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ON A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY FRACTIONAL CALCULUS OPERATOR

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ABSTRACT. The purpose of the present paper is to establish some results involving coefficient conditions, distortion bounds, extreme points, convolution, convex combinations and neighborhoods for a new class of harmonic univalent functions in the open unit disc. We also discuss a class preserving integral operator. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. See Clunie and Sheil-Small [4], (see also [7], [12], [13]).

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ |b_1| < 1. \tag{1}$$

Note that S_H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be exprssed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

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A function f of the form (1) is said to be harmonic starlike of order $\alpha, (0 \le \alpha < 1)$ for $|z| = r < 1$, if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > \alpha.$$

The class of all harmonic starlike functions of order α is denoted by $S_H^*(\alpha)$ and extensively studied by Jahangiri [8]. The case $\alpha = 0$ and $\alpha = b_1 = 0$ were studied by Silverman and Silvia [17] and Silverman [16], (see also [3]). In [8] Jahangiri proved that the coefficient condition

$$\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k| + \sum_{k=1}^{\infty}\frac{k+\alpha}{1-\alpha}|b_k| \le 1$$

is sufficient condition for functions $f = h + \overline{g}$ to be harmonic starlike of order α. If we put $\alpha = 0$ in above inequalities then we obtain sufficient condition for function $f = h + \overline{g}$ belonging to the class S_H^* of harmonic starlike functions.

Further, we denote by V_H the subclass of S_H consisting of functions of form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{k=2}^{\infty}|a_k|z^k,\ g(z) = (-1)^n\sum_{k=1}^{\infty}|b_k|z^k,\ |b_1| < 1. \tag{3}$$

2. Fractional Calculus

Let $L(a, b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a, b]$:

$$L(a,b) = \left\{f : ||f||_1 = \int_a^b |f(t)|dt < +\infty\right\}.$$

Definition 1 (see [10], page 79). Let $f(x) \in L(a, b)$, $\alpha \in C$, $Re(\alpha) > 0$, then

$${}_aI_x^{\alpha}f(x) = {}_aD_x^{-\alpha}f(x) = I_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a,$$

is called the Riemann-Liouville left-sided fractional integral of order α.

Definition 2 (see [10], page 84). The left-sided Riemann-Liouville fractional derivative of order $\alpha \in C$, $Re(\alpha) \ge 0$ of the function $f(x)$ is defined by

$$\left({}_aD_x^{\alpha}f\right)(x) = \left(D_{a+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt,\ n = [Re(\alpha)]+1;\ x > a,$$

where $[Re(\alpha)]$ means the integral part of $Re(\alpha)$.

The following definitions of fractional derivatives and fractional integrals are due to Owa [11] and Srivastava and Owa [18].

Definition 3. The fractional integral of order λ is defined for a function $f(z)$ of the form (2) by

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}}d\xi,$$

where $\lambda > 0, f(z)$ is an analytic functions in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z - \xi)$ to be real when $(z - \xi) > 0$.

It is easy to see that the Definition 3 is a particular case of Definition 1 for $a = 0$.

Definition 4. The fractional derivative of order λ is defined for a function $f(z)$ of the form (2) by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi,$$

where $0 \leq \lambda < 1, f(z)$ is an analytic functions in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 3 above.

It is easy to see that the Definition 4 is a particular case of Definition 2 for $a = 0$ and $0 \leq \alpha < 1$.

Very recently, Dixit and Porwal [5] introduce a new fractional derivative operator for function of the form (2) as follows

$$\Omega^0 f(z) = f(z)$$
$$\Omega^1 f(z) = \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Omega^n f(z) = \Omega(\Omega^{n-1} f(z)).$$

Thus, we note that

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k, \tag{4}$$

where

$$\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}.$$

It is interesting to note that for $\lambda = 0, \Omega^n f(z)$ reduces to familiar Salagean operator introduced by Salagean in [15].

