THE UNIFIED PATHWAY FRACTIONAL INTEGRAL FORMULAE

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ABSTRACT. The aim of the present paper is to study unified pathway fractional integral formulae, whose model was introduced by Mathai [11]. Here, first we obtain pathway fractional integral formula (PFIF) for the $t^{\rho-1}(b-ct)^{-\sigma}$ and further apply this result to establish the second PFIF which involves the product of a general class of polynomials and Fox H- function. These formulae, besides being of very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. Our findings provide interesting unifications and extensions of a number of (new and known) results. The results obtained by Nair [16] follow as simple special cases of our main findings. In the end, we record some new PFIF involving the product of the Wright generalized hypergeometric functions and Mittag-Leffer functions as a special case of our second formula.

1. INTRODUCTION

The fractional integral operator involving various special functions, have found significant importance and applications in various subfield of applicable mathematical analysis. Since last four decades, a number of workers like Love [10], McBride [14], Kalla [2, 3], Kalla and Saxena [4], Saigo [18, 19, 20], Kilbas [5], Kilbas and Sebastian [6], Kiryakova [8, 9] and Machado et.al [25, 26] etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Smako, Kilbas and Marichev [21], Miller and Ross [15], Kiryakova [8, 9], Kilbas, Srivastava and Trujillo [7] and Debnath and Bhatta [1].

The aim of the present investigation is to obtain new pathway fractional integral formulae (PFIF), using series expansion method, for the product of a general class of polynomials and the Fox H-function. The pathway fractional integral operator further developed in the present paper is based on the pathway model of Mathai and Haubold. The importance of the present study lies in the fact. Then it can

be related and applicable to a wide variety of statistical densities. For more details on the pathway model, the readers are referred to the recent papers of Mathai and Haubold [12, 13].

Let $f(x) \in L(a,b)$, $\eta \in \mathbb{C}, \Re(\eta) > 0$, $a > 0$ and let us take a "pathway parameter" $\alpha < 1$. The pathway fractional integration operator studied in the paper is defined and represented as follows ([16], p.239):

$$\left(P_{0+}^{(\eta,\alpha)}f\right)(x) = x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} [1 - \frac{a(1-\alpha)t}{x}]^{\frac{\eta}{(1-\alpha)}} f(t)dt \qquad (1)$$

The pathway model related to above operator was introduced by Mathai [11] and studied further by Mathai and Haubold [12, 13]. For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.):

$$f(x) = c |x|^{\gamma-1} \left[1 - a(1-\alpha) |x|^\delta\right]^{\frac{\beta}{(1-\alpha)}} \qquad (2)$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, [1 - a(1-\alpha) |x|^\delta] > 0, \gamma > 0$, where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2} \frac{\delta [a(1-\alpha)]^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{1-\alpha} + 1\right)}, \text{ for } \alpha < 1 \qquad (3)$$

$$= \frac{1}{2} \frac{\delta [a(1-\alpha)]^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta}\right)}, \text{ for } \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0, \ \alpha > 1 \qquad (4)$$

$$= \frac{1}{2} \frac{\delta (a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \alpha \to 1. \qquad (5)$$

We observe that for $\alpha < 1$ it is a finite range density with $[1 - a(1-\alpha) |x|^\delta] > 0$ and (2) remains in the extended generalized type-1 beta family. The pathway density in (2), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p. d. f.

For $\alpha > 1$, we have

$$f(x) = c |x|^{\gamma-1} \left[1 + a(\alpha-1) |x|^\delta\right]^{-\frac{\beta}{(\alpha-1)}} , \qquad (6)$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \alpha > 1$, which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the Student-t density, the Cauchy density and many more.

Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (2) and (6) take the exponential from, since

$$\lim_{\alpha \to 1} c |x|^{\gamma-1} \left[1 - a(1-\alpha) |x|^\delta\right]^{\frac{\eta}{(1-\alpha)}} = \lim_{\alpha \to 1} c |x|^{\gamma-1} \left[1 + a(\alpha-1) |x|^\delta\right]^{-\frac{\eta}{(\alpha-1)}}$$

$$= c |x|^{\gamma-1} \exp(-a\eta |x|^\delta) \qquad (7)$$

This include the generalized Gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities.

when $\alpha \to 1_-$, $\left[1 - \frac{a(1-\alpha)}{x}\right]^{\frac{\eta}{(1-\alpha)}} \to e^{-\frac{a\eta}{x}t}$. Then, operator (1) reduces to the Laplace integral transform of f with parameter $\frac{a\eta}{x}$:

$$\left(P_{0+}^{(\eta,1)}f\right)(x) = x^\eta \int_0^\infty e^{-\frac{a\eta}{x}} f(t)dt = x^\eta L_f(\frac{a\eta}{x}) \tag{8}$$

when $\alpha = 0, a = 1$, then replacing η by $\eta - 1$ in (1) the integral operator reduces to the Riemann-Liouville fractional integral operator (For more details, we refer to :[7], [8], [17], [18] and [21]).

