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CARATHÈODORY THEOREM FOR QUADRATIC INTEGRAL EQUATIONS OF ERDÉLYI-KOBER TYPE

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ABSTRACT. We present the existence of maximal and minimal at least one continuous solution for a nonlinear quadratic integral equation of Erdélyi-Kober type. Also, some special cases will considered.

1. Introduction and Preliminaries

It is well-known that a useful mathematical tool for physical investigation and description of non-local and anomalous diffusion is Fractional Calculus, which is that branch of mathematical analysis dealing with pseudo-differential operators interpreted as integrals and derivatives of non-integer order (see [1], [24] [29] and [30]).
the generalized grey Brownian motion is an anomalous diffusion process driven by a fractional integral equation in the sense of Erdélyi-Kober, and for this reason it is proposed to call such family of diffusive processes as Erdélyi- Kober fractional diffusion [27].

An Erdélyi-Kober operator is a fractional integration operation introduced by Arthur Erdélyi (1940) and Hermann Kober (1940).
The Erdélyi-Kober fractional integral is given by ([1] and [12]-[14])

$$I_m^\alpha f(t) = \int_0^t \frac{(t^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \ m \ s^{m-1} \ f(s) \ ds, \ m > 0.$$

which generalizes the Riemann fractional integral ($m = 1$) and its generalized fractional derivative of order $\alpha > 0$, like:

$$D_m^\alpha f(t) = D_m I_m^{1-\alpha}, \ m > 0, \ \alpha \in (0,1).$$

For the properties of Erdélyi-Kober operators see [1], [24] and [30] for example.

Quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and

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the traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. [2]-[9] and [11]-[22]. However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness we use Tychonoff fixed point theorem.
Let $\mathbb{R}$ be the set of real numbers whereas $I = [0, 1]$, $L_1 = L_1[0, 1]$ be the space of Lebesgue integrable functions on I.

Here, we prove the existence of at least one continuous solution for the quadratic integral equation of fractional order

$$x(t) = a(t) + g(t, x(t)) \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m\, s^{m-1} f(s, x(s))\, ds, \; t \in I, \; \alpha > 0, \; m > 0 \tag{1}$$

and the existence of a continuous solution of the nonlinear differential equation of fractional-order

$$ {}_R D^{\alpha}_{m} x(t) = f(t, x(t)), \; t \in I \text{ and } x(0) = 0, \; \alpha \in (0, 1) \tag{2}$$

(where ${}_R D^{\alpha}_{m}$ is the Erdélyi-Kober fractional order derivative) will be given as an application. Also, the results concerning the existence of continuous solution of the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \; x(0) = x_0, \tag{3}$$

will be given as another application.
Finally, the existence of maximal and minimal solutions of (1) will be proved.
For $m = 1$, J. Banaś (see [9]) proved the existence of a nondecreasing continuous solution of (1) by using the technique of measure of noncompactness. The existence of continuous solutions for some quadratic integral equations was proved by using Schauder-Tychonoff fixed point theorem [31].

The existence results will be based on the following fixed-point theorems and definitions.

Theorem 1. *Tychonoff fixed-point Theorem* [10]
Suppose B is a complete, locally convex linear space and S is a closed convex subset of B. Let the mapping $T : B \to B$ be continuous and $T(S) \subset S$. If the closure of $T(S)$ is compact, then T has a fixed point in S.

2. Existence of continuous solutions

Now, equation (1) will be investigated under the assumptions:

(i) $a : I \to \mathbb{R}$ is continuous and bound with $k_1 = \sup_{t \in I} |a(t)|$.

(ii) $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded with $k_2 = \sup_{(t,x) \in I \times \mathbb{R}} |g(t, x)|$.

(iii) There exist two constants l_i, $i = 1, 2$ satisfying

$$|g(t, x) - g(s, y)| \leq l_1 \, |t - s| + l_2 \, |x - y|$$

for all t, $s \in I$ and x, $y \in \mathbb{R}$.

(iv) $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathèodory condition (i.e. measurable in t for all $x : I \to \mathbb{R}$ and continuous in x for all $t \in I$).

(v) There exists a function $\phi \in L_1$ such that $|f(t,x)| \leq \phi(t)$ $(\forall\ (t,x) \in I \times \mathbb{R})$ and $k_3 = \sup_{t \in I} I_m^\beta\ \phi(t)$ for any $\beta \leq \alpha$.

Theorem 2. *Let the assumptions (i)-(v) be satisfied, then the quadratic functional integral equation (1) has at least one solution in the space* $x \in C(I)$.

