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EXISTENCE OF WEIGHTED PSEUDO ALMOST AUTOMORPHIC MILD SOLUTIONS TO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the existence of weighted pseudo almost automorphic mild solutions of integro-differential equations with fractional order $1 < \alpha < 2$, here A is a linear densely defined operator of sectorial type on a complex Banach space X. This paper also deals with existence of weighted pseudo almost automorphic mild solutions of semilinear integro-differential eqautions with A is the generator of the C_0-semigroup. The main results are obtained by suitable fixed point theorems.

1. Introduction

The origin of fractional calculus goes back to Newton and Leibnitz in the seventieth century. We observe that fractional order can be complex in viewpoint of pure mathematics and there is much interest in developing the theoretical analysis and numerical methods to fractional equations, because they have recently proved to be valuable in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, biology and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [11, 63] or to model activator-inhibitor dynamics with anomalous diffusion [35]. For details, including some applications and recent results, see the monographs of Ahn and MacVinisch [9], Gorenflo and Mainardi [27], Hilfer [36], Kilbas et al. [39], Kiryakova [40], Miller and Ross [52], Podlubny [58] and Samko et al. [59] and the papers of Agarwal et al. [3, 4, 5, 6], Benchohra et al. [12], Diethelm et al. [23, 24], El-Borai [14, 15, 16], El-Sayed [60, 61, 62], Chen et al. [18], Gaul et al. [26], Hu and Wang [37], Mophou et al. [53, 54, 56], Nieto et al. [7, 10], G.M.N'Guerekata [33], Lakshmikantham [41], Lakshmikantham et al. [42, 43, 44], Mainardi [48] and the references therein. Mainardi [48] and Mainardi et al. [49, 50] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process

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changes from slow diffusion to classical diffusion then to diffusion-wave and finally to classical wave when α increases from 0 to 2. The fundamental solutions of the Cauchy problems associated to these generalized diffusion equation ($0 < \alpha \leq 2$) are studied in [28, 50, 51].

The study of almost automorphic solutions to fractional differential equation were initiated by Araya and Lizama [8]. In their work, the authors investigated the existence and uniqueness of an almost automorphic mild solution of the fractional differential equation. In [20], the authors Cuevas and Lizama studied the existence and uniqueness of an almost automorphic mild solution of the fractional differential equation with A as a linear operator of sectorial negative type on a complex Banach space. Mophou et al. [55] prove the existence and uniqueness of pseudo almost automorphic mild solution to autonomous evolution equation. Also Mophou [57] studied weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Agarwal et al. [2] studied the existence and uniqueness of a weighted pseudo-almost periodic mild solution to the semilinear fractional differential equation. Recently, Abbas [1] studied pseudo almost automorphic solutions of some nonlinear integro-differential equations.

The concept of weighted pseudo almost periodic functions introduced by Diagana [22] and was generalized by G.M.N'Guerekata et al. [13] to the concept of weighted pseudo almost automorphic functions. By constructing counterexamples Liang et al. [47] showed that the decomposition of such functions is not unique in general. Actually, they proved that the decomposition of weighted pseudo almost periodic functions as well as weighted pseudo almost automorphic functions is unique if the space of the ergodic components is translation invariant.

To the best of our knowledge, there is no work reported in the literature on weighted pseudo almost automorphic fractional integro-differential equations in $1 < \alpha < 2$. To close this gap, motivated by the above mentioned works, the purpose of this paper is to study existence of weighted pseudo almost automorphic mild solutions to the following fractional integro-differential equation:

$$D_t^{\alpha}x(t) = Ax(t) + D_t^{\alpha-1}f(t, x(t), Kx(t)), \quad t \in \mathbb{R},\, 1 < \alpha < 2, \tag{1.1}$$
$$\text{and} \quad Kx(t) = \int_{-\infty}^{t} k(t-s)h(s, x(s))ds,$$

where $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $(X, \|.\|)$, K is a bounded linear operator and k satisfy $|k(t)| \leq c_k e^{-bt}$ for $t \geq 0$ and c_k, b are positive constants, $f : \mathbb{R} \times X \times X \to X$ is a weighted pseudo almost automorphic function in t for each $x, y \in X$ satisfying suitable conditions and $h : \mathbb{R} \times X \to X$ is a given function. The fractional derivative D_t^{α} is to be understood in Riemann-Liouville sense.

