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MAXIMAL AND MINIMAL POSITIVE SOLUTIONS FOR A BOUNDARY VALUE PROBLEM WITH A NONLOCAL CONDITIONS

BIN-TAHER, E.O

ABSTRACT. In this paper we study the existence of positive solution for the ordinary differential equation $u''(t) + f(t, u(t)) = 0$, $t \in (0,1)$, with the nonlocal conditions $u(0) = 0$, $u(1) + D^{\alpha} u(t)|_{t=1} = 0$, $\alpha \in (0,1)$ where f is L^1-Carathèodory. The existence of the maximal and minimal solutions are also studied.

1. INTRODUCTION

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1]-[3]), ([5]-[12]), ([14]-[19]) and ([22]-[24]), and references therein.
In this work we study the existence of at least one positive solution for the nonlocal boundary value problem of the ordinary differential equation

$$u''(t) + f(t, u(t)) = 0, \; t \in (0,1), \tag{1}$$

with the nonlocal conditions

$$u(0) = 0, \; u(1) + {}^{R}D^{\alpha} u(t)|_{t=1} = 0. \tag{2}$$

where f is L^1-Carathèodory and ${}^{R}D^{\alpha}$ is the Riemann-Liouville fractional-order derivative of order $\alpha \in (0,1)$.
The maximal and minimal solutions of the problem (1)-(2) is studied when the function f is nondecreasing in the second argument.

2. PRELIMINARIES

Let $C(I)$ denotes the class of continuous function on I and $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function.

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Definition 1.1 The Riemann-Liouville fractional-order derivative of f of order $\beta \in (0,1)$ is defined as (see [20] and [21])

$$D_a^{\beta} f(t) = \frac{d}{dt}\int_a^t \frac{(t - s)^{-\beta}}{\Gamma(1 - \beta)} f(s)\, ds.$$

Definition 1.2 The function $f : [0,1] \times R \to R$ is called L^1-Carathéodory if
(i) $t \to f(t,x)$ is measurable for each $x \in R$,
(ii) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$,
(iii) there exists $m \in L^1[0,1]$ such that $|f| \leq m$.

The following theorem will be needed
Theorem 2.1 (Schauder fixed point theorem [4])
Let E be a Banach space and Q be a convex subset of E, and $T : Q \longrightarrow Q$ is compact, continuous map, Then T has at least one fixed point in Q.

3. Existence of solution

Lemma 3.1 The solution of the problem (1)-(2) can be represent by the integral equation

$$u(t) = A\, t\left\{\int_0^1 (1-s)\, f(s,u(s))\, ds + \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\, f(s,u(s))\, ds\right\} - \int_0^t (t-s)\, f(s,u(s))\, ds. \quad (3)$$

where $A = \left(\frac{\Gamma(2-\alpha)}{1 + \Gamma(2-\alpha)}\right)$.
proof. Integral the both sides of equation (1) twice, we obtain

$$u(t) = C_2 + tC_1 - \int_0^t (t-s) f(s,\ u(s))\, ds.$$

From the relation $u(0) = 0$, we have

$$C_2 = 0$$

Then,we have

$$u(t) = tC_1 - \int_0^t (t-s) f(s,\ u(s))\, ds.$$

Operating on both sides of the above equation by $I^{1-\alpha}$, we obtain

$$I^{1-\alpha} u(t) = C_1 \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - I^{3-\alpha} f(t,\ u(t))$$

Differentiating the last relation , we obtain

$$D^{\alpha} u(t) = \frac{d}{dt} I^{1-\alpha} u(t) = C_1 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - I^{2-\alpha} f(t,\ u(t))$$

Also from the relation $u(1) + D^{\alpha}\, u(t)|_{t=1} = 0$, we have

$$C_1 - \int_0^1 (1-s) f(s,\ u(s))\, ds + \frac{C_1}{\Gamma(2-\alpha)} - \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,\ u(s))\, ds = 0$$

$$C_1\left(1 + \frac{1}{\Gamma(2-\alpha)}\right) = \int_0^1 (1-s) f(s,\ u(s))\, ds + \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,\ u(s))\, ds$$

then

$$C_1 = \left(\frac{\Gamma(2-\alpha)}{1 + \Gamma(2-\alpha)}\right)\left\{\int_0^1 (1-s)f(s,\ u(s))\ ds \ + \ \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,\ u(s))\ ds\right\}$$

and

$$u(t) = \left(\frac{t\ \Gamma(2-\alpha)}{1 + \Gamma(2-\alpha)}\right)\left\{\int_0^1 (1-s)f(s,\ u(s))\ ds \ + \ \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,\ u(s))\ ds\right\} - \int_0^t (t-s)\ f(s, u(s))\ ds,$$

then we get

$$u(t) = A\, t\ \left\{\int_0^1 (1-s)\ f(s,u(s))\ ds + \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds\right\} - \int_0^t (t-s)\ f(s,u(s))\ ds.$$

where $A = \left(\frac{\Gamma(2-\alpha)}{1 + \Gamma(2-\alpha)}\right)$.