Proof.
Let $C = C(I)$ be the space of all continuous functions on I. It can be verified that $C(I)$ is a complete locally convex linear space [10].
Define a subset S of $C(I)$ by

$$S = \{\ x \in C : |\ x(t)\ | \leq r\ \},\ t \in I,$$

then

$$|\ x(t)\ | \leq |a(t)| + |g(t,x(t))| \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\ m\ s^{m-1}\ |f(s,x(s))|\ ds$$

$$|x(t)| \leq k_1 + k_2\ I_m^{\alpha-\beta}\ I_m^\beta\ \phi(t).$$

Also from assumption (v) we obtain

$$|x(t)| \leq k_1 + k_2\ k_3\ m \int_0^t \frac{(t^m - s^m)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} s^{m-1}\ ds.$$

Then

$$|x(t)| \leq k_1 + \frac{k_2\ k_3}{\Gamma(\alpha - \beta + 1)}.$$

From the last estimate we deduce that $r = k_1 + \frac{k_2\ k_3}{\Gamma(\alpha - \beta + 1)}$.
It is clear that the set S is closed and convex.
Let T be an operator defined by

$$(Tx)(t) = a(t) + g(t,x(t)) \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\ m\ s^{m-1}\ f(s,x(s))\ ds,\ x \in S$$

Assumptions (ii) and (iv) imply that $T: S \to C(I)$ is a continuous operator in x. We shall prove that $TS \subset S$.
For every $x \in S$ we have

$$\begin{aligned}|(Tx)(t)| &\leq k_1 + k_2\ I_m^{\alpha-\beta}\ I_m^\beta\ \phi(t)\ ds\\ &\leq k_1 + \frac{k_2\ k_3}{\Gamma(\alpha - \beta + 1)} = r.\end{aligned}$$

Then, $Tx \in S$ and hence $TS \subset S$.
Now for t_1 and $t_2 \in I$ (without loss of generality assume that $t_1 < t_2$), we have

$$\begin{aligned}(Tx)(t_2) - (Tx)(t_1) &= a(t_2) - a(t_1)\\ &+ g(t_2,x(t_2))\ I_m^\alpha\ f(t_2,x(t_2)) - g(t_1,x(t_1))\ I_m^\alpha\ f(t_1,x(t_1))\\ &+ g(t_1,x(t_1))\ I_m^\alpha\ f(t_2,x(t_2)) - g(t_1,x(t_1))\ I_m^\alpha\ f(t_2,x(t_2))\\ &\leq a(t_2) - a(t_1) + [g(t_2,x(t_2)) - g(t_1,x(t_1))]\ I_m^\alpha\ f(t_2,x(t_2))\\ &+ g(t_1,x(t_1))\ [\ I_m^\alpha\ f(t_2,x(t_2)) - I_m^\alpha\ f(t_1,x(t_1))\ ],\end{aligned}$$

but

$$I_m^\alpha\ f(t_2,x(t_2)) - I_m^\alpha\ f(t_1,x(t_1)) = \int_0^{t_1} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\ m\ s^{m-1}\ f(s,x(s))\ ds$$

$$+ \int_{t_1}^{t_2} \frac{(t_2^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \, m\, s^{m-1} \, f(s,x(s)) \, ds - \int_0^{t_1} \frac{(t_1^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \, m\, s^{m-1} \, f(s,x(s)) \, ds$$
$$\leq \int_0^{t_1} \frac{(t_1^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \, m\, s^{m-1} \, f(s,x(s)) \, ds + \int_{t_1}^{t_2} \frac{(t_2^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \, m\, s^{m-1} \, f(s,x(s)) \, ds$$
$$- \int_0^{t_1} \frac{(t_1^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \, m\, s^{m-1} \, f(s,x(s)) \, ds \ = \ \int_{t_1}^{t_2} \frac{(t_2^m \ - \ s^m)^{\alpha-1}}{\Gamma(\alpha)} \, m\, s^{m-1} \, f(s,x(s)) \, ds.$$

Then

$$\begin{aligned} |I_m^{\alpha} \, f(t_2,x(t_2)) \ - \ I_m^{\alpha} \, f(t_1,x(t_1))| &\leq \ I_{m,t_1}^{\alpha} |f(t_2,x(t_2))| \\ &\leq \ I_{m,t_1}^{\alpha} \phi(t_2) \ = \ I_{m,t_1}^{\alpha-\beta} I_{m,t_1}^{\beta} \phi(t_2) \\ &\leq \ k_3 \, \frac{(t_2^m - t_1^m)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. \end{aligned}$$

