Journal of Fractional Calculus and Applications,
Vol. 4(1) Jan. 2013, pp. 82-98.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

EXISTENCE AND UNIQUENESS RESULT OF SOLUTIONS TO INITIAL VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE-ORDER

SHUQIN ZHANG

Abstract. In this work, an initial value problem is discussed for a fractional differential equation of variable-order. By means of some analysis techniques and Arzela-Ascoli theorem, existence result of solution is obtained; Using the upper solutions and lower solutions and monotone iterative method, uniqueness existence results of solutions are obtained.

1. Introduction

The fractional operators (fractional derivatives and integrals) refer to the differential and integral operators of arbitrary order, and fractional differential equations refer to those containing fractional derivatives. The former are the generalization of integer-order differential and integral operators and the latter, the generalization of differential equations of integer order. The fractional operators of variable order, which fall into a more complex category, are the derivatives and integrals whose order is the function of certain variables. In recent years, fractional operators and fractional differential equations of variable order have been applied in engineering more and more frequently, For the examples and details, see [1]-[14] and the references therein. Their extensive applications urgently call for systematic studies on the existence, uniqueness of solutions to initial value problems of these equations. Research in this area is at the trailblazing stage and has so far but produced a very limited number of published papers dealing with relatively simple problems with limited methods, such as [15], [16].

2000 *Mathematics Subject Classification.* 26A33, 34A12.

Key words and phrases. fractional calculus of variable-order, fractional differential equation, Arzela-Ascoli theorem, existence of solution, lower solution, upper solution, monotone iterative, uniqueness of solution.

Submitted July. 6, 2012. Published Jan. 3, 2013.

Research supported by the NNSF of China (10971221), the Ministry of Education for New Century Excellent Talent Support Program(NCET-10-0725) and the Fundamental Research Funds for the Central Universities(2009QS06).

The following are several definitions of fractional integral and fractional derivatives of variable-order, which can be founded in [14]:

$$I_{a+}^{p(t)}f(t)=\int_a^t\frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}f(s)ds,\quad p(t)>0,t>a, \tag{1}$$

where $\Gamma(\cdot)$ denotes the Gamma function, $-\infty<a<+\infty$, provided that the right-hand side is pointwise defined.

$$I_{a+}^{p(t)}f(t)=\int_a^t\frac{(t-s)^{p(s)-1}}{\Gamma(p(s))}f(s)ds,\quad p(t)>0,t>a, \tag{2}$$

provided that the right-hand side is pointwise defined.

$$I_{a^+}^{p(t)}f(t)=\int_a^t\frac{(t-s)^{p(t-s)-1}}{\Gamma(p(t-s))}f(s)ds,\quad p(t)>0,t>a, \tag{3}$$

provided that the right-hand side is pointwise defined.

$$D_{a+}^{p(t)}f(t)=\frac{d^n}{dt^n}I_{a+}^{n-p(t)}f(t)=\frac{d^n}{dt^n}\int_a^t\frac{(t-s)^{n-1-p(t)}}{\Gamma(n-p(t))}f(s)ds,\quad t>a, \tag{4}$$

where $n-1<p(t)<n, t>a, n\in N$, provided that the right-hand side is pointwise defined.

$$D_{a+}^{p(t)}f(t)=\frac{d^n}{dt^n}I_{a+}^{n-p(t)}f(t)=\frac{d^n}{dt^n}\int_a^t\frac{(t-s)^{n-1-p(s)}}{\Gamma(n-p(s))}f(s)ds,\quad t>a, \tag{5}$$

where $n-1<p(t)<n, t>a, n\in N$, provided that the right-hand side is pointwise defined.

$$D_{a^+}^{p(t)}f(t)=\frac{d^n}{dt^n}I_{a+}^{n-p(t)}f(t)=\frac{d^n}{dt^n}\int_a^t\frac{(t-s)^{n-1-p(t-s)}}{\Gamma(n-p(t-s))}f(s)ds,\quad t>a, \tag{6}$$

where $n-1<p(t)<n, t>a, n\in N$, provided that the right-hand side is pointwise defined.

In particular, when $p(t)$ is a constant function, i.e. $p(t)\equiv q$(q is a finite positive constant), then $I_{a+}^{p(t)}, D_{a+}^{p(t)}$ are the usual Riemann-Liouville fractional calculus[**?**]. It is well known that fractional calculus D_{a+}^q, I_{a+}^q have some very important properties, which play very important role in considering existence of solutions of fractional differential equation denoted by D_{a+}^q, by means of nonlinear analysis method. Such as, the following some properties, which can be founded in [17]:

Proposition 1.1.([17]**)** The equality $I_{a+}^{\gamma}I_{a+}^{\delta}f(t)=I_{a+}^{\gamma+\delta}f(t)$, $\gamma>0,\delta>0$ holds for $f\in L(a,b)$.

Proposition 1.2.([17]**)** The equality $D_{a+}^{\gamma}I_{a+}^{\gamma}f(t)=f(t)$, $\gamma>0$ holds for $f\in L(a,b)$.

Proposition 1.3.([17]**)** Let $\alpha>0$. Then the differential equation

$$D_{a+}^{\alpha}u=0$$

has unique solution

$$u(t)=c_1(t-a)^{\alpha-1}+c_2(t-a)^{\alpha-2}+\cdots+c_n(t-a)^{\alpha-n},$$

$c_i\in R, i=1,2,\cdots,n$, here $n-1<\alpha\le n$.

Proposition 1.4.([17]**)** Let $\alpha>0$, $u\in L(a,b)$, $D_{a+}^{\alpha}u\in L(a,b)$. Then the following equality holds

$$I_{a+}^{\alpha}D_{a+}^{\alpha}u(t)=u(t)+c_1(t-a)^{\alpha-1}+c_2(t-a)^{\alpha-2}+\cdots+c_n(t-a)^{\alpha-n},$$

$c_i \in R, i = 1, 2, \cdots, n$, here $n-1 < \alpha \leq n$.

But, in general, these properties don't hold for fractional calculus of variable-order $D_{a+}^{p(t)}, I_{a+}^{p(t)}$ defined by (1)-(6). For example,

$$I_{a+}^{p(t)} I_{a+}^{q(t)} f(t) \neq I_{a+}^{p(t)+q(t)} f(t), p(t) > 0, q(t) > 0, f \in L(a,b), \tag{7}$$

where $I_{a+}^{p(t)}$ denote one of fractional integrals defined by (1)-(3).

Example 1.1. Let $p(t) = t, 0 \leq t \leq 6$, $q(t) = \begin{cases} 2, & 0 \leq t \leq 2 \\ 1, & 2 < t \leq 3, \\ t, & 3 < t \leq 6, \end{cases}$ $f(t) = 1, 0 \leq t \leq 6$. We calculate $I_{0+}^{p(t)} f(t)$ and $I_{0+}^{p(t)+q(t)}$ defined by (1).