This work is organized as follows. In Section 2, we recall some basic definitions and preliminary facts of the standard properties of sectorial operators, on almost, pseudo-almost, weighted pseudo-almost automorphic functions and compactness criterion in $C_h(X)$ (see Lemma 2.6). In Section 3, we obtain very general results on the existence of weighted pseudo almost automorphic mild solutions for semilinear fractional integro-differential equations. Finally, in section 4, an example is provided and in section 5, conclusion is given.

2. Preliminaries and Basic Results

In this section, we introduce notations, definitions, lemmas and preliminary facts which are used throughout this work.

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two complex Banach spaces. Let $\mathcal{BC}(\mathbb{R}, X)$, (respectively $\mathcal{BC}(\mathbb{R}\times Y, X)$) denote the collection of all X-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $f : \mathbb{R}\times Y \to X$). The space $\mathcal{BC}(\mathbb{R}, X)$ equipped with the sup norm defined by

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$$

is a Banach space. Let also $\mathcal{L}(X)$ be the Banach space of all bounded linear operators from X into itself endowed with the norm:

$$\|T\|_{\mathcal{L}(X)} = \sup\{\|Tx\| : x \in X, \|x\| \le 1\}.$$

The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

also, the fractional derivative of function f of order $\alpha > 0$ is defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1} f(s)ds.$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2.1. [30, 64]. *Let $f : \mathbb{R} \to X$ be a bounded continuous function. We say that f is almost automorphic if for every sequence of real numbers $\{s_n\}_{n=1}^\infty$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that:*

$$g(t) := \lim_{n\to\infty} f(t+\tau_n)$$

is well-defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t \in \mathbb{R}$.

Definition 2.2. [30, 64]. *A continuous function $f : \mathbb{R}\times Y\times Y \to X$ is said to be almost automorphic if $f(t, x, y)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in M_2$, where M_2 is any bounded subset of $Y \times Y$.*

Clearly when the convergence above is uniform in $t \in \mathbb{R}$, f is almost periodic. The function g is measurable, but not continuous in general. Denote by $AA(X)$(respectively $AA(\mathbb{R}\times Y \times Y, X)$), the set of all almost automorphic function $f : \mathbb{R} \to X$,(respectively $f : \mathbb{R}\times Y\times Y \to X$). Obviously $AA(\mathbb{R}, X)$ is a subspace of $\mathcal{BC}(\mathbb{R}, X)$. Furthermore $AA(\mathbb{R}, X)$ endowed with the sup norm $\sup_{t\in\mathbb{R}} \|f(t)\|$ is a Banach space [30, 32].

Definition 2.3. *We define by*

$$AA_0(\mathbb{R}, X) = \Big\{\phi \in \mathcal{BC}(\mathbb{R}, X) : \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r} \|\phi(s)\|ds = 0\Big\}.$$

and by $AA_0(\mathbb{R}\times Y\times Y,X)$ the set of all continuous functions $f:\mathbb{R}\times Y\times Y\to X$ which belong to $\mathcal{BC}(\mathbb{R}\times Y\times Y,X)$ and satisfy

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|\phi(s,x,y)\|ds=0,$$

uniformly for (x,y) in any bounded subset of $Y\times Y$.

Definition 2.4. *A function $f\in\mathcal{BC}(\mathbb{R},X)$ is called pseudo almost automorphic function if it can be written as $f=g+\phi$, where $g\in AA(\mathbb{R},X)$ and $\phi\in AA_0(\mathbb{R},X)$.*

The functions g and ϕ are called respectively the principle and the ergodic terms of f.

