-; -; z) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(\alpha m + \beta)} z^m \quad (13)$$

- The generalized Mittag- Leffler function (2), is obtained from (12) for  $p = q = 1; a = \gamma \in C; b = 1$ :

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(\alpha m + \beta)} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{(1)_m} \frac{z^m}{\Gamma(\alpha m + \beta)} = {}_1^{\alpha} M_1^{\beta} (\gamma; 1; z) \quad (14)$$

Chouhan and Saraswat [3] established the following result for M-series

$$\left( I_{0+}^{\alpha} \left[ t^{\gamma-1} {}_p^{\beta} M_q^{\gamma} (\omega t^{\beta}) \right] \right) (x) = x^{\alpha+\gamma-1} {}_p^{\beta} M_q^{\alpha+\gamma} (\omega x^{\beta}) \quad (15)$$

where,  $\alpha > 0, \beta > 0, \gamma > 0, \omega \in R$ .

### 3. FRACTIONAL KINETIC DIFFERINTEGRAL EQUATIONS

In this section we have investigated the solution of certain fractional kinetic differintegral equations using Laplace transform technique and an alternative method is also suggested in subsection 3.1.

Let  $N(t)$  denotes the number density of a given species at time t,  $N_0 = N(0)$  is the number density of that species at time t = 0.

**Theorem 3.1** If  $\min \{ \text{Re}(\alpha), \text{Re}(v) \} > 0, c > 0$ , and  $f \in L(0, \infty)$ , then for the solution of the fractional kinetic differintegral equation

$$(D_{0+}^{\alpha} N)(t) - N_0 f(t) = -c^v (D_{0+}^{-v} N)(t), \quad (16)$$

with the initial condition

$$D_{0+}^{\alpha-k-1} N_0 = b_k, (k = 0, 1, 2, \dots, n - 1) \quad (17)$$

there holds the formula

$$N(t) = N_0 (\mathbf{E}_{\alpha+v,\alpha,-c^v;0+} f)(t) + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha+v,\alpha-k} [-c^v t^{\alpha+v}] \quad (18)$$

where,  $n = [\alpha] + 1$ .

**Proof.** Applying Laplace transform on both side of (16) and using (10), we get

$$\bar{N}(s) = N_0 \frac{s^v}{s^{\alpha+v} + c^v} \bar{f}(s) + \sum_{k=0}^{n-1} \frac{s^{v+k}}{s^{\alpha+v} + c^v} D_{0+}^{\alpha-k-1} N_0$$

making use of (7) and (8), we obtain

$$\begin{aligned} \bar{N}(s) = N_0 \mathcal{L} \left[ \int_0^t (t-x)^{\alpha-1} E_{\alpha+v, \alpha} \left( -c^v (t-x)^{\alpha+v} \right) f(x) dx \right] \\ + \sum_{k=0}^{n-1} \mathcal{L} \left[ t^{\alpha-k-1} E_{\alpha+v, \alpha-k} \left( -c^v t^{\alpha+v} \right) \right] D_{0+}^{\alpha-k-1} N_0 \end{aligned}$$

finally taking Laplace inverse both side, using (17) and by virtue of (6) (for  $\gamma = 1$ ), we arrive at the solution (18) asserted by Theorem 3.1.

**Lemma 3.1** (Kilbas et al. [7]). Let  $\alpha, \beta, \rho, \omega \in C, (Re(\alpha), Re(\rho), Re(\beta) > 0)$ , then

$$\int_0^t (t-x)^{\beta-1} E_{\alpha, \beta} [\omega(t-x)^\alpha] x^{\rho-1} dx = \Gamma(\rho) t^{\beta+\rho-1} E_{\alpha, \beta+\rho}(\omega t^\alpha) \quad (19)$$

If we set  $f(t) = t^{\rho-1}$  in (16), then

**Corollary 3.1** If  $\min\{Re(\alpha), Re(v), Re(\rho)\} > 0, c > 0$ , then for the solution of the equation

$$(D_{0+}^\alpha N)(t) - N_0 t^{\rho-1} = -c^v (D_{0+}^{-v} N)(t) \quad (20)$$

holds the relation

$$N(t) = N_0 \Gamma(\rho) t^{\alpha+\rho-1} E_{\alpha+v, \alpha+\rho}(-c^v t^{\alpha+v}) + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha+v, \alpha-k}(-c^v t^{\alpha+v}). \quad (21)$$