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = & \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\ = & \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\ = & \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{2-1}}{\Gamma(2)} d\tau ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\ = & \int_0^2 \frac{(t-s)^{p(t)-1} s^2}{2\Gamma(p(t))} ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds, \end{aligned}$$

$$I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} f(s) ds,$$

we see that

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} = & \int_0^2 \frac{(3-s)^{3-1} s^2}{2\Gamma(3)} ds + \int_2^3 \frac{(3-s)^{3-1}}{\Gamma(3)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds \\ = & \frac{8}{5} + \int_2^3 \frac{(3-s)^{3-1} s}{\Gamma(3)} ds = \frac{5}{8} + \frac{9}{24} = 1, \end{aligned}$$

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=3} = \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(s)+q(3))} f(s) ds = \int_0^3 \frac{(3-s)^{3+1-1}}{\Gamma(3+1)} ds = \frac{27}{8}$$

we see easily that

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}.$$

According to (7), we can see that Proposition $1.1 - 1.4$ don't hold for $D_{a+}^{p(t)}$ and $I_{a+}^{p(t)}$ defined by (1)-(6).

Example 1.2. Let $0 < p(t) < 1, t > 0$. By calculating, we have that

$$I_{0+}^{p(t)} 1 = \frac{1}{\Gamma(p(t))} \int_0^t (t-s)^{p(t)-1} ds = \frac{t^{p(t)}}{(p(t))\Gamma(p(t))}, t > 0,$$

$$D_{0+}^{p(t)} I_{0+}^{p(t)} 1 = \frac{d}{dt} I_{0+}^{1-p(t)} I_{0+}^{p(t)} 1 = \frac{d}{dt} \frac{1}{\Gamma(1-p(t))} \int_0^t \frac{(t-s)^{-p(t)} s^{p(s)}}{p(s)\Gamma(p(s))} ds \neq 1,$$

and

$$I_{0+}^{p(t)}D_{0+}^{p(t)}1 = I_{0+}^{p(t)}\frac{d}{dt}I_{0+}^{1-p(t)}1 = I_{0+}^{p(t)}(\frac{d}{dt}\frac{t^{1-p(t)}}{(1-p(t))\Gamma(1-p(t))}) \neq 1.$$

Remark 1.1. For fractional integral of variable-order defined by (5)-(6), we can't easily calculate out fractional integral $I_{a+}^{p(t)}$ of some functions $f(t)$, for example, we don't know that what $I_{a+}^{p(t)}1 = \int_a^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))}ds$ and $I_{a+}^{p(t)}1 = \int_a^t \frac{(t-s)^{p(t-s)-1}}{\Gamma(p(t-s))}ds$ equal.

Therefore, without those properties like as Propositions 1.1, 12, 1.3 and 1.4, ones can not transform a fractional differential equation of variable-order into an equivalent integral equation, so that one can consider existence of solutions of a fractional differential equation of variable-order, by means of nonlinear functional analysis method.

There also has more complex fractional calculus of variable-order, whose order function $p(t)$ of (1)-(6) are replaced by $p(t, f(t))$, please see [1], [15], [16]. In [15], authors considered the solution existence of the following variable order fractional differential equations

$$\begin{cases} D_{c+}^{p(t,x(t))}x(t) = f(t, x(t)), \quad c < t \leq b \\ x(c) = x_0, \end{cases} \tag{8}$$

where $D_{c+}^{p(t,x(t))}$ is a fractional derivative of variable-order defined by

$$D_{c+}^{p(t,x(t))}x(t) = \frac{d}{dt}\int_c^t \frac{(t-s)^{-p(s,x(s))}}{\Gamma(1-p(s,x(s)))}x(s)ds, t > c. \tag{9}$$

In [15], authors transformed (8) into one equivalent integral equation

$$x(t) = I_{c+}^{p(t,x(t))}f(t, x(t)) = \int_c^t \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))}f(s, x(s))ds, c \leq t \leq b, \tag{10}$$

and then, using approximated method, authors obtained existence result of solution for problem (8) with $\frac{1}{2} \leq q(t, x) \leq 1, c \leq t \leq b, x \in R$.

In my opinion, the problem and analysis techniques are interesting and meaning, but, there had a critical wrong, that is, the transformation(from (8) to (10)) is unsuitable, because fractional calculus $D_{c+}^{p(t,x(t))}, I_{c+}^{p(t,x(t))}$ don't usually have properties like Propositions 1.3 and 1.4. Among of those analysis, the sequence (6) has little mistakes, since ones can't know whether such sequence exists. As well, the initial value condition $x(c) = x_0$ isn't suitable when $x_0 \neq 0$, because, when $p(t, x(t))$ is a constant function, i.e. $p(t) \equiv q(0 < q < 1$ is a finite positive constant), then $D_{c+}^{p(t,x(t))}$ is the usual Riemann-Liouville fractional derivative D_{c+}^q. From [17], we know that Riemann-Liouville fractional derivative of order $0 < q < 1$ of constant x_0 is not zero, but is $\frac{x_0(t-c)^{-q}}{\Gamma(1-q)}, t > c$, as a result, fractional differential equation(involving Riemann-Liouville fractional derivative) can not have such $x(c) = x_0$ initial value condition, but is initial value condition $(t-c)^{1-q}x(t)|_{t=c}$ or $I_{c+}^{1-q}x(t)|_{t=c}$ ($(t-c)^{1-q}x(t)|_{t=c}$ and $I_{c+}^{1-q}x(t)|_{t=c}$ can transform each other, see [17]). Hence, in some degree, problem (8) is not a suitable problem(expect $x_0 = 0$).

In this paper, based on characters of fractional derivative of variable-order, by means of some analysis techniques, we will consider existence and uniqueness of solution to initial value problems for fractional differential equation of variable-order(IVP for short)

$$\begin{cases} D_{0+}^{q(t,x(t))}x(t) = f(t,x), \quad 0 < t \leq T, 0 < T < +\infty \\ x(0) = 0, \end{cases} \tag{11}$$

where $D_{0+}^{q(t,x(t))}$ denotes fractional derivative of variable-order defined by (9), $0 < q(t,x(t)) \leq q^* < 1, 0 \leq t \leq T, x \in R$, and $f : [0,T] \times R \to R$ is a continuous function.

The rest of this paper is organized as follows. In Section 2, we give some preparation results. In Section 3, the existence results of solutions for IVP (11) are presented. In section 4, uniqueness existence result od solution for a particular case of IVP(11).

2. Preliminaries

We assume that

(H_1) $q : [0,T] \times R \to (0,q^*], 0 < q^* < 1$, is a continuous function;

(H_2) $f : [0,T] \times R \to R$ is a continuous function.

It follows from the continuity of compose functions that $\Gamma(q(t,x(t)))$ is continuous on $[0,T]$, when q satisfies assumption condition (H_1). The following results will play very important role in proving our existence result of solution to problems (11).

Let δ is an arbitrary small positive constant.

Lemma 2.1. Let (H_1) hold. And let $x_n, x \in C[0,T]$, assume that $x_n(t) \to x(t), t \in [0,T]$ as $n \to \infty$, then

$$\int_0^{t-\delta} \frac{(t-s)^{-q(s,x_n(s))}}{\Gamma(1-q(s,x_n(s)))} x_n(s)ds \to \int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds, t \in [\delta,T], \tag{12}$$

as $n \to \infty$.