Now we can write equation (3) in the formula

$$u(t) = \int_0^1 G(t,\ s)\ f(t,\ u(s))\ ds. \tag{4}$$

where

$$G(t,\ s) = \begin{cases} \frac{-(1+\Gamma(2-\alpha))(t-s) + \ t\Gamma(2-\alpha)(1-s) + t(1-s)^{1-\alpha}}{1+\Gamma(2-\alpha)}, & 0 \leq \ s \leq \ t \leq \ 1 \\ \frac{t\Gamma(2-\alpha)(1-s) + t(1-s)^{1-\alpha}}{1+\Gamma(2-\alpha)}, & 0 \leq \ t \leq \ s \leq \ 1. \end{cases}$$

Lemma 2.2 The function $G(t,\ s)$ satisfies $G(t,\ s) > 0$, for $t,\ s \in \ (0,\ 1)$.

Proof. For $0 \leq \ s \leq \ t \leq \ 1$, let

$$g(t,s) = -(1+\Gamma(2-\alpha))(t-s) + \ t\Gamma(2-\alpha)(1-s) + t(1-s)^{1-\alpha}$$

then we have

$$\begin{aligned} t\Gamma(2-\alpha)(1-s) + t(1-s)^{1-\alpha} \ &\geq \ t\Gamma(2-\alpha)(1-s) + t(1-s) \\ &= \ (1+\Gamma(2-\alpha))(t-ts) > (1+\Gamma(2-\alpha))(t-s) \end{aligned}$$

Thus, $g(t,\ s) \ > \ 0$.

For $0 \leq \ t \leq \ s \leq \ 1$, $G(t,\ s) \geq \ 0$ holds clearly.

Then we get that $G(t,\ s) > 0$ for $t,\ s \in \ (0,\ 1)$.

Definition 2.1 The function u is called a solution of the fractional-order functional integral equation (3), if $u \in \ C[0,1]$ and satisfies (3).

For the existence of the solution we have the following theorem

Theorem Assume that the the function $f : [0,1] \times R^+ \to R^+$ is L^1-Carathèodory. Then the nonlocal boundary value problem (1)-(2) has at least one positive continuous solution $u \in C[0,1]$.

Proof. Define a subset $Q_r^+ \subset C[0,1]$ by

$$Q_r^+ = \{u(t) > 0, \ \text{for each } t \in [0,1], \ \|u\| \leq r\},\ r = 2||m||_{L^1} \ .$$

The set Q_r^+ is nonempty, closed and convex.
Let $T : Q_r^+ \to Q_r^+$ be an operator defined by

$$Tu(t) = A\ t\ \left\{\int_0^1 (1-s)\ f(s,u(s))\ ds - \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds\right\}$$

$$- \int_0^t (t-s)\ f(s,u(s))\ ds.$$

For $u \in Q_r^+$, let $\{u_n(t)\}$ be a sequence in Q_r^+ converges to $u(t)$, $u_n(t) \to u(t)$, $\forall t \in [0,1]$, then

$$\lim_{n\to\infty} Tu_n(t) = A\,t \lim_{n\to\infty} \int_0^1 (1-s)\ f(s,u_n(s))\ ds - A\,t \lim_{n\to\infty} \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u_n(s))\ ds$$

$$- \lim_{n\to\infty} \int_0^t (t-s)\ f(s,u_n(s))\ ds$$

Since f is L^1-Carathèodory, then by applying Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty} (Tu_n)(t) = (Tu)(t).$$

Then T is continuous.
Now, let $u \in Q_r^+$, then

$$\begin{aligned}
(Tu)(t) &\le A\ t \int_0^1 (1-s) f(s,\ u(s))\ ds + A\ t \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,\ u(s))\ ds \\
&+ \int_0^t (t-s)\ f(s,u(s))\ ds \\
&\le A \int_0^1 (1-s) f(s,\ u(s))\ ds + A \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,\ u(s))\ ds \\
&+ \int_0^1 (1-s)\ f(s,u(s))\ ds \\
&\le (A + \frac{A}{\Gamma(2-\alpha)} + 1) \int_0^1 (1-s)^{1-\alpha}\ f(s,u(s))\ ds \\
&\le \frac{A\Gamma(2-\alpha) + A + \Gamma(2-\alpha)}{\Gamma(2-\alpha)} \int_0^1 (1-s)^{1-\alpha}\ m(s)\ ds \\
&\le \frac{\Gamma(2-\alpha) + \Gamma(2-\alpha)}{\Gamma(2-\alpha)} \int_0^1 m(s)\ ds \\
&\le 2\ ||m||_{L^1} = r
\end{aligned}$$

Then $\{Tu(t)\}$ is uniformly bounded in Q_r^+.
In what follows we show that T is a completely continuous operator.