**Lemma 3.2** (Kilbas et al. [7]). Let  $\alpha, \beta, \rho, \omega \in C, (Re(\alpha), Re(\rho), Re(\beta) > 0)$ , then

$$\int_0^t (t-x)^{\beta-1} E_{\alpha, \beta} [\omega(t-x)^\alpha] x^{\rho-1} E_{\alpha, \rho}(\omega x^\alpha) dx = t^{\beta+\rho-1} E_{\alpha, \beta+\rho}^2(\omega t^\alpha) \quad (22)$$

If we set  $f(t) = t^{\rho-1} E_{\alpha+v, \rho}(-c^v t^{\alpha+v})$  in (16), then

**Corollary 3.2** If  $\min\{Re(\alpha), Re(v), Re(\rho)\} > 0, c > 0$ , then for the solution of the equation

$$(D_{0+}^\alpha N)(t) - N_0 t^{\rho-1} E_{\alpha+v, \rho}(-c^v t^{\alpha+v}) = -c^v (D_{0+}^{-v} N)(t) \quad (23)$$

holds the relation

$$N(t) = N_0 t^{\alpha+\rho-1} E_{\alpha+v, \alpha+\rho}^2(-c^v t^{\alpha+v}) + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha+v, \alpha-k}(-c^v t^{\alpha+v}). \quad (24)$$

**Lemma 3.3** Let  $\alpha, \mu, v, \omega \in C, (Re(\alpha), Re(v), Re(\mu) > 0)$ , then

$$\int_0^t (t-x)^{\beta-1} E_{\alpha, \beta} [\omega(t-x)^\alpha] x^{\rho-1} E_{\alpha, \rho}^\gamma(\omega x^\alpha) dx = t^{\beta+\rho-1} E_{\alpha, \beta+\rho}^{\gamma+1}[\omega t^\alpha] \quad (25)$$

For  $f(t) = t^{\rho-1} E_{\alpha+v, \rho}^\gamma(-c^v t^{\alpha+v})$  in (16), then

**Corollary 3.3** If  $\min\{Re(\alpha), Re(v), Re(\rho)\} > 0, c > 0$ , then for the solution of the equation

$$(D_{0+}^\alpha N)(t) - N_0 t^{\rho-1} E_{\alpha+v, \rho}^\gamma(-c^v t^{\alpha+v}) = -c^v (D_{0+}^{-v} N)(t) \quad (26)$$

holds the relation

$$N(t) = N_0 t^{\alpha+\rho-1} E_{\alpha+v, \alpha+\rho}^{\gamma+1}(-c^v t^{\alpha+v}) + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha+v, \alpha-k}(-c^v t^{\alpha+v}). \quad (27)$$

**Remark 1.** If we put  $\alpha = 0$  and  $b_k = 0$ , the above corollary give rise to the solution of generalized fractional kinetic equation as obtained by Saxena et al. ([13], Theorem 1).

Again if we set  $\alpha = 0$ , and  $b_k = 0$  in (18), we get the result obtained by Saxena et al. [18].

**Lemma 3.4** Let  $\alpha, \beta, \rho, \omega \in C, (Re(\alpha), Re(\rho), Re(\beta) > 0)$ , then

$$\mathbf{E}_{\alpha, \beta, \omega; 0+} \left[ t^{\rho-1} M_p^\alpha M_q^\rho(\omega t^\alpha) \right] = t^{\beta+\rho-1} \sum_{r=0}^{\infty} (\omega t^\alpha)^r \left[ M_p^{\alpha r + \beta + \rho} M_q^\rho(\omega t^\alpha) \right]. \quad (28)$$

**Proof.** Using the operator (6) (for  $\gamma = 1$ ) and (12), we get

$$\begin{aligned} & \mathbf{E}_{\alpha, \beta, \omega; 0+} \left[ t^{\rho-1} M_p^\alpha M_q^\rho(\omega t^\alpha) \right] \\ &= \sum_{m=0}^{\infty} \frac{(a_1)_m \dots (a_p)_m}{(b_1)_m \dots (b_q)_m} \frac{\omega^m}{\Gamma(\alpha m + \rho)} \int_0^t (t-x)^{\beta-1} E_{\alpha, \beta}[\omega(t-x)^\alpha] x^{\alpha m + \rho - 1} dx \end{aligned}$$

now using (19), (1) and finally by virtue of (12), we obtain (28).