Proof. We see that

$$\text{if } \ 0 < T \leq 1, \ \text{ then } \ T^{-q(s,x(s))} \leq T^{-q^*}, \tag{13}$$

$$\text{if } \ 1 < T < +\infty, \ \text{ then } \ T^{-q(s,x(s))} < 1. \tag{14}$$

Thus, for $0 < T < +\infty$, we let

$$T^* = \max\{T^{-q^*}, 1\}. \tag{15}$$

We let

$$M = \max_{0\leq t\leq T} |x(t)| + 1, M_1 = \max_{0\leq t\leq T} |x_n(t)| + 1, L = \max_{0\leq t\leq T, \|x_n\|\leq M_1} |\frac{1}{\Gamma(1-q(t,x_n(t)))}| + 1.$$

By the convergence of x_n, for $\frac{(1-q^*)\varepsilon}{3LT^*T}$($\varepsilon$ is arbitrary small positive number), there exists $N_0 \in N$ such that

$$|x_n(t) - x(t)| < \frac{(1-q^*)\varepsilon}{3LT^*T}, t \in [0,T], n \geq N_0. \tag{16}$$

Since $(t-s)^{-q(s,x(s))}, \delta \leq t-s \leq T$, is continuous with respect to its exponent $-q(s,x(s))$, for $\frac{\varepsilon}{3MLT}$, when $n \geq n_0$, it holds

$$|(t-s)^{-q(s,x_n(s))} - (t-s)^{-q(s,x(s))}| < \frac{\varepsilon}{3MLT}, \delta \leq t-s \leq T, \tag{17}$$

also, by continuity of $\frac{1}{\Gamma(1-q(s,x(s)))}$, for $\frac{(1-q^*)\varepsilon}{3MT^*T}$, when $n \geq n_0$, it holds

$$|\frac{1}{\Gamma(1-q(s,x_n(s)))} - \frac{1}{\Gamma(1-q(s,x(s)))}| < \frac{(1-q^*)\varepsilon}{3MT^*T}, 0 \leq s \leq T. \tag{18}$$

Hence, from (13)-(18), we have that

$$\begin{aligned}
& |\int_0^{t-\delta} \frac{(t-s)^{-q(s,x_n(s))}}{\Gamma(1-q(s,x_n(s)))} x_n(s)ds - \int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds| \\
\leq & \int_0^{t-\delta} |\frac{(t-s)^{-q(s,x_n(s))}}{\Gamma(1-q(s,x_n(s)))}||x_n(s)-x(s)|ds \\
& + \int_0^{t-\delta} |\frac{(t-s)^{-q(s,x_n(s))} - (t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x_n(s)))}||x(s)|ds \\
& + \int_0^{t-\delta} |(t-s)^{-q(s,x(s))}||\frac{1}{\Gamma(1-q(s,x_n(s)))} - \frac{1}{\Gamma(1-q(s,x(s)))}||x(s)|ds \\
\leq & \frac{L(1-q^*)\varepsilon}{3LT^*T} \int_0^{t-\delta} (t-s)^{-q(s,x_n(s))} ds + \frac{ML\varepsilon}{3MLT} \int_0^{t-\delta} ds \\
& + \frac{M(1-q^*)\varepsilon}{3MT^*T} \int_0^{t-\delta} (t-s)^{-q(s,x(s))} ds \\
= & \frac{(1-q^*)\varepsilon}{3T^*T} \int_0^{t-\delta} T^{-q(s,x_n(s))} (\frac{t-s}{T})^{-q(s,x_n(s))} ds + \frac{\varepsilon}{3T} \int_0^{t-\delta} ds \\
& + \frac{(1-q^*)\varepsilon}{3T^*T} \int_0^{t-\delta} T^{-q(s,x(s))} (\frac{t-s}{T})^{-q(s,x(s))} ds \\
\leq & \frac{(1-q^*)\varepsilon}{3T^*T} \int_0^{t-\delta} T^* (\frac{t-s}{T})^{-q^*} ds + \frac{\varepsilon}{3T} \int_0^{t-\delta} ds + \frac{(1-q^*)\varepsilon}{3T^*T} \int_0^{t-\delta} T^* (\frac{t-s}{T})^{-q^*} ds \\
= & \frac{(1-q^*)\varepsilon}{3T^{1-q^*}} \int_0^{t-\delta} (t-s)^{-q^*} ds + \frac{\varepsilon}{3T} \int_0^{t-\delta} ds + \frac{(1-q^*)\varepsilon}{3T^{1-q^*}} \int_0^{t-\delta} (t-s)^{-q^*} ds \\
= & \frac{\varepsilon}{3T^{1-q^*}} (t^{1-q^*} - \delta^{1-q^*}) + \frac{\varepsilon}{3T}(t-\delta) + \frac{\varepsilon}{3T^{1-q^*}} (t^{1-q^*} - \delta^{1-q^*}) \\
< & \frac{\varepsilon T^{1-q^*}}{3T^{1-q^*}} + \frac{T\varepsilon}{3T} + \frac{\varepsilon T^{1-q^*}}{3T^{1-q^*}} \\
= & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which implies that (12) holds.

By the similar arguments, we can know that

Lemma 2.2. Let (H_2) hold. And let $x_n, x \in C[0,T]$, assume that $x_n(t) \to x(t), t \in [0,T]$ as $n \to \infty$, then

$$\int_0^{t-\delta} f(s,x_n(s))ds \to \int_0^{t-\delta} f(s,x(s))ds, t \in [\delta, T], \tag{19}$$

as $n \to \infty$.

Proof. By the convergence of x_n, for $\zeta > 0$, there exists $N_0 \in N$ such that

$$|x_n(t) - x(t)| < \zeta, t \in [0,T], n \geq N_0,$$

by the continuity of f, for $\frac{\varepsilon}{T}$ (where ε is arbitrary small number), when $n \geq N_0$, it holds

$$|f(s, x_n(s)) - f(s, x(s))| < \frac{\varepsilon}{T}, s \in [0, T].$$

Thus, we have that

$$\begin{aligned} & |\int_0^{t-\delta} (f(s, x_n(s)) - f(s, x(s)))ds| \\ \leq & \int_0^{t-\delta} |f(s, x_n(s)) - f(s, x(s))|ds \\ < & \frac{\varepsilon}{T} \int_0^{t-\delta} ds \\ = & \frac{\varepsilon}{T}(t-\delta) \\ \leq & \frac{\varepsilon T}{T} = \varepsilon, \end{aligned}$$

which implies that (19) holds.

Lemma 2.3. Assume that (H_1) hold. Then for arbitrary fixed $x \in C[0, T]$, the following expression holds

$$\lim_{\delta \to 0} \int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1 - q(s, x(s)))} x(s)ds = \int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1 - q(s, x(s)))} x(s)ds. \quad (20)$$

Proof. For arbitrary fixed $x \in C[0, T]$, we let

$$M = \max_{0 \leq t \leq T} |x(t)| + 1, L = \max_{0 \leq t \leq T, \|x\| \leq M} \frac{1}{\Gamma(1 - q(t, x(t)))} + 1.$$

Thus, for arbitrary fixed function $x \in C[0, T]$, for $\forall\ \varepsilon > 0$, take $\delta_0 = (\frac{\varepsilon(1-q^*)}{MLT^*T^{q^*}})^{\frac{1}{1-q^*}}$, then, when $0 < \delta < \delta_0$, by (14, (15),(16), we have that

$$\begin{aligned} & |\int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1 - q(s, x(s)))} x(s)ds - \int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1 - q(s, x(s)))} x(s)ds| \\ = & |\int_{t-\delta}^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1 - q(s, x(s)))} x(s)ds| \\ = & |\int_{t-\delta}^t \frac{T^{-q(s,x(s))}}{\Gamma(1 - q(s, x(s)))} (\frac{t-s}{T})^{-q(s,x(s))} x(s)ds| \\ \leq & ML \int_{t-\delta}^t T^* (\frac{t-s}{T})^{-q^*} ds \\ = & \frac{MLT^*T^{q^*}}{1-q^*} \delta^{1-q^*} \\ < & \frac{MLT^*T^{q^*}}{1-q^*} \delta_0^{1-q^*} = \varepsilon, \end{aligned}$$

which implies that (20) holds.