Definition 2.5. *A function $f\in\mathcal{BC}(\mathbb{R}\times Y\times Y,X)$ is called pseudo almost automorphic in $t\in\mathbb{R}$ uniformly in $(x,y)\in Y\times Y$ if it can be written as $f=g+\phi$, where $g\in AA(\mathbb{R}\times Y\times Y,X)$ and $\phi\in AA_0(\mathbb{R}\times Y\times Y,X)$.*

We denote by $PAA(\mathbb{R},X)$(respectively $PAA(\mathbb{R}\times Y\times Y,X)$), the set of all pseudo almost automorphic function $f:\mathbb{R}\to X$,(respectively $f:\mathbb{R}\times Y\times Y\to X$).

Lemma 2.1. [64]. *$PAA(\mathbb{R},X)$ equipped with the supremum norm is a Banach space.*

We also refer to [21, 45, 46, 65] for more details on pseudo almost automorphic functions.

Now, let V be the set of all functions $\rho:\mathbb{R}\to(0,\infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $r>0$, set

$$m(r,\rho):=\int_{-r}^{r}\rho(x)dx$$

for each $\rho\in V$. Define

$$V_\infty:=\{\rho\in V:\lim_{r\to\infty}m(r,\rho)=\infty\}$$

and

$$V_b:=\{\rho\in V_\infty:\rho\,\text{is bounded and}\inf_{x\in\mathbb{R}}\rho(x)>0\}.$$

It is clear that $V_b\subset V_\infty\subset V$.

Now for $\rho\in V_\infty$ define

$$PAA_0(X,\rho):=\{f\in\mathcal{BC}(\mathbb{R},X):\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|f(s)\|\rho(s)ds=0\}.$$

Similarly we define $PAA_0(\mathbb{R}\times Y\times Y,\rho)$ as the collection of all functions $f:\mathbb{R}\times Y\times Y\to X$ which are jointly continuous and satisfy

$$\begin{cases} f(\cdot,x,y)\text{ is bounded for each }(x,y)\in Y\times Y,\\ \lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|f(s,x,y)\|\rho(s)ds=0\text{ uniformly in }(x,y)\in Y\times Y.\end{cases}$$

Definition 2.6. *Let $\rho\in V_\infty$. A function $f\in\mathcal{BC}(\mathbb{R},X)$ (respectively $f\in\mathcal{BC}(\mathbb{R}\times Y\times Y,X)$) is called weighted pseudo almost automorphic if it can be decomposed as $f=g+\phi$, where $g\in AA(\mathbb{R},X)$(respectively $AA(\mathbb{R}\times Y\times Y,X)$) and $\phi\in PAA_0(X,\rho)$(respectively $PAA_0(\mathbb{R}\times Y\times Y,\rho)$.*

We denote by $WPAA(\mathbb{R},\rho)$(respectively $WPAA(\mathbb{R}\times Y\times Y,\rho)$), the set of all such functions.

Remark 2.1. [13]. *When $\rho = 1$, we obtain the standard spaces $PAA(X)$ and $PAA(\mathbb{R} \times Y \times Y, X)$.*

Definition 2.7. *A subset P of $BC(\mathbb{R}, X)$ is said to be translation-invariant if for any $\phi(\cdot) \in P$ we have $\phi(\cdot + \tau) \in P$ for any $\tau \in \mathbb{R}$.*

Example 1. *$PAA_0(X, \rho)$ is translation-invariant for any $\rho \in V_b$.*

Theorem 2.1. [47] *Let $\rho \in V_\infty$. Assume that $PAA_0(X, \rho)$ is translation-invariant. Then the decomposition of a weighted pseudo almost automorphic function is unique.*

Remark 2.2. *Note that Theorem 2.1 does not hold in general without the assumption "$PAA_0(X, \rho)$ is translation-invariant" see* [47]*[Remark 3.3].*

From now on, we assume that $PAA_0(X, \rho)$ is translation-invariant in the sense that if $\phi(\cdot) \in PAA_0(X, \rho)$, then $\phi(\cdot + \tau) \in PAA_0(X, \rho)$ for any fixed τ.