From the motivation of the definition of modified Salagean operator defined by Jahangiri et al. [9] for function of the form $f = h + \overline{g}$, where h and g are the form (1) as follows

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}.$$

Now, we define

$$\Omega^n f(z) = \Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)}$$

where

$$\Omega^n h(z) = z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k$$

and

$$\Omega^n g(z) = \sum_{k=1}^{\infty} [\phi(k,\lambda)]^n b_k z^k.$$

Now, we let $R_H(n,\beta,\lambda)$ denote the subclass S_H consisting of functions $f = h + \overline{g}$ of the form (1) that satisfy the condition

$$Re\left\{\frac{\Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)}}{z}\right\} < \beta, \tag{5}$$

for some $\beta(1 < \beta \leq 2)$, $\lambda(0 \leq \lambda \leq 1)$, $n \in N$ and $z \in U$.

We further let $\overline{R_H}(n,\beta,\lambda)$ denote the subclass of $R_H(n,\beta,\lambda)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form (3).

We note that for $n = 1, \lambda = 0$ and $g \equiv 0$ the class $R_H(n, \beta, \lambda)$ reduces to the class $R(\beta)$ studied by Uralegaddi et al. [19], (see also [6]).

In the present paper, we study the coefficient bounds, distortion bounds, extreme points, convolution condition, convex combinations, neighborhood problems and discuss a class preserving integral operator.

3. Main Results

First, we give a sufficient coefficient condition for functions in $R_H(n, \beta, \lambda)$.

Theorem 1. Let $f = h + \overline{g}$ be such that h and g are given by (1). Furthermore, let

$$\sum_{k=2}^{\infty} [\phi(k,\lambda)]^n |a_k| + \sum_{k=1}^{\infty} [\phi(k,\lambda)]^n |b_k| \leq \beta - 1. \tag{6}$$

Then f is sense-preserving, harmonic univalent in U and $f \in R_H(n, \beta, \lambda)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)}\right| \\
&> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k|} \\
&\geq 0,
\end{aligned}$$

which proves univalence.

Note that f is sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
&> 1-\sum_{k=2}^{\infty} k|a_k| \\
&\geq 1-\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k| \\
&\geq \sum_{k=1}^{\infty} k|b_k| \\
&> \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \\
&\geq |g'(z)|.
\end{aligned}$$

Now, we show that $f \in R_H(n,\beta,\lambda)$. Using the fact that $Re\ \omega < \beta$, if and only if, $|\omega - 1| < |\omega + 1 - 2\beta|$, it suffices to show that

$$\left| \frac{\dfrac{\Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)}}{z} - 1}{\dfrac{\Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)}}{z} - (2\beta - 1)} \right| < 1, \quad z \in U.$$

We have

$$\begin{aligned}
&\left| \frac{\dfrac{z+\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty}\overline{[\phi(k,\lambda)]^n b_k z^k}}{z} - 1}{\dfrac{z+\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty}\overline{[\phi(k,\lambda)]^n b_k z^k}}{z} - (2\beta-1)} \right| \\
= &\left| \frac{\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^{k-1} + (-1)^n \dfrac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{[\phi(k,\lambda)]^n b_k z^{k-1}}}{2(\beta-1) - \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^{k-1} - (-1)^n \dfrac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{[\phi(k,\lambda)]^n b_k z^k}} \right| \\
\leq &\ \frac{\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n |a_k| |z|^{k-1} + \sum_{k=1}^{\infty}[\phi(k,\lambda)]^n |b_k| |z|^{k-1}}{2(\beta-1) - \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n |a_k| |z|^{k-1} - \sum_{k=1}^{\infty}[\phi(k,\lambda)]^n |b_k| |z|^{k-1}} \\
\leq &\ \frac{\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n |a_k| + \sum_{k=1}^{\infty}[\phi(k,\lambda)]^n |b_k|}{2(\beta-1) - \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n |a_k| - \sum_{k=1}^{\infty}[\phi(k,\lambda)]^n |b_k|}
\end{aligned}$$

which is bounded above by 1 by using (6) and so the proof is complete.