Remark 2.1. Assume that $(H_1), (H_2)$ hold. Obviously, we can know that: for the arbitrary fixed function $x \in C[0, T]$,

$$\lim_{\delta \to 0} \int_0^{t-\delta} x(s)ds = \int_0^t x(s)ds, \quad \lim_{\delta \to 0} \int_0^{t-\delta} f(s, x(s))ds = \int_0^t f(s, x(s))ds. \quad (21)$$

Lemma 2.4.([17]) Let $[a,b](\infty < a < b < +\infty)$ be a finite interval and let $AC[a,b]$ be the space of functions which are absolutely continuous on $[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:

$$f(t) \in AC[a,b] \Leftrightarrow f(t) = c + \int_0^t \varphi(s)ds, \varphi \in L(a,b), c \in R, \tag{22}$$

and therefore an absolutely continuous function $f(t)$ has a summable derivative $f^{'}(t) = \varphi(t)$ almost everywhere on $[a,b]$.

3. Existence result

In this section, we will consider the existence of solution for IVP (11), by means of some analysis techniques and Arzela-Ascoli theorem. By the definition of fractional derivative defined by (9), we see that problem (11) is equivalent to $x(0) = 0$ and the following expression

$$\int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds = c + \int_0^t f(s,x(s))ds, t \in [0,T], \tag{23}$$

where $c \in R$.

Theorem 3.1. Assume that $(H_1), (H_2)$ hold. Then IVP (11) exists one solution $x^* \in C[0,T]$.

Proof. In order to obtain the existence result of solution IVP (11), we firstly verify the following sequence has convergent subsequence,

$$x_k(t) = \begin{cases} 0, \quad 0 \le t \le \delta, \\ x_{k-1}(t) + \int_0^{t-\delta} \frac{(t-s)^{-q(s,x_{k-1}(s))}}{\Gamma(1-q(s,x_{k-1}(s)))} x_{k-1}(s)ds \\ -\int_0^{t-\delta} f(s,x_{k-1}(s))ds, \quad \delta < t \le T, \end{cases} \tag{24}$$

$k = 1, 2, \cdots$, where $x_0(t) = 0, t \in [\delta, T]$, δ is arbitrary small number.

In order to apply Arzela-Ascoli theorem to consider the existence of convergent subsequence of sequence x_k defined by (24), firstly, we prove the uniformly bounded of sequence x_k on $[0,T]$.

We find that x_k is uniformly bounded on $[0,\delta]$. Now, we will verify sequence x_k is uniformly bounded on $[\delta, T]$. Let $M = \max_{0\le t\le T} |f(s,0)| + 1$. Since $x_0 = 0$ is uniformly bounded on $[0,T]$, then, for $t \in [\delta, T]$, we have that

$$\begin{aligned} |x_1(t)| = & \quad |x_0(t) + \int_0^{t-\delta} \frac{(t-s)^{-q(s,x_0(s))}}{\Gamma(2-q(s,x_0(s)))} x_0(s)ds - \int_0^{t-\delta} f(s,0)ds \\ = & \quad |\int_0^{t-\delta} f(s,0)ds| \\ \le & \quad M \int_0^{t-\delta} ds \\ \le & \quad MT \doteq M_1, \end{aligned}$$

which implies that x_1 is uniformly bounded on $[\delta, T]$, together with $x_1(t) = 0$ for $t \in [0,\delta]$, we obtain that x_1 is uniformly bounded on $[0,T]$.

Let $M_f = \max_{0\le t\le T, \|x_1\|\le M_1} |f(t,x_1)|+1$, $L = \max_{0\le t\le T, \|x_1\|\le M_1} |\frac{1}{\Gamma(1-q(t,x_1(t)))}|+1$. From (13), (14), (15), for $t\in[\delta,T]$, we have that

$$\begin{aligned}
|x_2(t)| \le\ & |x_1(t)| + \int_0^{t-\delta} |\frac{(t-s)^{-q(s,x_1(s))}}{\Gamma(1-q(s,x_1(s)))}||x_1(s)|ds + \int_0^{t-\delta} |f(s,x_1(s))|ds \\
\le\ & M_1 + M_1 L \int_0^{t-\delta} T^{-q(s,x_1(s))}(\frac{t-s}{T})^{-q(s,x_1(s))}ds + M_f(T-\delta) \\
\le\ & M_1 + M_1 L \int_0^{t-\delta} T^*(\frac{t-s}{T})^{-q^*}ds + M_f T \\
=\ & M_1 + \frac{M_1 L T^* T^{q^*}}{1-q^*}(t^{1-q^*} - \delta^{1-q^*}) + M_f T \\
\le\ & M_1 + \frac{M_1 L T^* T}{1-q^*} + M_f T \\
\doteq\ & M_2,
\end{aligned}$$

which implies that x_2 is uniformly bounded on $[\delta,T]$, together with $x_2(t)=0$ for $t\in[0,\delta]$, we obtain that x_2 is uniformly bounded on $[0,T]$. Continuous this process, we can obtain that sequence x_k is uniformly bounded on $[0,T]$.

Now, we consider the equicontinuous of sequence x_k on $[0,T]$. Obviously, x_0 is equicontinuous on $[0,T]$. Firstly, we can know that function $k(t) = a^t - b^t$, $t\in(-1,0)$, $0<a<b<1$, is decreasing. Indeed, since $\ln a < \ln b < 0$ and $a^t > b^t > 0$, we have that

$$k'(t) = a^t \ln a - b^t \ln b < b^t \ln a - b^t \ln b = b^t(\ln a - \ln b) < 0,$$

which implies that $k(t)$ is a decreasing. Thus, for $l(s) = (\frac{t_1-s}{T})^{-q(s,x(s))} - (\frac{t_2-s}{T})^{-q(s,x(s))}$ (where $0 < \frac{t_1-s}{T} < \frac{t_2-s}{T} < 1$), we may look $l(s)$ as the same type as $k(s)$, then $l(s)$ is decreasing with respect to its exponent $-q(s,x(s))$.

As well, in the next analysis, we will use the Minkowsk's inequality: for a,b non negative, and any $R\ge 0$, it holds

$$(a+b)^R \le c_R(a^R + b^R), \quad \text{where} \quad c_R = \max\{1, 2^{R-1}\}. \tag{25}$$

As a result, for a,b non negative, and any $0<r<1$, it follows from (25) that

$$(a+b)^r \le c_r(a^r+b^r) = \max\{1,2^{r-1}\}(a^r+b^r) = a^r + b^r. \tag{26}$$

We let $M = \max_{0\le t\le T}|f(s,0)|+1$. For $\forall\ \varepsilon>0$, $\forall\ t_1,t_2\in[0,T]$, $t_1<t_2$, we consider result in two cases.