Lemma 2.2. [64]. *Assume that $g : \mathbb{R} \to X$ is an almost automorphic function, fix $t_0 \in \mathbb{R}$, $\epsilon > 0$ and write*

$$B_\epsilon = \{\tau \in \mathbb{R}, \|g(t_0 + \tau) - g(t_0)\| < \epsilon\}.$$

Then, there exists $s_1, s_2, ..., s_m \in \mathbb{R}$ such that

$$\cup_{i=1}^{m}(s_i + B_\epsilon) = \mathbb{R}.$$

Lemma 2.3. [57]. *Let $\rho \in V_\infty$. If $f = g + \phi$ with $g \in AA(\mathbb{R}, X)$ and $\phi \in PAA_0(X, \rho)$, then $g(\mathbb{R}) \subset \overline{f(\mathbb{R})}$.*

Lemma 2.4. [57]. *Let $\rho \in V_\infty$ and $f \in BC(\mathbb{R}, X)$. Then $f \in PAA_0(X, \rho)$ if and only if for every $\epsilon > 0$,*

$$\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{M_{r,\epsilon}(f)} \rho(t) dt = 0,$$

where $M_{r,\epsilon}(f) := \{t \in [-r, r]/\|f(t)\| \geq \epsilon\}$.

Theorem 2.2. [57]. *Let $\rho \in V_\infty$. Then $(WPAA(X, \rho), \|\cdot\|_{WPAA(X,\rho)})$ is a Banach space with the supremum norm given by*

$$\|f\|_{WPAA(X,\rho)} = \sup_{t \in \mathbb{R}} \|f(t)\|.$$

Definition 2.8. [19]. *A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in a Banach space X is said to be sectorial of type ω and angle θ if there are constants $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, $M > 0$ such that its resolvent exists outside the sector*

$$\omega + \Sigma_\theta := \{\lambda + \omega : \lambda \in \mathcal{C}, |\arg(-\lambda)| < \theta\}, \tag{2.1}$$

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + \Sigma_\theta. \tag{2.2}$$

Definition 2.9. *Let $1 < \alpha < 2$. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We say that A is the generator of a solution operator if there exist $\omega \in R$ and a strongly continuous functions $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that $\{\lambda^\alpha : Re\ \lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad Re\ \lambda > \omega, \quad x \in X.$$

In [19]*, Cuesta proves that if A is sectorial of type $\omega \in \mathbb{R}$ with $0 \leq \theta < \pi(1-\alpha/2)$, then A is a generator of a solution operator given by*

$$S_\alpha(t) := \frac{1}{2\pi i}\int_{\mathbb{G}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda, \quad t \geq 0$$

with $\mathbb{G}$ a suitable path lying outside the sector $\omega + \Sigma_0$. Furthermore he shows that the following Lemma holds.

Lemma 2.5. [19]*[Theorem 1]. Let $A : D(A) \subset X \to X$ be a sectorial operator in a complex Banach space X, satisfying hypothesis (2.1) and (2.2), for some $M > 0, \omega < 0$ and $0 \leq \theta < \pi(1-\alpha/2)$. Then there exists $C(\theta,\alpha) > 0$ depending solely on θ and α, such that*

$$\|S_\alpha(t)\|_{\mathcal{L}(X)} \leq \frac{C(\theta,\alpha)M}{1+|\omega|t^\alpha}, \quad t \geq 0. \tag{2.3}$$

Now, we recall a useful compactness criterion.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(X) = \left\{u \in C(\mathbb{R}, X) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0\right\}.$$

Endowed with the norm $\|u\|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$, it is a Banach space (see[34]).