Also, $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [22, p.1, eqn. (1)]:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k (n = 0, 1, 2, \dots), \tag{9}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n, k \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel's polynomials and several others (see [24, pp. 158-161].

The generalized hypergeometric function of one variable defined and represented as follows is also required here:

$$_pF_q\left[\begin{array}{c} (a_p); \\ (b_q); \end{array} z\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j)_n}{\prod_{j=1}^q (b_j)_n} \frac{z^n}{n!}, \text{ provided } p \leq q \, or \, p = q + 1 \, and \, |z| < 1. \tag{10}$$

The H-function of two variables occurring in the paper is defined and represented in the following form ([23, p.82, eq. (6.1.1)]) :

$$H[x,y] = H_{P_1,Q_1:P_2,Q_2;P_3,Q_3}^{0,N_1:M_2,N_2;M_3,N_3}\left[\begin{array}{c} x \\ y \end{array}\middle|\begin{array}{c} (a_j;\alpha_j,A_j)_{1,P_1}: (c_j,\gamma_j)_{1,P_2};(e_j,E_j)_{1,P_3} \\ (b_j;\beta_j,B_j)_{1,Q_1}: (d_j,\delta_j)_{1,Q_2};(f_j,F_j)_{1,Q_3} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi,\eta)\theta_1(\xi)\theta_2(\eta)x^\xi y^\eta d\xi d\eta \tag{11}$$

where $\omega = \sqrt{(-1)}$,

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{N_1} \Gamma(1 - a_j + \alpha_j\xi + A_j\eta)}{\prod_{j=N_1+1}^{P_1} \Gamma(a_j - \alpha_j\xi - A_j\eta) \prod_{j=1}^{Q_1} \Gamma(1 - b_j + \beta_j\xi + B_j\eta)}, \tag{12}$$

$$\theta_1(\xi) = \frac{\prod_{j=1}^{M_2} \Gamma(d_j - \delta_j\xi) \prod_{j=1}^{N_2} \Gamma(1 - c_j + \gamma_j\xi)}{\prod_{j=M_2+1}^{Q_2} \Gamma(1 - d_j + \delta_j\xi) \prod_{j=N_2+1}^{P_2} \Gamma(c_j + \gamma_j\xi)} \tag{13}$$

and

$$\theta_2(\eta) = \frac{\prod_{j=1}^{M_3} \Gamma(f_j - F_j\eta) \prod_{j=1}^{N_3} \Gamma(1 - e_j + E_j\eta)}{\prod_{j=M_3+1}^{Q_3} \Gamma(1 - f_j + F_j\eta) \prod_{j=N_3+1}^{P_3} \Gamma(e_j - E_j\eta)}, \tag{14}$$

The nature of contour L_1, L_2 in (11), the various special cases and other details of the above function can be found in the book referred to above.

It may be remarked here that all the Greek letter occurring in the left-hand side of (11) are assumed to be positive real numbers for standization purposes. The definition of this function will, however, be meaningful even if some of these quantities are zero.

Again, it is assumed that the various H-functions of one and two variables occurring in the paper always satisfy their appropriate conditions of convergences ([23, pp.10-11 and 82-83]).

2. Main Results

2.1. Pathway Fractional Integral Formula 1.

$$\left\{ P_{0+}^{(\eta,\alpha)} \left(t^{\rho-1} \left(b - ct\right)^{-\sigma} \right) \right\}(x) = x^{\eta+\rho} \frac{b^{-\sigma}}{[a(1-\alpha)]^{\rho}} \frac{\Gamma(1 + \frac{\eta}{1-\alpha})\Gamma(\rho)}{\Gamma(1 + \frac{\eta}{1-\alpha} + \rho)}$$

$$\times \ _2F_1 \left[\begin{array}{c} \sigma, \rho \\ 1 + \frac{\eta}{1-\alpha} + \rho \end{array} ; \frac{c}{ba(1-\alpha)}x \right], \tag{15}$$

provided that

(i) $\eta, \rho, \sigma \in \mathbb{C}, \Re(\eta) > 0$, and $\alpha < 1$.