Case I: $0\le t_1\le\delta<t_2\le T$. We take $\eta_{1,I} = \frac{\varepsilon}{M}$, when $t_2-t_1<\eta_{1,I}$, we have that

$$\begin{aligned}
|x_1(t_2) - x_1(t_1)| =\ & |\int_0^{t_2-\delta} f(s,0)ds| \\
\le\ & M\int_0^{t_2-\delta} ds \\
=\ & M(t_2-\delta) \\
\le\ & M(t_2-t_1) \\
<\ & M\eta_{1,I} \\
=\ & \varepsilon.
\end{aligned}$$

Case II: $\delta \le t_1 < t_2 \le T$. We take $\eta_{1,II} = \frac{\varepsilon}{M}$, when $t_2 - t_1 < \eta_{1,II}$, we have that

$$\begin{aligned}
|x_1(t_2) - x_1(t_1)| = & \quad |\int_0^{t_1-\delta} f(s,0)ds - \int_0^{t_2-\delta} f(s,0)ds| \\
\le & \quad \int_{t_1-\delta}^{t_2-\delta} |f(s,0)|ds \\
\le & \quad M\int_{t_1-\delta}^{t_2-\delta} ds \\
= & \quad M(t_2-t_1) \\
< & \quad M\eta_{1,II} \\
= & \quad \varepsilon.
\end{aligned}$$

These imply that $x_1(t)$ is equicontinuous on $[0,T]$, the same result can be obtained when $t_2 < t_1$.

We let $M_f = \max_{0\le t\le T, \|x_1\|\le M_1} |f(s,x_1)| + 1$, $L = \max_{0\le t\le T, \|x_1\|\le M_1} |\frac{1}{\Gamma(1-q(s,x_1(s)))}| + 1$. For $\forall\ \frac{3\varepsilon}{2} > 0$ (ε is arbitrary small number), $\forall\ t_1, t_2 \in [0,T]$, $t_1 < t_2$, we consider result in two cases.

Case I: $0 \le t_1 \le \delta < t_2 \le T$. We take $\eta_{2,I} = \min\{\eta_{1,I}, (\frac{(1-q^*)\varepsilon}{4M_1LT^*T^{q^*}})^{\frac{1}{1-q^*}}, \frac{\varepsilon}{4M_f}\}$, when $t_2 - t_1 < \eta_{2,I}$, by (13), (14), (15), (26), we have that

$$\begin{aligned}
& |x_1(t_2) - x_1(t_1)| \\
= \quad & |x_1(t_2) + \int_0^{t_2-\delta} \frac{(t_2-s)^{-q(s,x_1(s))}}{\Gamma(1-q(s,x_1(s)))} x_1(s)ds - \int_0^{t_2-\delta} f(s,x_1)ds| \\
\le \quad & |x_1(t_2)| + M_1L\int_0^{t_2-\delta} (t_2-s)^{-q(s,x_1(s))}ds + M_f\int_0^{t_2-\delta} ds \\
\le \quad & |x_1(t_2)| + M_1L\int_0^{t_2-\delta} T^{-q(s,x_1(s))}(\frac{t_2-s}{T})^{-q(s,x_1(s))}ds + M_f(t_2-\delta) \\
\le \quad & |x_1(t_2)| + M_1L\int_0^{t_2-\delta} T^*(\frac{t_2-s}{T})^{-q^*}ds + M_f(t_2-\delta) \\
= \quad & |x_1(t_2)| + \frac{M_1LT^*T^{q^*}}{1-q^*}(t_2^{1-q^*} - \delta^{1-q^*}) + M_f(t_2-\delta) \\
= \quad & |x_1(t_2) - x_1(t_1)| + \frac{M_1LT^*T^{q^*}}{1-q^*}((t_2-\delta+\delta)^{1-q^*} - \delta^{1-q^*}) + M_f(t_2-\delta) \\
\le \quad & |x_1(t_2) - x_1(t_1)| + \frac{M_1LT^*T^{q^*}}{1-q^*}((t_2-\delta)^{1-q^*} + \delta^{1-q^*} - \delta^{1-q^*}) + M_f(t_2-\delta) \\
= \quad & |x_1(t_2) - x_1(t_1)| + \frac{M_1LT^*T^{q^*}}{1-q^*}(t_2-t_1)^{1-q^*} + M_f(t_2-t_1) \\
< \quad & |x_1(t_2) - x_1(t_1)| + \frac{M_1LT^*T^{q^*}}{1-q^*}\eta_{2,I}^{1-q^*} + M_f\eta_{2,I} \\
< \quad & \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
= \quad & \frac{3\varepsilon}{2}.
\end{aligned}$$

Case II: $\delta \le t_1 < t_2 \le T$. We take $\eta_{2,II} = \min\{\eta_{1,II}, (\frac{(1-q^*)\varepsilon}{8M_1LT^*T^{q^*}})^{\frac{1}{1-q^*}}, \frac{\varepsilon}{4M_f}\}$, when $t_2 - t_1 < \eta_{2,I}$, by (13), (14), (15), (26) we have that

$$
\begin{aligned}
& |x_1(t_2) - x_1(t_1)| \\
= \quad & |x_1(t_2) - x_1(t_1) + \int_0^{t_2-\delta} \frac{(t_2-s)^{-q(s,x_1(s))}}{\Gamma(1-q(s,x_1(s)))} x_1(s) ds \\
& - \int_0^{t_1-\delta} \frac{(t_1-s)^{-q(s,x_1(s))}}{\Gamma(1-q(s,x_1(s)))} x_1(s) ds - \int_0^{t_2-\delta} f(s,x_1) ds + \int_0^{t_1-\delta} f(s,x_1) ds| \\
\le \quad & |x_1(t_2) - x_1(t_1)| + \int_{t_1-\delta}^{t_2-\delta} |\frac{(t_2-s)^{-q(s,x_1(s))}}{\Gamma(1-q(s,x_1(s)))}||x_1(s)| ds \\
& + \int_0^{t_1-\delta} |\frac{1}{\Gamma(1-q(s,x_1(s)))}||(t_2-s)^{-q(s,x_1(s))} - (t_1-s)^{-q(s,x_1(s))}||x_1(s)| ds \\
& + \int_{t_1-\delta}^{t_2-\delta} |f(s,x_1(s))| ds \\
\le \quad & |x_1(t_2) - x_1(t_1)| + M_1 L \int_0^{t_1-\delta} ((t_1-s)^{-q(s,x_1(s))} - (t_2-s)^{-q(s,x_1(s))}) ds + \\
& M_1 L \int_{t_1-\delta}^{t_2-\delta} (t_2-s)^{-q(s,x_1(s))} ds + M_f \int_{t_1-\delta}^{t_2-\delta} ds \\
= \quad & |x_1(t_2) - x_1(t_1)| + M_1 L \int_0^{t_1-\delta} T^{-q(s,x_1(s))} ((\frac{t_1-s}{T})^{-q(s,x_1(s))} - (\frac{t_2-s}{T})^{-q(s,x_1(s))}) ds \\
& + M_1 L \int_{t_1-\delta}^{t_2-\delta} T^{-q(s,x_1(s))} (\frac{t_2-s}{T})^{-q(s,x_1(s))} ds + M_f(t_2-t_1) \\
\le \quad & |x_1(t_2) - x_1(t_1)| + M_1 L \int_0^{t_1-\delta} T^* ((\frac{t_1-s}{T})^{-q^*} - (\frac{t_2-s}{T})^{-q^*}) ds \\
& + M_1 L \int_{t_1-\delta}^{t_2-\delta} T^* (\frac{t_2-s}{T})^{-q^*} ds + M_f(t_2-t_1) \\
= \quad & |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^* T^{q^*}}{1-q^*} (t_1^{1-q^*} - \delta^{1-q^*} + 2(t_2 - t_1 + \delta)^{1-q^*} - t_2^{1-q^*} - \delta^{1-q^*}) \\
& + M_f(t_2-t_1) \\
\le \quad & |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^* T^{q^*}}{1-q^*} (t_2^{1-q^*} - 2\delta^{1-q^*} + 2(t_2 - t_1)^{1-q^*} + 2\delta^{1-q^*} - t_2^{1-q^*}) \\
& + M_f(t_2-t_1) \\
= \quad & |x_1(t_2) - x_1(t_1)| + \frac{2M_1 L T^* T^{q^*}}{1-q^*} (t_2 - t_1)^{1-q^*} + M_f(t_2-t_1) \\
< \quad & |x_1(t_2) - x_1(t_1)| + \frac{2M_1 L T^* T^{q^*}}{1-q^*} \eta_{2,II}^{1-q^*} + M_f \eta_{2,II} \\
< \quad & \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
= \quad & \frac{3\varepsilon}{2}.
\end{aligned}
$$