For $t_1, t_2 \in (0,1),\ t_1 < t_2$ such that $|t_2 - t_1| < \delta$ we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= |A\ t_2 \int_0^1 (1-s) f(s,u(s))\ ds + A t_2 \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,u(s)) ds \\
&- \int_0^{t_2} (t_2 - s) f(s,u(s)) ds \\
&- A t_1 \int_0^1 (1-s) f(s,u(s)) ds - A t_1 \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,u(s)) ds \\
&+ \int_0^{t_1} (t_1 - s) f(s,u(s))\ ds|
\end{aligned}$$

$$\begin{aligned}
&\leq |\int_0^{t_2} (t_2 - s) f(s,u(s)))\ ds - \int_0^{t_1} (t_1 - s) f(s,u(s)))\ ds\ | \\
&+ A\ |t_2\ -\ t_1| \int_0^1 (1-s) |f(s,u(s))|\ ds \\
&+ A\ |t_2\ -\ t_1| \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,u(s)) ds \\
&\leq |\int_0^{t_1} ((t_2 - t_1))\, f(s,u(s)))\ ds \\
&+ \int_{t_1}^{t_2} (t_2 - s) f(s,u(s)))\ ds\ | \\
&+ A\ |t_2\ -\ t_1| \int_0^1 (1-s) |f(s,u(s))|\ ds \\
&+ A\ |t_2\ -\ t_1| \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} |f(s,u(s))| ds.
\end{aligned}$$

Hence the class of functions $\{TQ_r^+\}$ is equi-continuous. By Arzela-Ascolis Theorem $\{TQ_r^+\}$ is relatively compact. Since all conditions of Schauder Theorem are hold, then T has a fixed point in Q_r^+.
Therefor the integral equation (3) has at least one positive continuous solution $u \in C(0,1)$.
Now,

$$\begin{aligned}
\lim_{t\to 0} u(t) &= A \lim_{t\to 0} t \int_0^1 (1-s) f(s,u(s)) ds + A \lim_{t\to 0} t \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,u(s)) ds \\
&- \lim_{t\to 0} \int_0^t (t-s)\ f(s,u(s)))\ ds = u(0) = 0,
\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 1} u(t) &= A \lim_{t\to 1} t \int_0^1 (1-s) f(s,u(s))ds + A \lim_{t\to 1} t \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,u(s))ds \\ &- \lim_{t\to 1} \int_0^t (t-s)\ f(s,u(s)))\ ds = u(1).\end{aligned}$$

Then the integral equation (3) has at least one positive continuous solution $u \in C[0,1]$.

To complete the proof differentiating equation (3) twice we obtain the differential equation (1). Operating on both sides of equation (3) by $I^{1-\alpha}$, we obtain

$$\begin{aligned}I^{1-\alpha}u(t) &= \frac{A\ t^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^1 (1-s)\ f(s,u(s))\ ds + \frac{A\ t^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds \\ &- \int_0^t \frac{(t-s)^{2-\alpha}}{\Gamma(3-\alpha)}\ f(s,u(s))\ ds\end{aligned}$$

differentiating the above relation, we get

$$\begin{aligned}D^{\alpha}u(t) &= \frac{d}{dt} I^{1-\alpha}u(t) = \frac{At^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^1 (1-s) f(s,u(s))\ ds + \frac{At^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} f(s,u(s))\ ds \\ &- \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds\end{aligned}$$

Let $t = 1$ in equation (3) and in the above equation, we get

$$\begin{aligned}u(1) &+ D^{\alpha}u(t)|_{t=1} = A \int_0^1 (1-s)\ f(s,u(s))\ ds + A \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds \\ &- \int_0^1 (1-s)\ f(s,u(s))\ ds + \frac{A}{\Gamma(2-\alpha)} \int_0^1 (1-s)\ f(s,u(s))\ ds \\ &+ \frac{A}{\Gamma(2-\alpha)} \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds - \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds \\ &= \left(A(1+\frac{1}{\Gamma(2-\alpha)}) - 1\right) \int_0^1 (1-s)\ f(s,u(s))\ ds \\ &+ \left(A(1+\frac{1}{\Gamma(2-\alpha)}) - 1\right) \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds \\ &= \left(A(\frac{\Gamma(2-\alpha)+1}{\Gamma(2-\alpha)}) - 1\right) \int_0^1 (1-s)\ f(s,u(s))\ ds \\ &+ \left(A(\frac{\Gamma(2-\alpha)+1}{\Gamma(2-\alpha)}) - 1\right) \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds \\ &= \left((\frac{\Gamma(2-\alpha)}{1\ +\ \Gamma(2-\alpha)})(\frac{\Gamma(2-\alpha)+1}{\Gamma(2-\alpha)}) - 1\right) \int_0^1 (1-s)\ f(s,u(s))\ ds \\ &+ \left((\frac{\Gamma(2-\alpha)}{1\ +\ \Gamma(2-\alpha)})(\frac{\Gamma(2-\alpha)+1}{\Gamma(2-\alpha)}) - 1\right) \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\ f(s,u(s))\ ds\ =\ 0.\end{aligned}$$

The proof is complete.■

4. Maximal and minimal solutions

Here we study the existence of the maximal and minimal solutions of the fractional-order integral equation (3).