The harmonic univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} \frac{\beta-1}{[\phi(k,\lambda)]^n} x_k z^k + \sum_{k=1}^{\infty} \frac{\beta-1}{[\phi(k,\lambda)]^n} \overline{y_k z^k}, \tag{7}$$

where $1 < \beta \leq 2,\ 0 \leq \lambda \leq 1,\ n \in N$ and $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (6) is sharp. It is worthy to note that the function of the form (7) belongs to the class $R_H(n,\beta,\lambda)$ for all $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\le 1$ because coefficient inequality (6) holds.

Theorem 2. Let f_n be given by (3). Then $f_n\in\overline{R_H}(n,\beta,\lambda)$ if and only if

$$\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n|a_k|+\sum_{k=1}^{\infty}[\phi(k,\lambda)]^n|b_k|\le\beta-1.$$

Proof. Since $\overline{R_H}(n,\beta,\lambda)\subset R_H(n,\beta,\lambda)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (3), we notice that the condition

$$\mathrm{Re}\left\{\frac{\Omega^n h(z)+(-1)^n\overline{\Omega^n g(z)}}{z}\right\}<\beta$$

is equivalent to

$$\begin{aligned}&\mathrm{Re}\left\{1+\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^{k-1}+(-1)^n\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{[\phi(k,\lambda)]^n b_k z^{k-1}}\right\}\\ \le\;& 1+\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n|a_k||z|^{k-1}+\sum_{k=1}^{\infty}[\phi(k,\lambda)]^n|b_k||z|^{k-1}<\beta,\quad z\in U.\end{aligned}$$

The above condition must hold for all values of z, $|z|=r<1$. Upon choosing the values of z to be real and let $z\to1^-$, we obtain

$$\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n|a_k|+\sum_{k=1}^{\infty}[\phi(k,\lambda)]^n|b_k|\le\beta-1,$$

which is the required condition.

The harmonic univalent functions of the form

$$f_n(z)=z+\sum_{k=2}^{\infty}\frac{\beta-1}{[\phi(k,\lambda)]^n}x_k z^k+(-1)^n\sum_{k=1}^{\infty}\frac{\beta-1}{[\phi(k,\lambda)]^n}y_k\overline{z^k},\tag{8}$$

where $1<\beta\le2,\ 0\le\lambda\le1,\ n\in N,\ x_k\ge0,\ y_k\ge0$ and $\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k\le1$ belongs to the class $\overline{R_H}(n,\beta,\lambda)$.

Theorem 3. If $f\in\overline{R_H}(n,\beta,\lambda)$, then

$$|f(z)|\le(1+|b_1|)r+\left(\frac{1-\lambda}{2}\right)^n(\beta-1-|b_1|)r^2,\ |z|=r<1$$

and

$$|f(z)|\ge(1-|b_1|)r-\left(\frac{1-\lambda}{2}\right)^n(\beta-1-|b_1|)r^2,\ |z|=r<1.$$

Proof. Let $f \in \overline{R_H}(n,\beta,\lambda)$. Taking the absolute value of f, we have

$$\begin{aligned} |f(z)| \leq & (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\ & \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\ & \leq (1+|b_1|)r + \left(\tfrac{1-\lambda}{2}\right)^n \sum_{k=2}^{\infty}\left(\frac{2}{1-\lambda}\right)^n (|a_k|+|b_k|)r^2 \\ & \leq (1+|b_1|)r + \left(\tfrac{1-\lambda}{2}\right)^n \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n(|a_k|+|b_k|)r^2 \\ & \leq (1+|b_1|)r + \left(\tfrac{1-\lambda}{2}\right)^n (\beta-1-|b_1|)r^2 \end{aligned}$$

and

$$\begin{aligned} |f(z)| & \geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\ & \geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\ & \geq (1-|b_1|)r - \left(\tfrac{1-\lambda}{2}\right)^n \sum_{k=2}^{\infty}\left(\frac{2}{1-\lambda}\right)^n (|a_k|+|b_k|)r^2 \\ & \geq (1-|b_1|)r - \left(\tfrac{1-\lambda}{2}\right)^n \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n(|a_k|+|b_k|)r^2 \\ & \geq (1-|b_1|)r - \left(\tfrac{1-\lambda}{2}\right)^n (\beta-1-|b_1|)r^2. \end{aligned}$$