These imply that $x_2(t)$ is equicontinuous on $[0,T]$, the same result can be obtained when $t_2 < t_1$. Continue these process, we can obtain that x_k, $k = 1, 2\cdots$, is equicontinuous on $[0,T]$.

As well, by the arguments of equicontinuity of x_k, we can know that $x_k \in C[0,T]$, $k = 1, 2, \cdots$. Then, from Arzela-Ascoli theorem, sequence x_k exists a convergent subsequence x_{m_k}. From (24), x_{m_k} should satisfy

$$x_{m_k}(t) = \begin{cases} 0, \quad 0 \le t \le \delta, \\ x_{m_{k-1}}(t) + \int_0^{t-\delta} \frac{(t-s)^{-q(s,x_{m_{k-1}}(s))}}{\Gamma(1-q(s,x_{m_{k-1}}(s)))} x_{m_{k-1}}(s)ds \\ -\int_0^{t-\delta} f(s, x_{m_{k-1}}(s))ds, \quad \delta < t \le T. \end{cases} \tag{27}$$

Now, we will prove that the continuous limit of x_{m_k}, denoted by x^* is one solution of IVP (11).

Let $k \to +\infty$ in (27), by Lemmas 2.1, 2.2, we have that

$$x^*(t) = \begin{cases} 0, \quad 0 \le t \le \delta, \\ x^*(t) + \int_0^{t-\delta} \frac{(t-s)^{-q(s,x^*(s))}}{\Gamma(1-q(s,x^*(s)))} x^*(s)ds \\ -\int_0^{t-\delta} f(s, x^*(s))ds, \quad \delta < t \le T. \end{cases} \tag{28}$$

Thus, we find that,

$$x^*(t) = 0, 0 \le t \le \delta; \quad \int_0^{t-\delta} \frac{(t-s)^{-q(s,x^*(s))}}{\Gamma(1-q(s,x^*(s)))} x^*(s)ds - \int_0^{t-\delta} f(s, x^*(s))ds = 0, \tag{29}$$

$\delta < t \le T$. In order to verify x^* is one solution of IVP (11), we let $\delta \to 0$ in (29), by (20), (21), we obtain that

$$x^*(0) = 0; \quad \int_0^{t} \frac{(t-s)^{-q(s,x^*(s))}}{\Gamma(1-q(s,x^*(s)))} x^*(s)ds = \int_0^{t} f(s, x^*(s))ds, 0 < t \le T. \tag{30}$$

It follows from the continuity of f and Lemma 2.4 that $\int_0^t f(s, x^*(s))ds \in AC[0,T]$, consequently, from (30), we get

$$\int_0^{t} f(s, x^*(s))ds = \int_0^{t} \frac{(t-s)^{-q(s,x^*(s))}}{\Gamma(1-q(s,x^*(s)))} x^*(s)ds \in AC[0,T].$$

As a result, differential on both sides of the second expression in (30), we get

$$D_{0+}^{q(t,x^*(t))} x^*(t) = f(t, x^*), 0 < t \le T, \tag{31}$$

together with $x^*(0) = 0$, we see that x^* is one solution of IVP (11). Thus, we complete this proof.

4. Unique results

In this section, using the monotone iterative method, we will consider the existence and unique result of solution to the following particular case of IVP (11),

$$\begin{cases} D_{0+}^{p(t)} x(t) = f(t, x), \quad 0 < t \le T, 0 < T < +\infty \\ x(0) = 0, \end{cases} \tag{32}$$

where $D_{0+}^{p(t)}$ denotes fractional derivative of variable-order defined by (5), where $p : [0,T] \to (0, p^*]$, $0 < p^* < 1$, is a continuous function.

We assume that

(H_3) $p : [0,T] \to (0, p^*]$ is continuous, here $0 < p^* < 1$.

The following result will play very important role in our next analysis.

Lemma 4.1. Let (H_3) hold. If $x \in C[0,T]$ and satisfies the relations

$$\begin{cases} D_{0+}^{p(t)}x + d_0 x \geq 0, t \in (0,T] \\ x(0) \geq 0 \end{cases} \tag{33}$$

where $D_{0+}^{p(t)}$ denotes fractional derivative of variable-order defined by (5), $d_0 > 0$ is a constant. Then $x \geq 0$ for $t \in [0,T]$.

Proof. We assume that $x(t) \geq 0, t \in [0,T]$ is false. Then from $x(0) \geq 0$, there exists points $t_0 \in [0,T], t_0^{'} \in (0,T]$ such that, $x(t_0) = 0$, $x(t_0^{'}) < 0$; and that $x(t) \geq 0$ for $t \in [0, t_0]$, $x(t) < 0$ for $t \in (t_0, t_0^{'}]$, and assume that t_1 is the first minimal point of $x(t)$ on $[t_0, t_0^{'}]$.

It follows from the inequality of (33) that

$$D_{0+}^{p(t)}x(t) \geq 0, \quad t \in [t_0, t_0^{'}],$$

hence, we have

$$\int_{t_0}^{t} D_{0+}^{p(s)}x(s)ds \geq 0, \quad t \in [t_0, t_0^{'}],$$

from the definition of fractional derivative of variable order defined by (5), we can obtain that

$$\int_{t_0}^{t} \frac{d}{ds}(I_{0+}^{1-p(s)}x(s))ds = I_{0+}^{1-p(t)}x(t) - I^{1-p(t)}x(t_0) \geq 0, \quad t \in [t_0, t_0^{'}]. \tag{34}$$

On the other hand, for $t \in [t_0, t_0^{'}]$, we have

$$\begin{aligned} I_{0+}^{1-p(t)}x(t) - I^{1-p(t)}x(t_0) &= \int_0^t \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))}x(s)ds - \int_0^{t_0} \frac{(t_0-s)^{-p(s)}}{\Gamma(1-p(s))}x(s)ds \\ = & \int_0^{t_0} \frac{(t-s)^{-p(s)} - (t_0-s)^{-p(s)}}{\Gamma(1-p(s))}x(s)ds + \int_{t_0}^{t} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))}x(s)ds \\ < & \quad 0 + 0 = 0, \end{aligned}$$

which contradicts (34). Therefore, we obtain that $x(t) \geq 0, t \in [0,T]$. Thus we complete this proof.

For problem (32), we have the following definitions of upper and lower solutions.