(ii) $\Re(\rho) > 0, \Re(\sigma) > 0$ and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$,

(iii) $\left|\frac{c}{b}x\right| < 1$.

2.2. Pathway Fractional Integral Formula 2.

$$\left\{ P_{0+}^{(\eta,\alpha)} \left(t^{\rho-1} \left(b - ct\right)^{-\sigma} S_n^m \left(e_1 t^{\rho_1} \left(b - ct\right)^{-\sigma_1}\right) H_{P,Q}^{M,N} \left[e_2 t^{\rho_2} \left(b - ct\right)^{-\sigma_2}\right] \right) \right\}(x)$$

$$= x^{\eta+\rho} \frac{b^{-\sigma}}{[a(1-\alpha)]^{\rho}} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} \left(e_1 b^{-\sigma_1}\right)^k \left(\frac{x}{a(1-\alpha)}\right)^{\rho_1 k} \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)$$

$$H_{2,1:P+1,Q;0,1}^{0,2:M,N+1;1,0} \left[\begin{array}{c} \frac{b^{-\sigma_2}e_2 x^{\rho_2}}{[a(1-\alpha)]^{\rho_2}} \\ \frac{-xc}{[ab(1-\alpha)]} \end{array} \middle| \begin{array}{c} (1 - \rho - \rho_1 k; \rho_2, 1), (1 - \sigma - \sigma_1 k; \sigma_2, 1): \\ (-\rho - \rho_1 k - \frac{\eta}{1-\alpha}; \rho_2, 1), -: \end{array} \right.$$

$$\left. \begin{array}{c} (a_1, \alpha_1), ..., (a_P, \alpha_P), (1 - \sigma - \sigma_1 k; \sigma_2); - \\ (b_1, \beta_1), ..., (b_Q, \beta_Q), -; (0, 1) \end{array} \right] \tag{16}$$

provided that

(i) $\eta, \rho, \rho_1, \rho_2, \sigma, \sigma_1, \sigma_2 \in \mathbb{C}, \Re\left(\frac{\eta}{1-\alpha}\right) > -1, \Re(\eta, \rho, \rho_1, \rho_2, \sigma, \sigma_1, \sigma_2) > 0$, and $\alpha < 1, e_1, e_2 \in \mathbb{R}$.

(ii) $\left|\frac{c}{b}x\right| < 1$.

Proof of (15): To prove the pathway fractional integral formula (PFIF) 1, we first express the $(b - ax)^{-\sigma}$ occurring on its left-hand side in the following binomial expansion form

$$(b - cx)^{-\sigma} = b^{-\sigma} \sum_{l=0}^{\infty} \frac{(\sigma)_l}{l!} \left(\frac{c}{b}x\right)^l ; \qquad \left|\frac{c}{b}x\right| < 1 \tag{17}$$

and then by using (1), taking $f(x) = x^{\rho+l-1}$, we have:

$$\left\{ P_{0+}^{(\eta,\alpha)} \left(t^{\rho-1} \left(b - ct\right)^{-\sigma} \right) \right\}(x)$$

$$= b^{-\sigma} \sum_{l=0}^{\infty} \frac{(\sigma)_l}{l!} \left(\frac{c}{b}\right)^l x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} t^{\rho+l-1} dt$$

$$= b^{-\sigma} x^{\eta+\rho} \frac{\Gamma(1 + \frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}} \sum_{l=0}^{\infty} \frac{(\sigma)_l}{[a(1-\alpha)]^l l!} \frac{\Gamma(\rho+l)}{\Gamma(1 + \frac{\eta}{1-\alpha} + \rho + l)} \left(\frac{c}{b}x\right)^l \tag{18}$$

Finally, interpreting the hypergeometric series of x in terms of the $_pF_q$-function, we easily arrive at the desired formula (15) after a little simplification.