**Theorem 3.2** If  $\min\{Re(\alpha), Re(v)\} > 0, c > 0$ , then for the solution of the fractional kinetic differintegral equation

$$(D_{0+}^\alpha N)(t) - N_0 t^{\rho-1} [{}^{\alpha+v} M_p^\rho(-c^v t^{\alpha+v})] = -c^v (D_{0+}^{-v} N)(t). \quad (29)$$

with the initial condition (17), there holds the formula

$$\begin{aligned} N(t) &= N_0 t^{\alpha+\rho-1} \sum_{r=0}^{\infty} (-c^v t^{\alpha+v})^r [{}^{\alpha+v} M_p^{(\alpha+v)r + \alpha + \rho}(-c^v t^{\alpha+v})] \\ &+ \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha+v, \alpha-k}[-c^v t^{\alpha+v}] \end{aligned} \quad (30)$$

where,  $n = [\alpha] + 1$ .

**Proof.** The proof follows from Theorem 3.1, and via Lemma 3.4.

**Remark 2.** For  $p = q = 0$  in Theorem 3.2 we obtain Corollary 3.2.

**Remark 3.** For  $p = q = 1, a_1 = \gamma$  and  $b_1 = 1$  in Theorem 3.2 we obtain Corollary 3.3.

**3.1. An Alternative Method.** This is employed by Babenko [1] for solving various types of fractional integral and differential equations and further described by Podlubny [11].

Applying fractional integral operator  $D_{0+}^{-\alpha}$  to the both side of (16) and using the formula from Samko et al. [11],

$$D_{0+}^{-\alpha} D_{0+}^\alpha N(t) = N(t) - \sum_{k=0}^{n-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} D_{0+}^{\alpha-k-1} N_0 \quad (31)$$

we have,

$$\left(1 + c^v D_{0+}^{-(\alpha+v)}\right) N(t) = N_0 D_{0+}^{-\alpha} f(t) + \sum_{k=0}^{n-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} D_{0+}^{\alpha-k-1} N_0$$

using (17) and binomial expansion, we get

$$N(t) = N_0 \sum_{r=0}^{\infty} (-c^v)^r D_{0+}^{-r(\alpha+v)-\alpha} f(t) + \sum_{k=0}^{n-1} b_k \sum_{r=0}^{\infty} (-c^v)^r D_{0+}^{-r(\alpha+v)} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)}$$

On making use of equation (2.35) from Samko et al. [11], i.e.

$$D_{0+}^{-r(\alpha+v)} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} = \frac{t^{\alpha+r(\alpha+v)-k-1}}{\Gamma[r(\alpha+v) + (\alpha-k)]}$$

we have,

$$N(t) = N_0 \sum_{r=0}^{\infty} (-c^v)^r D_{0+}^{-r(\alpha+v)-\alpha} f(t) + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} \sum_{r=0}^{\infty} \frac{[-c^v t^{\alpha+v}]^r}{\Gamma[r(\alpha+v) + (\alpha-k)]} \quad (32)$$

finally by virtue of (3), (1) and (6) (for  $\gamma = 1$ ) we obtained (16).

Moreover, the proof of Theorem 3.2 follows from (32) and (15).

#### 4. GENERALIZED FRACTIONAL KINETIC EQUATION WHICH INVOLVE AN INTEGRAL OPERATOR CONTAINING GENERALIZED MITTAG-LEFFLER FUNCTION IN ITS KERNEL

