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CERTAIN PROPERTIES OF THE I-FUNCTION OF R-VARIABLES & ITS MULTIPLE STIELTJES TRANSFORM

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ABSTRACT. The aim of this paper is to study certain properties of the I-function of r-variables and also obtain its multiple stieltjes transform.

1. Introduction and definitions

Recent development in the theory of I-functions have gained much interest due to introduction of multivariable I-function which has been introduced & studied by Prasad [3] and Prasad and Yadav [4]. Further Prasad and Singh [3] studied the Mellin and Laplace transform of multivariable I-function.

In this paper, certain properties, derivative formula & multiple Stieltjes Transform of I-function of r-variables have studied. The I-function which was introduced by Saxena [5] is an extension of Fox's H-function. On Specializing the parameters, I-function can be reduced almost all the known as well as unknown special functions.

DEFINITION 2.1:
The multivariable I-function represented by Prasad [5] as

$$I[z_1, ..., z_r] = I^{\{0,n_i\}_{2,r}:\left\{(m^{(i)},n^{(i)})\right\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\left\{(p^{(i)},q^{(i)})\right\}^{1,r}} \left[\begin{array}{c|c} z_1 & \\ \vdots & A:B \\ & C:D \\ z_r & \end{array} \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\zeta_1, ..., \zeta_r\right) \prod_{i=1}^{r} \left\{ \phi_i\left(\zeta_i\right) z_i^{\zeta_i} \right\} d\zeta_1 ... d\zeta_r \tag{1}$$

where $\omega = \sqrt{-1}$,

$$\psi\left(\zeta_1, ...\zeta_r\right) = \frac{\prod_{k=2}^{r} \left[\prod_{j=1}^{n_k} \Gamma\left(1 - a_{kj} + \sum_{i=1}^{k} \alpha_{kj}^{(i)} \zeta_i\right)\right]}{\prod_{k=2}^{r} \left[\prod_{j=n_k+1}^{p_k} \Gamma\left(a_{kj} - \sum_{i=1}^{k} \alpha_{kj}^{(i)} \zeta_i\right)\right]}$$

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$$\times \frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_k}\Gamma\left(1-b_{kj}+\sum_{i=1}^{k}\beta_{kj}^{(i)}\zeta_i\right)\right]} \tag{2}$$

$$\phi_i\left(\zeta_i\right)=\frac{\left[\prod_{k=1}^{m^{(i)}}\Gamma\left(b_k^{(i)}-\beta_k^{(i)}\zeta_i\right)\right]\left[\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}\zeta_i\right)\right]}{\left[\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}\zeta_i\right)\right]\left[\prod_{k=m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b_k^{(i)}+\beta_k^{(i)}\zeta_i\right)\right]} \tag{3}$$

$\forall i\in\{1,...,r\}$.

Also, $\{0,\ \mathrm{n}_i\}_{2,r} := 0,\ \mathrm{n}_2 : ... : 0,\ \mathrm{n}_r$,
$\{\mathrm{p}_i,\ \mathrm{q}_i\}_{2,r} := \mathrm{p}_2,\ \mathrm{q}_2 : ... : \mathrm{p}_r,\ \mathrm{q}_r$,

$$\left\{\left(m^{(i)},n^{(i)}\right)\right\}^{1,r} := \left(m^{(1)},n^{(1)}\right);...;\left(m^{(r)},n^{(r)}\right),$$

$$\left\{\left(p^{(i)},q^{(i)}\right)\right\}^{1,r} := \left(p^{(1)},q^{(1)}\right);...;\left(p^{(r)},q^{(r)}\right),$$

$$A :=: \left\{\left(a_{ij};\alpha_{ij}^{(1)},...,\alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}\right\} := \left(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)}\right)_{1,p_2};...;\left(a_{rj};\alpha_{rj}^{(1)},\alpha_{rj}^{(r)}\right)_{1,p_r}$$

$$B :=: \left\{\left(a_j^{(i)};\alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}\right\} := \left(a_j^{(1)},\alpha_j^{(1)}\right)_{1,p^{(1)}};...;\left(a_j^{(r)},\alpha_j^{(r)}\right)_{1,p^{(r)}}$$