Proof of (16): To prove PFIF 2, we first express $S_n^m[x]$ in its series form with the help of (9) and put the value of $H_{P,Q}^{M,N}[.]$ in terms of Mellin-Barnes contour integral by help of application of (11) (see also [23, p.10, eqn. (2.1.1)]). Interchanging the order of integration and summation (which is permissible under the conditions stated with PFIF 2) and using PFIF 1, then by using (1), taking $f(x) = x^{\rho+l+\rho_1 k+\rho_2 \xi-1}$ (16) assume the following form (Say I) after a little simplification:

$$I = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} (e_1)^k \left\{ \left(\frac{1}{2\pi i}\right)^2 \int_{L_1} \int_{L_2} \phi(\xi) e_2^\xi b^{-\sigma - \sigma_1 k - \sigma_2 \xi} \frac{\Gamma(\sigma + \sigma_1 k + \sigma_2 \xi)}{\Gamma(l+1)} \left(-\frac{c}{b}\right)^l \right.$$

$$\left. \times x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} t^{\rho + l + \rho_1 k + \rho_2 \xi - 1} dt d\xi dl \right\} \tag{19}$$

Finally, interpreting the ξ, l contour integrals in terms of the H-function of two variables with the help of (11), we shall be led fairly easily to our PFIF 2.

3. Special Cases and Applications

The pathway fractional integral formula 1 and 2 established earlier are unified in nature and act as key formulae. These results can be reduced to the Laplace integral transform formulae when the pathway parameter $\alpha \to 1$ in (15) and (16). Thus the general class of polynomial involved in PFIF 2 reduces to a large spectrum of polynomials listed by Srivastava and Singh [24] and so from formula 2 we can further obtain various pathway fractional integral formulas involving a number of simpler polynomials. Again, the H-function of one variable occurring in the PFIF 2 can be suitably specialized to a remarkably wide variety of useful functions which are expressible in terms of Wright hypergeometric functions and Mittag-Lafter functions of one variable. For example:
If in PFIF 1, we take $\alpha \to 1_-$, $\frac{\eta}{1-\alpha} \to \infty$ and expand the gamma-functions involved in (15) by using the Stirling formula, we have

$$\frac{\Gamma(1 + \frac{\eta}{1-\alpha})}{(1-\alpha)^{\rho+l} \Gamma(1 + \frac{\eta}{1-\alpha} + \rho + l)} \to \frac{1}{(\eta)^{\rho+l}}.$$

Hence

$$\lim_{\alpha \to 1_-} \left\{ P_{0+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma}\right)\right\} (x) = x^{\eta+\rho} b^{-\sigma} \sum_{l=0}^{\infty} \frac{(\sigma)_l}{l!} \frac{\Gamma(\rho+l)}{(a\eta)^{\rho+l}} \left(\frac{c}{b}x\right)^l, \tag{20}$$

provided that $\left|\frac{c}{b}x\right| < 1$.

The formula given by (20) can also be considered as the following Laplace transform formula with the help of (18):

$$\lim_{\alpha \to 1_-} \left\{ P_{0+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma}\right)\right\} (x) = x^\eta (b)^{-\sigma} \sum_{l=0}^{\infty} \frac{(\sigma)_l}{l!} \left(\frac{c}{b}\right)^l \int_0^\infty e^{-\frac{a\eta}{x}t} t^{\rho+l-1} dt$$

$$= x^{\eta+\rho} b^{-\sigma} \sum_{l=0}^{\infty} \frac{(\sigma)_l}{l!} \frac{\Gamma(\rho+l)}{(a\eta)^{\rho+l}} \left(\frac{c}{b}x\right)^l. \tag{21}$$

The conditions of validity of (21) can be easily obtained from those of (15).

Similarly, if we set $\alpha \to 1_-$, in the right hand side of PFIF 2, it (say R) tends to

$$R = x^{\eta+\rho} \frac{b^{-\sigma}}{[a(1-\alpha)]^\rho} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} \left(e_1 b^{-\sigma_1}\right)^k \left(\frac{x}{a(1-\alpha)}\right)^{\rho_1 k} \Gamma(1 + \frac{\eta}{1-\alpha})$$

$$\times \left(\frac{1}{2\pi i}\right)^2 \int_{L_1} \int_{L_2} \phi(\xi)(-\frac{c}{b})^l \frac{\Gamma(\sigma + \sigma_1 k + \sigma_2 \xi)\Gamma(\sigma + \sigma_1 k + \sigma_2 \xi + l)}{\Gamma(l+1)}$$

$$\times \frac{\Gamma(\rho + \rho_1 k + \xi \rho_2 + l)}{\Gamma(1 + \rho + \frac{\eta}{1-\alpha} + \rho_1 k + \xi \rho_2 + l)} \frac{(e_2)^\xi (x)^{\xi \rho_2 + l}}{[a(1-\alpha)]^{\xi \rho_2 + l}} d\xi dl$$

$$= x^{\eta+\rho} \frac{b^{-\sigma}}{(a\eta)^\rho} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} \left(e_1 b^{-\sigma_1}\right)^k \left(\frac{x}{a\eta}\right)^{\rho_1 k} \left(\frac{1}{2\pi i}\right)^2$$

$$\times \int_{L_1} \int_{L_2} \phi(\xi)(-\frac{c}{b})^l \frac{\Gamma(\sigma + \sigma_1 k + \sigma_2 \xi)\Gamma(\sigma + \sigma_1 k + \sigma_2 \xi + l)\Gamma(\rho + \rho_1 k + \xi \rho_2 + l)}{\Gamma(l+1)}$$

$$\times \frac{(e_2)^\xi (x)^{\xi \rho_2 + l}}{[a\eta]^{\xi \rho_2 + l}} \ d\xi dl.$$