Lemma 2.6. [66, 34]. *A subset $K' \subset C_h(X)$ is a relatively compact set if it verifies the following conditions:*

- (c-1) *The set $K'(t) = \{u(t) : u \in \mathbb{R}\}$ is relatively compact in X for each $t \in \mathbb{R}$.*
- (c-2) *The set K' is equicontinuous.*
- (c-3) *For each $\epsilon > 0$ there exists $L' > 0$ such that $\|u(t)\| \leq \epsilon h(t)$ for all $u \in \mathbb{R}$ and all $|t| > L'$.*

Lemma 2.7. [29]*[Leray-Schauder Alternative Theorem]. Let D be a closed convex subset of a Banach space X such that $0 \in D$. Let $F' : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F'(x), 0 < \lambda < 1\}$ is unbounded or the map F' has a fixed point in D.*

3. Weighted pseudo almost automorphic mild solutions

Before starting our main results in this section, we recall the definition of the mild solution to (1.1).

Definition 3.1. [2]. *Assume that A generates an integrable solution operator $S_\alpha(t)$. A continuous function $x : \mathbb{R} \to X$ satisfying the integral equation*

$$x(t) = \int_{-\infty}^{t} S_\alpha(t-s)f\big(s, x(s), Kx(s)\big)ds, \quad t \in \mathbb{R}$$

is called a mild solution on $\mathbb{R}$ to (1.1).

We make the following assumptions:

- (H1) $f(t,x,y)$ is uniformly continuous on any bounded subset $M_2 \subset X \times X$ uniformly in $t \in \mathbb{R}$.
- (H2) $g(t,x,y)$ is uniformly continuous on any bounded subset $M_2 \subset X \times X$ uniformly in $t \in \mathbb{R}$.

(H3) A is a sectorial operator of type $\omega < 0$.

(H4) There exist constant L_f such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_f \left[\|x_1 - x_2\| + \|y_1 - y_2\|\right]$$

for each $x_i, y_i \in X$, $i = 1, 2$.

(H5) The function $h : \mathbb{R} \times X \to X$ is a weighted pseudo almost automorphic in t uniformly in $x \in X$ and satisfies

$$\|h(t,x) - h(t,y)\| \leq L_f^{'} \|x - y\| \quad \text{for each } x, y \in X.$$

Lemma 3.1. *Let $\rho \in V_\infty$ and $f = g + \phi \in WPAA(R \times X \times X, \rho)$. Assume that (H1), (H2) are satisfied. Then the function defined by $L(\cdot) := f(\cdot, u(\cdot), v(\cdot)) \in WPAA(X, \rho)$ if $u, v \in WPAA(X, \rho)$.*

Proof. We have $f = g+\phi$, where $g \in AA(\mathbb{R}\times X\times X, X)$ and $\phi \in PAA_0(\mathbb{R}\times X\times X, \rho)$ and $u = u_1 + u_2,\ v = v_1 + v_2$ where $u_1, v_1 \in AA(\mathbb{R}, X)$ and $u_2, v_2 \in PAA_0(X, \rho)$.

Now, the function f can be decomposed as

$$\begin{aligned}
&f(t,u(t),v(t))\\
&= g(t,u_1(t),v_1(t)) + f(t,u(t),v(t)) - g(t,u_1(t),v_1(t))\\
&= g(t,u_1(t),v_1(t)) + f(t,u(t),v(t)) - f(t,u_1(t),v_1(t)) + \phi(t,u_1(t),v_1(t)).
\end{aligned}$$

Define

$$\begin{gathered}
G(t) = g(t,u_1(t),v_1(t)),\ F(t) = f(t,u(t),v(t)) - f(t,u_1(t),v_1(t)),\\
H(t) = \phi(t,u_1(t),v_1(t)).
\end{gathered}$$

Then $f(t,u(t),v(t)) = G(t) + F(t) + H(t)$.
Using the assumption (H2), $G(t) \in AA(\mathbb{R}, X)$ by [45] (Lemma 2.2).