Then we get

$$| \, (Tx)(t_2) - (Tx)(t_1) \, | \ \leq \ | \, a(t_2) - a(t_1) \, | + [l_1|t_2-t_1| + l_2|x(t_2)) - x(t_1))|] \, I_m^{\alpha} \, | \, f(t_2,x(t_2)) \, |$$
$$+ \ | \ g(t_1,x(t_1)) \ | \ k_3 \, \frac{(t_2^m - t_1^m)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$$

i.e.,

$$\begin{aligned} | \, (Tx)(t_2) \, - \, (Tx)(t_1) \, | \ &\leq \ | \, a(t_2) \, - \, a(t_1) \, | + [l_1|t_2-t_1| + l_2|x(t_2)-x(t_1)|] \, I_m^{\alpha} \, \phi(t_2) \\ &\quad + \, k_2 \, k_3 \, \frac{(t_2^m - t_1^m)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\ &\leq \ | \ a(t_2) \ - \ a(t_1) \ | \ + \ \frac{k_3 \, (t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \, [l_1|t_2^m - t_1^m| \ + \ l_2|x(t_2)-x(t_1)|] \\ &\quad + \frac{k_2 \, k_3}{\Gamma(\alpha-\beta+1)} \, (t_2^m - \ t_1^m)^{\alpha-\beta} \ \to 0 \text{ as } t_2 \to t_1. \end{aligned}$$

This means that the functions of TS are equi-continuous on I. Then by the Arzela-Ascoli Theorem [10] the closure of TS is compact .
Since all conditions of the Tychonoff Fixed-point Theorem hold, then T has a fixed point in S. ■

3. Spacial cases

Corollary 1. *Let the assumptions of Theorem 2 be satisfied (with* $m = 1$*), then the fractional-order quadratic integral equation*

$$x(t) \ = \ a(t) \ + \ g(t,x(t)) \, \int_0^t \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \, f(s,x(s)) \, ds$$

has at least one solution $x \in C$.

Corollary 2. *Let the assumptions of Theorem 2 be satisfied (with* $g(t,x) = 1$ *), then the fractional-order integral equation*

$$x(t) \ = \ a(t) \ + \ \int_0^t \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \, f(s,x(s)) \, ds$$

has at least one solution $x \in C$.

which is the same results obtained in [23].
Now letting $\alpha, \ \beta \to 1$, we obtain

Corollary 3. *Let the assumptions of Theorem 2 be satisfied (with* $g(t,x) = 1,\ a(t) = x_0$ *and letting* $\alpha,\ \beta \to 1$ *), then the integral equation*

$$x(t) \ = \ x_0 \ + \ \int_0^t \ f(s, x(s)) \ ds.$$

has at least one solution $x \in C$ *which is equivalent to the initial value problem (3).*

which is the Carathéodory Theorem (proved in [10]).

4. Fractional order functional differential equations

For the initial value problem of the nonlinear fractional-order differential equation (2) we have the following theorem.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied (with* $a(t) = 0$ *and* $g(t, x(t)) = 1$*), then the Cauchy type problem (2) has at least one solution* $x \in C$.

Proof. Integrating (2) we obtain the integral equation

$$x(t) \ = \ \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} \ m \ s^{m-1} \ f(s, x(s)) \ ds, \ t \ \in \ I \tag{4}$$

which by Theorem 2 has the desired solution.
Operating with ${}_RD_m^{\alpha}$ on (4) we obtain the initial value problem (2). So the equivalence between the initial value problem(2) and the integral equation (4) is proved and then the results follow from Theorem 2.■

5. Maximal and minimal solutions

Definition 1. *(see* [25]*) Let* $q(t)$ *be a solution of (1) Then* $q(t)$ *is said to be a maximal solution of (1) if every solution of (1) on* I *satisfies the inequality* $x(t) \ < \ q(t), \ t \in I$ *. A minimal solution* $s(t)$ *can be defined in a similar way by reversing the above inequality i.e.* $x(t) \ > \ s(t), \ t \in I$.

we need the following lemma to prove the existence of maximal and minimal solutions of (1).

Lemma 1. *Let* $g(t,x),\ f(t,x)$ *satisfy the assumptions in Theorem 2 and let* $x(t),\ y(t)$ *be continuous functions on I satisfying*

$$\begin{aligned} x(t) \quad &\leq \quad a(t) \ + \ g(t, x(t)) \ I_m^{\alpha} f(t, x(t)) \\ y(t) \quad &\geq \quad a(t) \ + \ g(t, y(t)) \ I_m^{\alpha} f(t, y(t)) \end{aligned}$$

where one of them is strict.
Suppose $f(t,x)$ *is nondecreasing function in* x. *Then*