Theorem 4. Let $f \in \mathrm{clco}\overline{R_H}(n,\beta,\lambda)$, if and only if

$$f(z) = \sum_{k=1}^{\infty}(\lambda_k h_k(z) + \gamma_k g_k(z)), \tag{9}$$

where $h_1(z) = z$

$$\begin{aligned} h_k(z) &= z + \tfrac{\beta-1}{[\phi(k,\lambda)]^n} z^k, \qquad (k = 2,3,...) \\ g_k(z) &= z + (-1)^n \tfrac{\beta-1}{[\phi(k,\lambda)]^n} \overline{z}^k, \qquad (k = 1,2,3,...) \end{aligned}$$

and $\sum_{k=1}^{\infty}(\lambda_k + \gamma_k) = 1,\ \lambda_k \geq 0$ and $\gamma_k \geq 0$.

In particular the extreme points of $\overline{R_H}(n,\beta,\lambda)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (9) we may write

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\{\lambda_k h_k(z) + \gamma_k g_k(z)\} \\ &= z + \sum_{k=2}^{\infty}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n}\right)\lambda_k z^k + (-1)^n \sum_{k=1}^{\infty}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n}\right)\gamma_k \overline{z}^k. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n}\lambda_k\right)+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n}\gamma_k\right)$$

$$=\sum_{k=2}^{\infty}\lambda_k+\sum_{k=1}^{\infty}\gamma_k$$

$$=1-\lambda_1\leq 1,$$

and so $f\in \text{clco}\ \overline{R_H}(n,\beta,\lambda)$.

Conversely, suppose that $f\in \text{clco}\ \overline{R_H}(n,\beta,\lambda)$.

Set

$$\lambda_k=\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k|,\ (k=2,3,4,...)$$

and

$$\gamma_k=\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k|,\ (k=1,2,3,...).$$

Then note that by Theorem 2,

$$0\leq\lambda_k\leq 1,\ (k=2,3,4,...)$$

and

$$0\leq\gamma_k\leq 1,\ (k=1,2,3,...).$$

We define $\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k-\sum_{k=1}^{\infty}\gamma_k$ and note that by Theorem 2, $\lambda_1\geq 0$. Consequently, we obtain $f(z)=\sum_{k=1}^{\infty}\{\lambda_k h_k(z)+\gamma_k g_k(z)\}$ as required.

Theorem 5. $\overline{R_H}(n,\beta,\lambda)\subseteq S_H^*$ where $n\in N, 1<\beta\leq 2, 0\leq\lambda<1$.

Proof. Let $f\in\overline{R_H}(n,\beta,\lambda)$.

Then by Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k|\leq 1. \tag{10}$$

Now

$$\begin{aligned}&\sum_{k=2}^{\infty}k|a_k|+\sum_{k=1}^{\infty}k|b_k|\\ \leq\ &\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k|\\ \leq\ &1,\ (\text{Using } (10)).\end{aligned}$$

Thus $f\in S_H^*$.

This completes the proof of the Theorem 5.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic function of the form

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F(z)=z+\sum_{k=2}^{\infty}|A_k|z^k+(-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$

we define their convolution

$$(f*F)(z)=f(z)*F(z)=z+\sum_{k=2}^{\infty}|a_kA_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_kB_k|\overline{z}^k, \qquad (11)$$

using this definition, we show that the class $\overline{R_H}(n,\beta,\lambda)$ is closed under convolution.
Theorem 6. For $1<\beta\le\alpha\le 2$, let $f\in\overline{R_H}(n,\beta,\lambda)$ and $F\in\overline{R_H}(n,\alpha,\lambda)$.
Then $(f*F)(z)\in\overline{R_H}(n,\beta,\lambda)\subseteq\overline{R_H}(n,\alpha,\lambda)$.