Next we prove that $F \in PAA_0(X, \rho)$.

For this, it is enough to show that $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\epsilon}(F)} \rho(t)dt = 0$.

By Lemma 2.3, $u_1(\mathbb{R}) \times v_1(\mathbb{R}) \subset \overline{u(\mathbb{R})} \times \overline{v(\mathbb{R})}$ which is a bounded set. Using hypothesis (H1) with $M_2 = \overline{u(\mathbb{R})} \times \overline{v(\mathbb{R})}$, we say that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|u - u_1\| + \|v - v_1\| < 2\delta \implies \|f(t,u(t),v(t)) - f(t,u_1(t),v_1(t))\| < \epsilon,\ \forall\ t \in \mathbb{R}.$$

Thus, we obtain,

$$\begin{aligned}
M_{r,\epsilon}(F) &= M_{r,\epsilon}\big(f(t,u(t),v(t)) - f(t,u_1(t),v_1(t))\big)\\
&\subset M_{r,\delta}(u - u_1) \cup M_{r,\delta}(v - v_1)\\
&= M_{r,\delta}(u_2) \cup M_{r,\delta}(v_2).
\end{aligned}$$

Consequently,

$$\frac{1}{m(r,\rho)} \int_{M_{r,\epsilon}(F)} \rho(t)dt \leq \frac{1}{m(r,\rho)} \int_{M_{r,\delta}(u_2)} \rho(t)dt + \frac{1}{m(r,\rho)} \int_{M_{r,\delta}(v_2)} \rho(t)dt.$$

By using Lemma 2.4, we have,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\delta}(u_2)} \rho(t)dt = \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\delta}(v_2)} \rho(t)dt = 0.$$

Since $u_2, v_2 \in PAA_0(X, \rho)$, then by Lemma 2.4,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\epsilon}(F)} \rho(t)dt = 0.$$

Thus, $F \in PAA_0(X,\rho)$.

Finally, it remains to show that $H \in PAA_0(X,\rho)$.

We have $u_1([-r,r]) \times v_1([-r,r])$ is compact since u_1 and v_1 are continuous on $\mathbb{R}$ as almost automorphic functions. So the function g being in $AA(\mathbb{R}\times X\times X, X)$, g is uniformly continuous on $[-r,r]\times u_1([-r,r])\times v_1([-r,r])$. Then it follows from (H1) that $\phi(t,x,y)$ is uniformly continuous in $(u_1,v_1) \in u_1([-r,r])\times v_1([-r,r])$ uniformly in $t \in [-r,r]$. Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $(x_1,y_1),(x_2,y_2) \in u_1([-r,r]) \times v_1([-r,r])$ and $\|x_1 - x_2\| + \|y_1 - y_2\| < \delta$ imply that

$$\|\phi(t,x_1,y_1) - \phi(t,x_2,y_2)\| < \frac{\epsilon}{2} \quad \forall \quad t \in [-r,r]. \tag{3.1}$$

On the other hand, since $u_1([-r,r]) \times v_1([-r,r])$ is compact, one can find balls O_k with $(\alpha_k,\beta_k) \in u_1([-r,r]) \times v_1([-r,r]),\ k = 1,2,\cdots,m$ and radius less than δ such that $u_1([-r,r]) \times v_1([-r,r]) \subset \cup_{k=1}^m O_k$.

Then the sets $U_k := \{t \in [-r,r]/(u_1(t),v_1(t)) \in O_k\},\ k = 1,2,\cdots,m$ are open in $[-r,r]$ and $[-r,r] = \cup_{k=1}^m U_k$.