Definition 4.1. A function $\alpha \in C[0,T]$ is called a upper solution of problem (32), if it satisfies

$$\begin{cases} D_{0+}^{p(t)}\alpha(t) \geq f(t,\alpha), t \in (0,T] \\ \alpha(0) \geq 0 \end{cases} \tag{35}$$

Analogously, function $\beta \in C[0,T]$ is called a lower solution of problem (32), if it satisfies

$$\begin{cases} D_{0+}^{p(t)}\beta(t) \leq f(t,\beta), t \in (0,T] \\ \beta(0) \leq 0 \end{cases} \tag{36}$$

In what follows we assume that

$$\alpha(t) \geq \beta(t), \quad t \in [0,T], \tag{37}$$

and define that sector

$$\langle \alpha, \beta \rangle = \langle u \in C[0,T]; \alpha(t) \leq u(t) \leq \beta(t), t \in [0,T]. \rangle$$

We also assume that f satisfies the following condition

$$f(t,x_1)-f(t,x_2)\geq -d_0(x_1-x_2), \alpha\leq x_2\leq x_1\leq\beta, \tag{38}$$

where $d_0 \geq 0$ is a constant and $\alpha, \beta \in C[0,T]$ are lower and upper solutions of problem (32). Clearly this condition is satisfied with $d_0 = 0$, when f is monotone nondecreasing in u. In view of (38), the function

$$F(t,x)=d_0x+f(t,x) \tag{39}$$

is monotone nondecreasing in x for $x \in \langle\alpha, \beta\rangle$.

We also suppose that there exists a constant $d_1 \leq 0$, such that

$$f(t,x_1)-f(t,x_2)\leq d_1(x_1-x_2), \alpha\leq x_2\leq x_1\leq\beta, \tag{40}$$

where $\alpha, \beta \in C[0,T]$ are lower and upper solutions of problem (32).

The following is existence and uniqueness theorem of solution for (32).

Theorem 4.1. Let $(H_2), (H_3)$ hold. Assume that $\alpha, \beta \in C[0,T]$ are lower and upper solutions of problem (32), such that (37) holds, f also satisfies (38). Then problem (32) exists one solution in the sector $\langle\alpha, \beta\rangle$, as well, if condition (40) holds, then (32) exists one unique solution in the sector $\langle\alpha, \beta\rangle$.

proof. We see that (32) is equivalent to the following problem

$$\begin{cases} D_{0+}^{p(t)}v+d_0v=d_0v+f(t,v), t\in(0,T], \\ v(0)=0, \end{cases} \tag{41}$$

where d_0 is the constant in (38). This proof consists of six steps.

Step 1. Constructing sequences $\{v^{(k)}\}, k = 1, 2, \cdots$ as following

$$\begin{cases} D_{0+}^{p(t)}v^{(k)}(t)+d_0v^{(k)}=d_0v^{(k-1)}+f(t,v^{(k-1)}), t\in(0,T], \\ v^{(k)}(0)=0. \end{cases} \tag{42}$$

Theorem 3.1 assures that the sequence $\{v^{(k)}\}$ is well defined, since for each k, we can obtain result from theorem 3.1 with $f(t, v^{(k)}) \doteq d_0v^{(k-1)} + f(t, v^{(k-1)}) - d_0v^{(k)}$ and $q(t, v^{(k)}) \doteq p(t)$. Of particular interest is the sequence obtained from (42) with a upper solution or lower solution of problem (32) as the initial iteration. Denote the sequence with the initial iteration $v^{(0)} = \beta$ by $\{\overline{v}^{(k)}\}$ and the sequence with $v^{(0)} = \alpha$ by $\{\underline{v}^{(k)}\}$.

Step 2. Monotone property of the two sequences.

The sequences $\{\overline{v}^{(k)}\}$, $\{\underline{v}^{(k)}\}$ constructed by (41) process the monotone property

$$\alpha\leq\underline{v}^{(k)}\leq\underline{v}^{(k+1)}\leq\overline{v}^{(k+1)}\leq\overline{v}^{(k)}\leq\beta, t\in(0,T], \tag{43}$$

for every $k = 1, 2, \cdots$.

In fact, let $r = \overline{v}^{(0)} - \overline{v}^{(1)}$. By (42), (36), (37), (38), and $\overline{v}^{(0)} = \beta$, there has

$$\begin{aligned} D_{0+}^{p(t)}r+d_0r= &\quad (D_{0+}^{p(t)}\beta+d_0\beta-(d_0\beta+f(t,\beta)) \\ = &\quad D_{0+}^{p(t)}\beta-f(t,\beta)\geq 0, t\in(0,T], \end{aligned}$$

$$r(0)\geq 0-0=0.$$

In view of Lemma 4.1, $r \geq 0$ for $t \in [0,T]$, which leads to $\overline{v}^{(1)} \leq \overline{v}^{(0)} = \beta$, $t \in [0,T]$. A similar argument using the property of a lower solution of (32) gives

$\underline{v}^{(1)} \geq \underline{v}^{(0)} = \alpha$, $t \in [0, T]$. Let $r^{(1)} = \overline{v}^{(1)} - \underline{v}^{(1)}$. By (42), (36), (37), (38), there has

$$\begin{aligned} D_{0+}^{p(t)} r^{(1)} + d_0 r^{(1)} = & \quad d_0 \overline{v}^{(0)} + f(t, \overline{v}^{(0)}) - (d_0 \underline{v}^{(0)} + f(t, \underline{v}^{(0)})) \\ = & \quad d_0(\beta - \alpha) + f(t, \beta) - f(t, \alpha) \\ \geq & \quad 0, t \in (0, T], \end{aligned}$$

$$r^{(1)}(0) = \overline{v}^{(1)}(0) - \underline{v}^{(1)}(0) = 0,$$

Again, in view of Lemma 4.1, $r^{(1)} \geq 0$ for $t \in [0, T]$, the above conclusion shows that

$$\alpha = \underline{v}^{(0)} \leq \underline{v}^{(1)} \leq \overline{v}^{(1)} \leq \overline{v}^{(0)} = \beta, \quad t \in [0, T]$$

Assume, by induction

$$\alpha \leq \underline{v}^{(k-1)} \leq \underline{v}^{(k)} \leq \overline{v}^{(k)} \leq \overline{v}^{(k-1)} \leq \beta, \quad t \in (0, T]. \tag{44}$$

Then by (42), (44), (38), the function $r^{(k)} = \overline{v}^{(k)} - \overline{v}^{(k+1)}$ satisfies the relations

$$\begin{aligned} D_{0+}^{p(t)} r^{(k)}(t) + d_0 r^{(k)} = & \quad d_0 \overline{v}^{(k-1)} + f(t, \overline{v}^{(k-1)}) - (d_0 \overline{v}^{(k)} + f(t, \overline{v}^{(k)})) \\ \geq & \quad 0, t \in (0, T], \end{aligned}$$

$$r^{(k)}(0) = 0,$$

In view of Lemma 4.1, $r^{(k)} \geq 0$, that is $\overline{v}^{(k+1)} \leq \overline{v}^{(k)}$ for $t \in [0, T]$. Similar reasoning gives $\underline{v}^{(k)} \leq \underline{v}^{(k+1)}$ and $\underline{v}^{(k+1)} \leq \overline{v}^{(k+1)}$ for $t \in [0, T]$. Hence, the monotone property (43) follows from the principle of induction.