$$x(t) \ < \ y(t). \tag{5}$$

proof
Let the conclusion (5) be false; then there exists t_1 such that

$$x(t_1) \ = \ y(t_1), \ t_1 \ > \ 0$$

and

$$x(t) \ < \ y(t), \ 0 \ < \ t \ < \ t_1.$$

From the monotonicity of the function f in x, we get

$$\begin{aligned} x(t_1) &\leq a(t_1) + g(t_1, x(t_1))\, I_m^{\alpha}\, f(t_1, x(t_1)) \\ &= a(t_1) + g(t_1, x(t_1)) \int_0^{t_1} \frac{(t_1^m - s^m)^{\alpha - 1}}{\Gamma(\alpha)}\, m\, s^{m-1}\, f(s, x(s))\, ds \\ &< a(t_1) + g(t_1, y(t_1)) \int_0^{t_1} \frac{(t_1^m - s^m)^{\alpha - 1}}{\Gamma(\alpha)}\, m\, s^{m-1}\, f(s, y(s))\, ds \\ &< y(t_1). \end{aligned}$$

This contradicts the fact that $x(t_1) = y(t_1)$; then

$$x(t) < y(t).$$

Theorem 4. *Let the assumptions of Theorem 2 be satisfied. Furthermore, if $f(t,x)$ is nondecreasing functions in x, then there exist maximal and minimal solutions of (1).*

Proof

Firstly, we shall prove the existence of maximal solution of (1). Let $\epsilon > 0$ be given. Now consider the fractional-order quadratic integral equation

$$x_\epsilon(t) = a(t) + g_\epsilon(t, x_\epsilon(t))\, I_m^{\alpha}\, f_\epsilon(t, x_\epsilon(t)), \tag{6}$$

where

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) + \epsilon$$

and

$$g_\epsilon(t, x_\epsilon(t)) = g(t, x_\epsilon(t)) + \epsilon.$$

Clearly the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ satisfy assumptions (ii), (iv) and

$$|\, g_\epsilon(t, x_\epsilon)\, | \leq M + \epsilon = M'.$$

$$|\, f_\epsilon(t, x_\epsilon)\, | \leq \phi(t) + \epsilon = \phi'(t) \in L_1.$$

Therefore, equation (6) has a continuous solution $x_\epsilon(t)$ according to Theorem 2. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$x_{\epsilon_1}(t) = a(t) + g_{\epsilon_1}(t, x_{\epsilon_1}(t))\, I_m^{\alpha}\, f_{\epsilon_1}(t, x_{\epsilon_1}(t)),$$

$$\begin{aligned} x_{\epsilon_1}(t) &= a(t) + (g(t, x_{\epsilon_1}(t)) + \epsilon_1)\, I_m^{\alpha}\, (f(t, x_{\epsilon_1}(t)) + \epsilon_1), \\ &> a(t) + (g(t, x_{\epsilon_1}(t)) + \epsilon_2)\, I_m^{\alpha}\, (f(t, x_{\epsilon_1}(t)) + \epsilon_2), \end{aligned} \tag{7}$$

$$x_{\epsilon_2}(t) = a(t) + (g(t, x_{\epsilon_2}(t)) + \epsilon_2)\, I_m^{\alpha}\, (f(t, x_{\epsilon_2}(t)) + \epsilon_2). \tag{8}$$

Applying Lemma 1, then (7) and (8) imply

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t)\ for\ t \in I.$$

As shown before in the proof of Theorem 2, the family of functions $x_\epsilon(t)$ defined by (6) is uniformly bounded and of equi-continuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in I and we denote this limit by $q(t)$. from the continuity of the functions f_{ϵ_n} and g_{ϵ_n} in the second argument, we get

$$q(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = a(t) + g(t, q(t))\, I_m^{\alpha}\, f(t, q(t))$$

which proves that $q(t)$ is a solution of (1).
Finally, we shall show that $q(t)$ is maximal solution of (1). To do this, let $x(t)$ be any solution of (1). Then

$$\begin{aligned} x_\epsilon(t) &= a(t) + g_\epsilon(t, x_\epsilon(t)) I_m^\alpha f_\epsilon(t, x_\epsilon(t)) \\ &> a(t) + g(t, x_\epsilon(t)) I_m^\alpha f(t, x_\epsilon(t)). \end{aligned}$$

and

$$x(t) = a(t) + g(t, x(t)) I_m^\alpha f(t, x(t)).$$

Applying Lemma 1, we get

$$x_\epsilon(t) > x(t) \; for \; t \in I.$$

from the uniqueness of the maximal solution (see [25], [28]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in I \; as \; \epsilon \to 0$.
By a similar way we can prove that s(t) is the minimal solution of (1). ■

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