Define V_k by

$$V_1 = U_1 \quad V_k = U_k - \cup_{k=1}^{k-1} U_k,\ 2 \le k \le m.$$

Then it is clear that,

$$V_i \cap V_j = \emptyset,\ \text{if } i \neq j,\ 1 \le i,j \le m.$$

So, we get

$$\begin{aligned}
L_1 :&= \{t \in [-r,r]/\|H(t)\| \ge \epsilon\} \\
&= \{t \in [-r,r]/\|\phi(t,u_1(t),v_1(t))\| \ge \epsilon\} \\
&\subset \cup_{k=1}^m \{t \in V_k/\|\phi(t,u_1(t),v_1(t)) - \phi(t,\alpha_k,\beta_k)\| + \|\phi(t,\alpha_k,\beta_k)\| \ge \epsilon\} \\
&\subset \cup_{k=1}^m \Big(\Big\{ t \in V_k/\|\phi(t,u_1(t),v_1(t)) - \phi(t,\alpha_k,\beta_k)\| \ge \frac{\epsilon}{2} \Big\} \\
&\qquad \cup \Big\{ t \in V_k/\|\phi(t,\alpha_k,\beta_k)\| \ge \frac{\epsilon}{2} \Big\} \Big).
\end{aligned}$$

From (3.1), it follows that,

$$\Big\{ t \in V_k/\|\phi(t,u_1(t),v_1(t)) - \phi(t,\alpha_k,\beta_k)\| \ge \frac{\epsilon}{2} \Big\} = \emptyset,\ k = 1,2,\cdots,m.$$

Thus, if we set $M_{r,\frac{\epsilon}{2}}(\phi_k) := M_{r,\frac{\epsilon}{2}}(\phi(t,\alpha_k,\beta_k))$, then

$$M_{r,\epsilon}(H) \subset \cup_{k=1}^m M_{r,\frac{\epsilon}{2}}(\phi_k)$$

and

$$\frac{1}{m(r,\rho)} \int_{M_{r,\epsilon}(H)} \rho(t)dt \le \sum_{k=1}^m \frac{1}{m(r,\rho)} \int_{M_{r,\frac{\epsilon}{2}}(\phi_k)} \rho(t)dt.$$

And since $\phi \in PAA_0(X \times X, \rho)$, we have

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\frac{\epsilon}{2}}(\phi_k)} \rho(t)dt = 0, \quad k = 1,2,\cdots,m,$$

it follows that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\epsilon(H)}} \rho(t)dt = 0.$$

According to Lemma 2.4, we have

$$H(t) = \phi(t, u_1(t), v_1(t)) \in PAA_0(X, \rho).$$

This completes the proof. □

Corollary 3.1. *$f = g + \phi \in WPAA(\mathbb{R} \times X \times X, \rho)$ where $\rho \in V_\infty$ assume both f and g are Lipschitzian in $(x, y) \in X \times X$ uniformly in $t \in \mathbb{R}$. Then the Nemytskii operator $L(\cdot) := f(\cdot, u(\cdot), v(\cdot)) \in WPAA(X, \rho)$ if $u, v \in WPAA(X, \rho)$.*

Lemma 3.2. *Let $\rho \in V_\infty$ and $f = g + \phi \in WPAA(\mathbb{R} \times X \times X, \rho)$. Assume that (H1), (H2) are satisfied. Then the function defined by $\phi(\cdot) := f(\cdot, x(\cdot), Kx(\cdot)) \in WPAA(X, \rho)$ if $x \in WPAA(X, \rho)$.*

Proof. Let us observe that if $x \in WPAA(X, \rho)$ then $x = x_1 + x_2$ where $x_1 \in AA(\mathbb{R}, X)$ and $x_2 \in PAA_0(X, \rho)$. Since K is a bounded and linear operator on X, it is easy to prove that $Kx = Kx_1 + Kx_2$ are also bounded and $Kx_2(\cdot) \in PAA_0(X, \rho)$. Therefore by [30], $Kx_1(\cdot) \in AA(\mathbb{R}, X)$, we deduce that $Kx(\cdot) \in WPAA(X, \rho)$. Hence in view of Lemma 3.1, we have $\phi(\cdot) \in WPAA(X, \rho)$. □