$$C :=: \left\{\left(b_{ij};\beta_{ij}^{(1)},...,\beta_{ij}^{(i)}\right)_{1,q_i}^{2,r}\right\} := \left(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)}\right)_{1,q_2};...;\left(b_{rj};\beta_{rj}^{(1)},\beta_{rj}^{(r)}\right)_{1,q_r} \tag{4}$$

$$D :=: \left\{\left(b_j^{(i)};\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}\right\} := \left(b_j^{(1)},\beta_j^{(1)}\right)_{1,q^{(1)}};...;\left(b_j^{(r)},\beta_j^{(r)}\right)_{1,q^{(r)}}$$

such that n_i, p_i, q_i, $m^{(i)}$, $n^{(i)}$, $p^{(i)}$ $q^{(i)}$ *are non-negative integers and all* a_{ij}, b_{ij}, α_{ij}, β_{ij}, $a_j^{(i)}$, $b_j^{(i)}$, $\alpha_j^{(i)}$, $\beta_j^{(i)}$ *are complex numbers and the empty product denotes unity.*
The contour integral (1) *converges, if*

$$|\arg z_i| < \frac{1}{2}U_i\pi, U_i>0, i=1,...,r \tag{5}$$

where

$$U_i=\sum_{j=1}^{n^{(i)}}\alpha_j^{(i)}-\sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)}+\sum_{j=1}^{m^{(i)}}\beta_j^{(i)}-\sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)}+\left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)}-\sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right)$$
$$+...+\left(\sum_{j=1}^{n_r}\alpha_{rj}^{(i)}-\sum_{j=n_r+1}^{p_r}\alpha_{rj}^{(i)}\right)-\left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)}+...+\sum_{j=1}^{q_r}\beta_{rj}^{(i)}\right) \tag{6}$$

and $\mathrm{I}[\mathrm{z}_1, ..., \mathrm{z}_r] = O\left(|z_1|^{\alpha_1},...,|z_r|^{\alpha_r}\right)$, $\max\{|\mathrm{z}_1|,...,|\mathrm{z}_r|\}\to 0$,
where $\alpha_i=\min_{1\le j\le m^{(i)}}\Re\left(b_j^{(i)}\big/\beta_j^{(i)}\right)$, *and* $\beta_i=\max_{1\le j\le n^{(i)}}\Re\left(a_j^{(i)}-1\big/\alpha_j^{(i)}\right)$, i = 1, ..., r.

For the condition of convergence and analyticity of multivariable I-function we refer [2, 4]. *Further A,B,C,D notations used throughout this paper, will be given by above equation* (4).

DEFINITION 2.2: Differentiation *formula for the I-function given by* [6] *as-*

$$\frac{d}{dx}\left(x^{\lambda} I^{m,n}_{p_i,q_i:r}\left[zx^{\sigma} \middle| \begin{array}{c} (a_j,\alpha_j):(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j):(b_{ji},\beta_{ji}) \end{array} \right]\right) = x^{\lambda-1} I^{m,n+1}_{p_i+1,q_i+1:r}\left[zx^{\sigma} \middle| \begin{array}{c} (-\lambda,\sigma)(a_j,\alpha_j):(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j):(b_{ji},\beta_{ji})(-\lambda+1,\sigma) \end{array} \right]$$

where σ & λ are real and positive.

DEFINITION 2.3:

The generalized Stieltjes transformation is taken in the form of [1] *as*

$$\Im\left(f(x);y\right) = \int_0^{\infty} x^{\rho-1}(x+y)^{-\sigma} f(x)dx. \tag{7}$$

where σ & ρ are real and positive.

2. Identities

IDENTITY 1:

$$\text{I}\,[\text{z}_1,\,\text{z}_2,\,\ldots\,\text{z}_r]:\; I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)})\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)},q^{(i)})\}^{1,r}}\left[\begin{array}{c} z_1^{k_1} \\ z_2^{k_2} \\ \vdots \\ z_r^{k_r} \end{array} \middle| \begin{array}{c} A_1:B_1 \\ C_1:D_1 \end{array} \right]$$

$$= \frac{1}{\prod_{i=1}^{r} k_i} I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)})\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)},q^{(i)})\}^{1,r}}\left[\begin{array}{c} z_1 \\ \vdots \\ \vdots \\ z_r \end{array} \middle| \begin{array}{c} A:B \\ C:D \end{array} \right] \tag{8}$$

where

$\text{B}_1 = \left\{ \left(a_j^{(i)} - k_i, \alpha_j^{(i)} \right)^{1,r}_{1,p^{(i)}} \right\}$, with the convergence conditions of I-function & provided $\left| 1 - \lambda_i^{\frac{-1}{\alpha_i}} \right| < 1$, $\text{n}^{(i)} \geq 1$, where i = 1, 2, ... r.