Interpreting the above expression in terms of the H-function of two variables, we finally get the following result after a little simplification:

$$\lim_{\alpha \to 1_-} \left\{ P_{0^+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma} S_n^m \left(e_1 t^{\rho_1} (b-ct)^{-\sigma_1} \right) H_{P,Q}^{M,N} \left[e_2 t^{\rho_2} (b-ct)^{-\sigma_2} \right] \right) \right\} (x)$$

$$= x^{\eta+\rho} \frac{b^{-\sigma}}{(a\eta)^\rho} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} \left(e_1 b^{-\sigma_1} \right)^k \left(\frac{x}{a\eta} \right)^{\rho_1 k}$$

$$\times H_{2,0:P+1,Q;0,1}^{0,2:M,N+1;1,0} \left[\begin{array}{c} \frac{e_2 x^{\rho_2}}{[a\eta]^{\rho_2}} \\ \frac{-xc}{[ba\eta]} \end{array} \middle| \begin{array}{c} (1-\rho-\rho_1 k; \rho_2, 1), (1-\sigma-\sigma_1 k; \sigma_2, 1) : \\ - : \end{array} \right.$$

$$\left. \begin{array}{c} (a_1,\alpha_1),...,(a_P,\alpha_P),(1-\sigma-\sigma_1 k;\sigma_2); - \\ (b_1,\beta_1),...,(b_Q,\beta_Q);(0,1) \end{array} \right]. \tag{22}$$

The result given by (22) can also be express in the following form:

$$\lim_{\alpha \to 1_-} \left\{ P_{0^+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma} S_n^m \left(e_1 t^{\rho_1} (b-ct)^{-\sigma_1} \right) H_{P,Q}^{M,N} \left[e_2 t^{\rho_2} (b-ct)^{-\sigma_2} \right] \right) \right\} (x)$$

$$= \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} (e_1)^k \left\{ \left(\frac{1}{2\pi i} \right)^2 \int_{L_1} \int_{L_2} \phi(\xi) e_2^\xi b^{-\sigma-\sigma_1 k - \sigma_2 \xi} \frac{\Gamma(\sigma + \sigma_1 k + \sigma_2 \xi)}{\Gamma(l+1)} \left(-\frac{c}{b} \right)^l \right.$$

$$\left. \times x^\eta \int_0^\infty e^{-\frac{a\eta}{x} t} t^{\rho+l+\rho_1 k + \rho_2 \xi - 1} dt d\xi dl \right\}. \tag{23}$$

The above result can be interpreted as a Laplace transform formula of the product of a general class of polynomials and Fox H-function

If in PFIF 2, we reduce the Fox H-function to the exponential function [23] and $S_n^M[.]$ to the Hermite polynomial [24] by setting:

$$S_n^2 \to x^{n/2} H_n \left[\frac{1}{\sqrt[2]{x}} \right]$$

which case $m \to 2, A_{n,k} \to (-1)^k$, we have the following interesting consequences of the (16) after little simplification.

$$\left\{ P_{0^+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma} t^{n/2} H_n \left[\frac{1}{\sqrt[2]{t}} \right] \beta_1^{-1} t^{b_1/\beta_1} \exp(-t^{1/\beta_1}) \right) \right\} (x)$$

$$= x^{\eta+\rho} \frac{b^{-\sigma}}{[a(1-\alpha)]^\rho} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2k}}{k!} (-1)^k \left(\frac{x}{a(1-\alpha)} \right)^k \Gamma(1 + \frac{\eta}{1-\alpha}) \Gamma(\sigma)$$

$$\times H_{1,1:0,1;1,1}^{0,1:1,0;1,1} \left[\begin{array}{c} \frac{-x}{[a(1-\alpha)]^{\rho_2}} \\ \frac{-xc}{[ab(1-\alpha)]} \end{array} \middle| \begin{array}{c} (1-\rho-k;1,1) : -; (1-\sigma,1) \\ (-\rho-k-\frac{\eta}{1-\alpha};1,1) : (b_1,\beta_1);(0,1) \end{array} \right]. \tag{24}$$

provided that $\left| \frac{c}{b} x \right| < 1$. The conditions of validity of (24) can be easily obtained from those of (16).