Definition 3.1 Let n be a solution of the integral equation (3) , then n is said to be a maximal solution of (3) if, for every solution u of (3) , the inequality $u(t) \leq n(t), t \in [0,1]$, holds.

A minimal solution may be define similarly by reversing the last inequality.

From Theorem 3 we get that the integral equation (3) has a positive solution $u \in C[0,1]$. Based on this criterion we can prove the following theorem.

Theorem Let the assumptions of Theorem 3 be satisfied. Furthermore, if $f(t,x)$ is non- decreasing functions in x, then there exist maximal and minimal solutions of the integral equation (3).

Proof: Consider the fractional-order integral equation

$$u_\epsilon(t) \ = \ \epsilon \ + \int_0^1 G(t,s) \ f(s, u_\epsilon(s)) \ ds, \ \epsilon \ > \ 0. \tag{5}$$

In the view of Theorem 3, it is clear that equation (5) has at least one positive solution $u(t) \in C[0,1]$. Now, let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$. Then, we have $u_{\epsilon_2}(0) \ < \ u_{\epsilon_1}(0)$ (from (3)-(5), we have $u_{\epsilon_2}(0) \ = \ \epsilon_2 \ < \ \epsilon_1 \ = \ u_{\epsilon_1}(0)$). We can prove

$$u_{\epsilon_2}(t) \ < \ u_{\epsilon_1}(t) \text{ for all } t \ \in \ [0,1]. \tag{6}$$

To prove conclusion (6), we assume that it is false, then there exist a t_1 such that

$$u_{\epsilon_2}(t_1) \ = \ u_{\epsilon_1}(t_1) \text{ and } u_{\epsilon_2}(t) \ < \ u_{\epsilon_1}(t) \text{ for all } t \ \in \ [0,t_1).$$

Since f is monotonic nondecreasing in u, it follows that $f(t, u_{\epsilon_2}(t)) \ \leq \ f(t, u_{\epsilon_1}(t))$ and consequently, using equation (5), we obtain

$$\begin{aligned} u_{\epsilon_2}(t_1) \ &= \ \epsilon_2 \ + \int_0^1 G(t_1,s) \ f(s, u_{\epsilon_2}(s)) \ ds \\ &< \ \epsilon_1 \ + \int_0^1 G(t_1,s) \ f(s, u_{\epsilon_1}(s)) \ ds \\ &= \ u_{\epsilon_1}(t_1). \end{aligned}$$

Which contradict the fact that $u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1)$. Hence the inequality (6) is true.

From the hypothesis, it follows as in the proof of Theorem 3 that the family of functions $\{u_\epsilon\}$ is relatively compact on $[0,1]$, hence, we can extract a uniformly convergent subsequence $\{u_{\epsilon p}\}$, that is, there exists a decreasing sequence $\{\epsilon_p\}$ such that $\epsilon_p \to 0$ as $p \to \infty$ and $\lim_{p\to\infty} u_{\epsilon p}(t)$ exists uniformly in $t \in [0,1]$, we denote this limiting value by $n(t)$.

Obviously, the uniform continuity of f and the equation

$$u_{\epsilon_p}(t) \ = \ \epsilon_p \ + \int_0^1 G(t,s) \ f(s, u_{\epsilon_p}(s)) \ ds, \ t \ \in \ [0,1],$$

yields n is a solution of equation (3). Finally, we show that the solution n is the maximal solution of equation (3). To achieve this goal, let u be any solution of (3) existing on the interval $[0,1]$. Then

$$u(t) \ < \ \epsilon \ + \ \int_0^1 \ G(t,s) \ f(s,u_\epsilon(s)) \ ds \ = \ u_\epsilon(t), \ t \ \in \ [0,1].$$

Since the maximal solution is unique (see [13]), it is clear that $u_\epsilon(t)$ tends to $n(t)$ uniformly in $t \in [0,1]$ as $\epsilon \to 0$. Which proves the existence of maximal solution to the integral equation (3). A similar argument holds for the minimal solution. ■

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Bin-Taher, E.O

Faculty of Science, Hadhramout University of Sci. and Tech., Hadhramout, Yemen

E-mail address: ebtsamsam@yahoo.com