$\text{A}_1 = \left\{ \left(a_{ij}, k_1\alpha_{ij}^{(1)};...;k_i\alpha_{ij}^{(i)} \right)^{2,r}_{1,p_i} \right\}$ $\text{B}_1 = \left\{ \left(a_j^{(i)}, k_i\alpha_j^{(i)} \right)^{1,r}_{1,p^{(i)}} \right\}$

$\text{C}_1 = \left\{ \left(b_{2j}; k_1\beta_{ij}^{(1)};...;k_i\beta_{ij}^{(i)} \right)^{2,r}_{1,q_i} \right\}$

$\text{D}_1 = \left\{ \left(b_j^{(i)}, k_i\beta_j^{(i)} \right)^{1,r}_{1,q^{(i)}} \right\}$

with the convergence and analyticity conditions as defined by [5,6]

Further $\text{k}_i > 0$, i = 1 ... r

IDENTITY 2:

$$I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)})\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)},q^{(i)})\}^{1,r}}\left[\begin{array}{c} \lambda_1 z_1 \\ \vdots \\ \vdots \\ \lambda_r z_r \end{array} \middle| \begin{array}{c} A:B \\ C:D \end{array} \right]$$

$$= \prod_{i=1}^{r} \left\{ \lambda_i^{\frac{a_i-1}{\alpha_i}} \sum_{k_i=0}^{\infty} \frac{\left(1-\lambda_i^{\frac{-1}{\alpha_i}}\right)^{k_i}}{(k_i)\,!} \right\} \times I^{\{0,n_i\}_{2,r}:\left\{\left(m^{(i)},n^{(i)}\right)\right\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\left\{\left(p^{(i)},q^{(i)}\right)\right\}^{1,r}} \left[\begin{array}{c|c} z_1 & \\ \vdots & A:B_1 \\ \vdots & C:D \\ z_r & \end{array} \right] \tag{9}$$

where

$B_1 = \left\{ \left(a_j^{(i)} - k_i, \alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r} \right\}$, with the convergence conditions of I-function & provided $\left|1-\lambda_i^{\frac{-1}{\alpha_i}}\right| < 1$, $n^{(i)} \geq 1$, where i = 1, 2, ... r.

IDENTITY 3:

$$I\left[\lambda_1 z_1, ..., \lambda_r z_r \left| \begin{array}{c} A:B \\ C:D \end{array} \right.\right] = \prod_{i=1}^{r} \left\{ \lambda_i^{\frac{b_i}{\beta_i}} \sum_{k_i=0}^{\infty} \frac{\left(1-\lambda_i^{\frac{1}{\beta_i}}\right)^{k_i}}{(k_i)\,!} I\left[z_1, z_2, ..., z_r \left| \begin{array}{c} A:B \\ C:D_1 \end{array} \right.\right] \right\} \tag{10}$$

where

$D_1 = \left(b_i^{(i)} + k_i, \beta_i\right)^{1,r} \left(b_j^i, \beta_j^i\right)$, $\left|1-\lambda_i^{\frac{1}{\beta_i}}\right| < 1$ & $m^{(i)} \geq 1$ ∀ i = 1, 2, ... r

with the convergence conditions of I-function as stated earlier.

IDENTITY 4:

$$\prod_{i=1}^{r} (b_i \beta_{R_i} - b_{R_i}\beta_i)\, I\left[z_1, z_2, ..., z_r \left| \begin{array}{c} A:B \\ C:D \end{array} \right.\right]$$

$$= \prod_{i=1}^{r} \beta_{R_i} I\left[z_1, z_2, ..., z_r \left| \begin{array}{c} A:B \\ C:D_1 \end{array} \right.\right]$$

$$+ \prod_{i=1}^{r} \beta_i\, I\left[z_1, z_2, ..., z_r \left| \begin{array}{c} A:B \\ C:D_2 \end{array} \right.\right] \tag{11}$$

where D_1 & D_2 are given as

$D_1 = \left\{ (b_i + 1, \beta_{R_i}) \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r} \right\}$

$D_2 = \left\{ (b_{Ri} + 1, \beta_{R_i}) \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r} \right\}$ Provided convergence and analyticity of multivariable I-function.