Proof. Let $f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $\overline{R_H}(n,\beta,\lambda)$ and $F(z)=z+\sum_{k=2}^{\infty}|A_k|z^k+(-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^k$ be in $\overline{R_H}(n,\alpha,\lambda)$. Then the convolution $(f*F)(z)$ is given by (11). We wish to show that the coefficients of $f*F$ satisfy the required condition given in Theorem 2. For $F(z)\in\overline{R_H}(n,\alpha,\lambda)$, we note that $|A_k|\le 1$ and $|B_K|\le 1$. Now, for the convolution function $(f*F)(z)$ we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_kA_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_kB_k|\\
\le\ &\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k|\\
\le\ &1,\ (\text{since } f\in\overline{R_H}(n,\beta,\lambda)).
\end{aligned}$$

Therefore $(f*F)(z)\in\overline{R_H}(n,\beta,\lambda)\subseteq\overline{R_H}(n,\alpha,\lambda)$.
Theorem 7. The class $\overline{R_H}(n,\beta,\lambda)$ is closed under convex combination.
Proof. For $i=1,2,3...$ let $f_i(z)\in\overline{R_H}(n,\beta,\lambda)$ where $f_i(z)$ is given by

$$f_i(z)=z+\sum_{k=2}^{\infty}|a_{k_i}|z^k+(-1)^n\sum_{k=1}^{\infty}|b_{k_i}|\overline{z}^k.$$

Then by Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_{k_i}|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_{k_i}|\le 1.$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\le t_i\le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{k_i}|\right)z^k+(-1)^n\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{k_i}|\right)\overline{z}^k.$$

Then by Theorem 2, we have

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}\left(\sum_{i=1}^{\infty}t_i|a_{k_i}|\right)+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}\left(\sum_{i=1}^{\infty}t_i|b_{k_i}|\right)\\ =\ &\sum_{i=1}^{\infty}t_i\left(\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_{k_i}|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_{k_i}|\right)\\ \le\ &\sum_{i=1}^{\infty}t_i=1. \end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty}t_if_i(z)\in\overline{R_H}(n,\beta,\lambda).$$

The δ-neighborhood of f is the set, (see [2], [14])

$$N_\delta(f)=\left\{F:F(z)=z+\sum_{k=2}^{\infty}|A_k|z^k+(-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^k\text{and}\sum_{k=1}^{\infty}k(|a_k-A_k|+|b_k-B_k|\le\delta)\right\}.$$

Theorem 8. Let $f\in\overline{R_H}(n,\beta,\lambda)$ and $\delta\le 2-\beta$. If $F\in N_\delta(f)$, then F is harmonic starlike function.

Proof. Let $F(z)=z+\sum_{k=2}^{\infty}|A_k|z^k+(-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^k$ belong to $N_\delta(f)$. We have

$$\begin{aligned} &\sum_{k=2}^{\infty}k|A_k|+\sum_{k=1}^{\infty}k|B_k|\\ \le&\sum_{k=2}^{\infty}k(|a_k-A_k|+|b_k-B_k|)+\sum_{k=2}^{\infty}k(|a_k|+|b_k|)+|b_1-B_1|+|b_1|\\ \le&\ \delta+\beta-1\\ \le&\ 1. \end{aligned}$$

Hence, $F(z)$ is harmonic starlike function.

4. A Family of Class Preserving Integral Operator

Let $f(z)=h(z)+\overline{g(z)}\in S_H$ be given by (1) then $F(z)$ defined by relation

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt},\ (c>-1). \tag{12}$$

Theorem 9. Let $f(z)=h(z)+\overline{g(z)}\in S_H$ be given by (3) and $f(z)\in\overline{R_H}(n,\beta,\lambda)$ then $F(z)$ be defined by (12) also belong to $\overline{R_H}(n,\beta,\lambda)$.

Proof. Let

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

be in $\overline{R_H}(n,\beta,\lambda)$ then by Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n}{\beta-1}|b_k|\le 1. \tag{13}$$

By definition of $F(z)$ we have

$$F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \overline{z}^k.$$

Now

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} \left(\frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} \left(\frac{c+1}{c+k} |b_k| \right) \\ \leq & \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} |b_k| \\ \leq & 1. \end{aligned}$$

Thus $F(z) \in \overline{R_H}(n,\beta,\lambda)$.

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