If, we put $n, b_1 \to 0$ and make suitable adjustment in the parameter in (24), we arrive at the known result given by Nair [16].

If in (16), we reduce the Fox H-function occurring therein to generalized Mittag-Laffer-function [23], we easily get new and interesting result of PFIF 2 after little simplification:

$$\left\{ P_{0^+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma} S_n^m \left(e_1 t^{\rho_1} (b-ct)^{-\sigma_1} \right) E_{\beta,\mu}^\nu \left[e_2 t^{\rho_2} \right] \right) \right\} (x)$$

$$= x^{\eta+\rho} \frac{b^{-\sigma}}{[a(1-\alpha)]^{\rho}} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} \left(e_1 b^{-\sigma_1}\right)^k \left(\frac{x}{a(1-\alpha)}\right)^{\rho_1 k} \Gamma(1 + \frac{\eta}{1-\alpha})\Gamma(\sigma + \sigma_1 k)$$

$$\times H_{1,1:1,2;1,1}^{0,1:1,1;1,1} \left[\begin{array}{c} \frac{e_2 x^{\rho_2}}{[a(1-\alpha)]^{\rho_2}} \\ \frac{-xc}{[ab(1-\alpha)]} \end{array} \middle| \begin{array}{c} (1-\rho-\rho_1 k; \rho_2, 1): (1-\nu, 1); (1-\sigma-\sigma_1 k, 1) \\ (-\rho-\rho_1 k - \frac{\eta}{1-\alpha}; \rho_2, 1): (0,1), (1-\mu, \beta); (0,1) \end{array} \right], \quad (25)$$

provided that $\left|\frac{c}{b}x\right| < 1$. The conditions of validity of (25) can be easily obtained from those of (16).

If, we set $S_n^m[.] \to 1$ and make suitable adjustment in the parameter in (25), we arrive at the known result given by Nair [16].

If in PFIF 2, we reduce the Fox H-function occurring therein to Wright generalized hypergeometric function [23, p.19, eqn. (2.6.11)], we easily get after little simplification the following new and interesting result:

$$\left\{ P_{0+}^{(\eta,\alpha)} \left(t^{\rho-1} (b-ct)^{-\sigma} S_n^m \left(e_1 t^{\rho_1} (b-ct)^{-\sigma_1}\right) {}_P\psi_Q [e_2 t^{\rho_2}]\right) \right\}(x)$$

$$= x^{\eta+\rho} \frac{b^{-\sigma}}{[a(1-\alpha)]^{\rho}} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} \left(e_1 b^{-\sigma_1}\right)^k \left(\frac{x}{a(1-\alpha)}\right)^{\rho_1 k} \Gamma(1 + \frac{\eta}{1-\alpha})\Gamma(\sigma + \sigma_1 k)$$

$$\times H_{1,1:P,Q+1;1,1}^{0,1:1,P;1,1} \left[\begin{array}{c} \frac{-e_2 x^{\rho_2}}{[a(1-\alpha)]^{\rho_2}} \\ \frac{-xc}{[ab(1-\alpha)]} \end{array} \middle| \begin{array}{c} (1-\rho-\rho_1 k; \rho_2, 1): \\ (-\rho-\rho_1 k - \frac{\eta}{1-\alpha}; \rho_2, 1): \end{array} \right.$$

$$\left. \begin{array}{c} (1-a_1, \alpha_1), ..., (1-a_P, \alpha_P); (1-\sigma-\sigma_1 k; 0, 1) \\ (0,1), (1-b_1, \beta_1), ..., (1-b_Q, \beta_Q); (0,1) \end{array} \right], \quad (26)$$

provided that $\left|\frac{c}{b}x\right| < 1$. The conditions of validity of (26) can be easily obtained from those of (16).

4. Conclusion

In this paper, we have presented two pathway fractional integral formulas (PFIF). The results have been developed in terms of the generalized hypergeometric functions and the product of H-functions and a general class of polynomials respectively in a compact and elegant form with the help of Nair [16] operator. Most of the results obtained are in a form besides being of very general character has been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. The result obtained in the present paper provides an extension of the results given by Nair [16] as mentioned earlier.

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