& $m^{(i)} \geq 1$, $q^{(i)} \geq 2$, $b_{q_i i} = b_{Ri}$, $\beta_{q_i i} = \beta_{Ri}$

for all values of i from 1 to r.

3. Differential formula

The multi differential formula for I-function of r variable is given as:-

$$\frac{\partial}{\partial x_1 \partial x_2 ... \partial x_r}\left\{\prod_{i=1}^{r} x_i^{\lambda_i} I\left[z_1 x_1^{\sigma_1}, ..., z_r x_r^{\sigma_r}\right]\right\}$$

$$= \prod_{i=1}^{r} x_i^{\lambda_i - 1}\, I^{\{0,n_i\}_{2,r}:\{m^i,n^i+1\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{p^i+1,q^i+1\}^{1,r}} \left[\begin{array}{c|c} z_1 x_1^{\sigma_1} & \\ \vdots & A' : B' \\ \vdots & C' : D' \\ z_r x_r^{\sigma_r} & \end{array}\right] \tag{12}$$

where
A' $= \left\{\left(a_{rj};\alpha_{rj}^{1},...,\alpha_{rj}^{r}\right)_{1,p_i}^{2,r}\right\}$ B' $= \left\{\left(-\lambda_i,\sigma_i\right)\left(a_j^i,...,\alpha_j^i\right)_{1,p^i}^{1,r}\right\}$
C' $= \left\{\left(b_{rj};\beta_{rj}^{1},...,\beta_{rj}^{r}\right)_{1,q_i}^{2,r}\right\}$ D' $= \left\{\left(-\lambda_i+1,\sigma_i\right)\left(b_j^i,...,\beta_j^i\right)_{1,q^i}^{1,r}\right\}$
Provided convergence condition of I-function (5) satisfied
& σ_i $\forall = 1$ ... r are real and positive.

4. Generalized stieltjes transform of the multivariable i- function

$$\Im^r(s) = \int_0^\infty ...... \int_0^\infty \prod_{i=1}^{r} t_i^{\rho_i - 1}\left(t_i+\beta_i\right)^{-\sigma_i} I\left[\alpha_1 t_1, \alpha_2 t_2 .... \alpha_r t_r\right].dt_1, dt_2....dt_r$$

$$= \prod_{i=1}^{r}\left(\frac{\beta_i^{(\rho_i-\sigma_i)}}{\Gamma_{\sigma_i}}\right) I^{\{0,n_i\}_{2,r}:\{m^i+1,n^i+1\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{p^i+1,q^i+1\}^{1,r}} \left[\begin{array}{c|c} \alpha_1\beta_1 & \\ \vdots & A" : B" \\ \vdots & C" : D" \\ \alpha_r\beta_r & \end{array}\right] \tag{13}$$

where
A" $= \left\{\left(a_{rj};\alpha_{rj}^{(1)},...,\alpha_{rj}^{r}\right)_{1,p_i}^{2,r}\right\}$ B" $= \left\{\left(1-\rho_i\right)\left(\alpha_j^i,...,\alpha_j^i\right)_{1,p^i}^{1,r}\right\}$
C" $= \left\{\left(b_{rj};\beta_{rj}^{1},...,\beta_{rj}^{r}\right)_{1,q_i}^{2,r}\right\}$ D" $= \left\{\left(\sigma_i-\rho_i\right)\left(b_j^i,...,\beta_j^i\right)_{1,q^i}^{1,r}\right\}$
& Re $(\sigma_i) > 0$, where i = 1, 2 ... r
provided the convergence condition of I-function (5) and involving integrals [1] are satisfied.

Proof : First we express I-function of r-variables on left side of equation (13) as a product of multiple Mellin-Barnes contour integral by using (1) and interchanging the order of integration and applied same approach as generalized Stieltjes transform of I-function of single variable [6] we can easily be arrived at R.H.S of (13).

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