**Theorem 4.1** If  $\min\{Re(\alpha), Re(\beta), Re(\rho)\} > 0, c > 0$ , then for the solution of the fractional kinetic equation

$$(D_{0+}^{\beta, v} N)(t) - N_0 t^{\rho-1} {}_p M_q^{\rho}(\omega t^{\alpha}) = -c^{\beta} \mathbf{E}_{\alpha, \beta, \omega; 0+}^{\gamma} N(t) \quad (33)$$

with initial condition,

$$\left(I_{0+}^{(1-v)(1-\beta)} N\right)(0+) = b_0 \quad (34)$$

there holds the formula

$$\begin{aligned} N(t) &= b_0 \sum_{k=0}^{\infty} (-c^{\beta})^k t^{2\beta k + \beta + v - \beta v - 1} E_{\alpha, 2\beta k + \beta + v - \beta v}^{\gamma k}(\omega t^{\alpha}) \\ &+ N_0 t^{\beta + \rho - 1} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (-c^{\beta} t^{2\beta})^k \frac{(\gamma k)_r}{r!} (\omega t^{\alpha})^r \left[ {}_p M_q^{2\beta k + \alpha r + \beta + \rho}(\omega t^{\alpha}) \right] \end{aligned} \quad (35)$$

**Proof.** Applying Laplace transform on both side of (33) and using (11) and (34), we get

$$\bar{N}(s) = \frac{b_0 s^{v(\beta-1)}}{\left[s^{\beta} + c^{\beta} \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\omega)^{\gamma}}\right]} + N_0 \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \omega^m \frac{s^{-\alpha m - \rho}}{\left[s^{\beta} + c^{\beta} \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\omega)^{\gamma}}\right]} \quad (36)$$

Again by virtue of (8), it is not difficult to see that

$$\frac{s^{v(\beta-1)}}{\left[s^{\beta} + c^{\beta} \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\omega)^{\gamma}}\right]} = \mathcal{L} \left[ \sum_{k=0}^{\infty} (-c^{\beta})^k t^{2\beta k + \beta + v - \beta v - 1} E_{\alpha, 2\beta k + \beta + v - \beta v}^{\gamma k}(\omega t^{\alpha}) \right] (s)$$

and,

$$\frac{s^{-\alpha m - \rho}}{\left[ s^\beta + c^\beta \frac{s^{\alpha\gamma - \beta}}{(s^\alpha - \omega)^\gamma} \right]} = \mathcal{L} \left[ \sum_{k=0}^{\infty} (-c^\beta)^k t^{2\beta k + \alpha m + \rho + \beta - 1} E_{\alpha, 2\beta k + \alpha m + \rho + \beta}^{\gamma k} (\omega t^\alpha) \right] (s)$$

Upon using these last two results in (36) and applying inverse Laplace transform, we obtain

$$N(t) = b_0 \sum_{k=0}^{\infty} (-c^\beta)^k t^{2\beta k + \beta + v - \beta v - 1} E_{\alpha, 2\beta k + \beta + v - \beta v}^{\gamma k} (\omega t^\alpha) + N_0 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \omega^m (-c^\beta)^k t^{2\beta k + \alpha m + \rho + \beta - 1} E_{\alpha, 2\beta k + \alpha m + \rho + \beta}^{\gamma k} (\omega t^\alpha)$$

finally expanding Mittag-Leffler function in second term of R.H.S. using (2) and applying (12) we arrive at the solution (35) asserted by Theorem 4.1.

If we set  $p = 0 = q$ , then we get the following particular case of the solution (35).

**Corollary 4.1** If  $\min \{Re(\alpha), Re(\beta), Re(\rho)\} > 0, c > 0$ , then for the solution of the fractional kinetic equation

$$(D_{0+}^{\beta, v} N)(t) - N_0 t^{\rho-1} E_{\alpha, \rho}(t) (\omega t^\alpha) = -c^\beta \mathbf{E}_{\alpha, \beta, \omega; 0+}^\gamma N(t) \tag{37}$$

with initial condition (34), there holds the formula

$$N(t) = b_0 \sum_{k=0}^{\infty} (-c^\beta)^k t^{2\beta k + \beta + v - \beta v - 1} E_{\alpha, 2\beta k + \beta + v - \beta v}^{\gamma k} (\omega t^\alpha) + N_0 t^{\beta + \rho - 1} \sum_{k=0}^{\infty} (-c^\beta t^{2\beta})^k E_{\alpha, 2\beta k + \beta + \rho}^{\gamma k + 1} (\omega t^\alpha). \tag{38}$$

A number of several special cases of Theorem 4.1 can also be obtained by taking suitable values for parameters in M-series.

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