Step 3. The two sequences constructed by (42) have pointwise limits and satisfy some relations, that is

$$\lim_{k \to \infty} \overline{v}^{(k)}(t) = v(t), \quad \lim_{k \to \infty} \underline{v}^{(k)}(t) = w(t), t \in [0, T] \tag{45}$$

exists and satisfy the relation

$$\alpha \leq \underline{v}^{(k)} \leq \underline{v}^{(k+1)} \leq w \leq v \leq \overline{v}^{(k+1)} \leq \overline{v}^{(k)} \leq \beta, t \in [0, T] \tag{46}$$

for every $k = 1, 2, \cdots$.

In fact, By (43), we see that the upper sequence $\{\overline{v}^{(k)}\}$ is monotone nonincreasing and is bounded from below and that the lower sequence $\{\underline{v}^{(k)}\}$ is monotone nondecreasing and is bounded from above. Therefore the pointwise limits exist and these limits are denoted by v and w as in (45). Moreover, by (43), the limits v, w satisfy (46).

Step 4. To prove that v and w are solutions of initial value problem (32).

Let $v^{(k)}$ be either $\overline{v}^{(k)}$ or $\underline{v}^{(k)}$ From the definition of fractional derivative $D_{0+}^{p(t)}$ defined by (5), we see that the equation of (42) may be expressed as and

$$I_{0+}^{1-p(t)} v^{(k)}(t) + d_0 I_{0+}^1 v^{(k)}(t)) = c + d_0 I_{0+}^1 v^{(k-1)}(t) + I_{0+}^1 f(t, v^{(k-1)}(t)),$$

where $c \in R$, that is

$$\int_0^t \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v^{(k)}(s) ds + d_0 \int_0^t v^{(k)}(s) ds = c + d_0 \int_0^t v^{(k-1)}(s) ds + \int_0^t f(s, v^{(k-1)}(s)) ds. \tag{47}$$

Now, we consider the expression

$$\int_0^{t-\delta} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v^{(k)}(s) ds + d_0 \int_0^{t-\delta} v^{(k)}(s) ds = c + d_0 \int_0^{t-\delta} v^{(k-1)}(s) ds + \int_0^{t-\delta} f(s, v^{(k-1)}(s)) ds. \tag{48}$$

where $\delta > 0$ is arbitrary small number. By Lemma 2.1, we know that $\int_0^{t-\delta} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v^{(k)}(s)ds \in C[\delta, T]$ with $v^{(0)} = \alpha$ or $v^{(0)} = \beta$.

Let $k \to \infty$ in (48) and apply the Lemmas 2.2, 2.3 and the dominated convergence theorem, v satisfies the integral equation

$$\int_0^{t-\delta} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v(s)ds + d_0 \int_0^{t-\delta} v(s)ds = c + d_0 \int_0^{t-\delta} v(s)ds + \int_0^{t-\delta} f(s, v(s))ds,$$

that is

$$\int_0^{t-\delta} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v(s)ds = c + \int_0^{t-\delta} f(s, v(s))ds. \tag{49}$$

Now let $\delta \to 0$ in (49), by (20), (21), we get

$$\int_0^{t} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v(s)ds = c + \int_0^{t} f(s, v(s))ds. \tag{50}$$

It follows from the continuity of f and Lemma 2.4 that $c + \int_0^t f(s, v(s))ds \in AC[0, T]$, consequently, from (50), we get

$$c + \int_0^t f(s, v(s))ds = \int_0^t \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} v(s)ds \in AC[0, T].$$

As a result, differential on two sides of (50), we have that

$$D_{0+}^{p(t)} v(t) = f(t, v(t)), t \in (0, T]. \tag{51}$$

We also Let $k \to \infty$ in the second expression of (42), it holds $v(0) = 0$, this together with (51), we know that $v(t)$ is a solution of (32). This proves that the upper sequence $\{\overline{v}^{(k)}\}$ converges to a solution v of problem (32), the lower sequence $\{\underline{v}^{(k)}\}$ converges to a solution w of problem (32), and satisfies relation $v(t) \geq w(t)$, $t \in [0, T]$.

Step 5 If condition (40) holds, then $v = w$ is unique solution of problem (32).

It is sufficient to prove $v(t) \leq w(t)$, $t \in [0, T]$, by $v(t) \geq w(t)$, $t \in [0, T]$ obtained in Step 4. In fact, by (32) and (40), the function $r = w - v$ satisfies the relations

$$\begin{cases} D_{0+}^{p(t)} r = -(f(t, v) - f(t, w)) \geq d_1 r, t \in (0, T], \\ r(0) = 0, \end{cases}$$

then, Lemma 4.1 implies that $r(t) \geq 0, t \in [0, T]$, this proves $w \geq v$, therefore, we obtain that $v = w$ is unique solution of problem (32). Thus, we complete this proof.

References

[1] H.G. Sun, W. Chen, Y.Q. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, Physica A, 388(2009) 4586-4592.

[2] C.F.M. Coimbra, Mechanics with variable-order differential operators, Annalen der Physik, 12(2003) 692-703.

[3] H. Sheng, H.G. Sun, Y.Q. Chen, T. Qiu, Synthesis ofmultifractional Gaussian noises based on variable-order fractional operators, Signal Processing, 91(2011) 1645-1650.

[4] C.C. Tseng, Design of variable and adaptive fractional order FIR differentiators, Signal Processing, 86(2006) 2554-2566.

[5] H. Sheng, H. G. Sun, C. Coopmans, Y.Q. Chen, G.W. Bohannan, Physical experimental study of variable-order fractional integrator and differentiator, in: Proceedings of the 4th IFAC Workshop on Fractional Differentiation and its Applications(FDA'10), 2010.

[6] H.G. Sun, W. Chen, H. Wei, Y.Q. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, Eur. Phys. J. Special Topics, 193(2011) 185-192.

[7] T.T. Hartley, C.F. Lorenzo, Fractional system identification: An approach using continuous order distributions, NASA/TM, 40(1999) 1999-2096.

[8] C.H. Chan, J.J. Shyu, R.H.H. Yang, A new structure for the design of wideband variable fractional-order FIR differentiator, Signal Processing, 90(2010) 2594-2604.

[9] C.C.Tseng, Series expansion design for variable fractional order integrator and Differentiator using logarithm, Signal Processing, 88(2008) 278-2292.

[10] C.F. Lorenzo, T.T. Hartley, Variabe order and distributed order fractional operators, Nonlinear Dynamics, 29(2002) 57-98.

[11] C.H. Chan, J.J.Shyu, R.H.H. Yang , Iterative design of variable fractional-order IIR differintegrators, Signal Processing, 90(2010) 670-678.

[12] C.H. Chan, J.J. Shyu, R. H.H. Yang, An iterative method for the design of variable fractional-order FIR differintegrators, Signal Processing, 89(2009) 320-327.

[13] B. ROSS, Fractional integration operator of variable-order in the Hölder space $H^{\lambda(x)}$, Internat. J. Math. and Math. Sci., 18(1995) 777-788.

[14] D. Valério, J.S. Costa, Variable-order fractional derivatives and their numerical approximations, Signal Processing, 91(2011) 470-483.

[15] A. Razminia, A.F. Dizaji, V.J. Majd, Solution existence for non-autonomous variable-order fractional differential equations, Mathematical and Computer Modelling, 55(2012) 1106-1117.

[16] R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, Applied Mathematics and Computation, 2(2009) 435-445.

[17] A . A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B. V., Amsterdam, 2006.

Shuin Zhang

Department of Mathematics, China University of Mining and Technology, Beijing, 100083, China

E-mail address: zsqjk@163.com