Lemma 3.3. *Let $f = g + \phi \in WPAA(X, \rho)$ where $\rho \in V_\infty$ with $g \in AA(\mathbb{R}, X)$, $\phi \in PAA_0(X, \rho)$. Then $Q(t) := \int_{-\infty}^{t} S_\alpha(t - s) f(s) ds \in WPAA(X, \rho)$.*

Proof. Let $Q(t) = R(t) + S(t)$, where

$$R(t) := \int_{-\infty}^{t} S_\alpha(t - s) g(s) ds$$
$$S(t) := \int_{-\infty}^{t} S_\alpha(t - s) \phi(s) ds.$$

Now, let $(s_n^{'})$ be an arbitrary sequence of real numbers. Since $g \in AA(\mathbb{R}, X)$ there exists a subsequence s_n of $(s_n^{'})$ such that

$$\overline{g}(t) := \lim_{n\to\infty} g(t + s_n) \text{ is well defined for each } t \in R$$

and

$$\lim_{n\to\infty} \overline{g}(t - s_n) = g(t), \text{ for all } t \in \mathbb{R}.$$

We define $\overline{R}(t) := \int_{-\infty}^{t} S_\alpha(t - s)\overline{g}(s) ds$.

Now, consider

$$\begin{aligned} R(t + s_n) &= \int_{-\infty}^{t+s_n} S_\alpha(t + s_n - s) g(s) ds \\ &= \int_{-\infty}^{t} S_\alpha(t - \sigma) g(\sigma + s_n) d\sigma \\ &= \int_{-\infty}^{t} S_\alpha(t - \sigma) g_n(\sigma) d\sigma \end{aligned}$$

where $g_n(\sigma) = g(\sigma + s_n),\ n = 1, 2, \cdots$

$$R(t + s_n) = \int_{0}^{\infty} S_\alpha(\sigma) g_n(t - \sigma) d\sigma.$$

Now, by inequality (2.3)

$$\|R(t+s_n)\| \le \int_0^\infty \frac{C(\theta,\alpha)M}{1+|\omega|\sigma^\alpha}\|g_n(t-\sigma)\|d\sigma$$
$$\le C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\|g\|_\infty$$

and by continuity of $S_\alpha(\cdot)x$ we have $S_\alpha(t-\sigma)g_n(\sigma) \to S_\alpha(t-\sigma)\overline{g}(\sigma)$ as $n\to\infty$ for each $\sigma \in \mathbb{R}$ fixed and any $t \ge \sigma$. Then by the Lebesgue dominated convergence theorem,

$$R(t+s_n) \to \overline{R}(t) \quad \text{as} \quad n\to\infty \quad \text{for all} \quad t\in\mathbb{R}.$$

In similar way we can show that

$$\overline{R}(t-s_n) \to R(t) \quad \text{as} \quad n\to\infty \quad \text{for all} \quad t\in\mathbb{R}.$$

This shows that $R(t) \in AA(\mathbb{R}, X)$.

Now let us show that $S(t) \in PAA_0(X,\rho)$. For $r>0$, we see that

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\|S(t)\|\rho(t)dt\\
&= \frac{1}{m(r,\rho)}\int_{-r}^{r}\Big\|\int_{-\infty}^{t} S_\alpha(t-s)\phi(s)ds\Big\|\rho(t)dt\\
&= \frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{t}\|S_\alpha(t-s)\|\|\phi(s)\|\rho(t)dsdt\\
&= \frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{0}^{\infty}\|S_\alpha(s)\|\|\phi(t-s)\|\rho(t)dsdt\\
&\le C(\theta,\alpha)M\int_0^\infty \frac{1}{1+|\omega|s^\alpha}\Big(\frac{1}{m(r,\rho)}\int_{-r}^{r}\|\phi(t-s)\|\rho(t)dt\Big)ds\\
&= C(\theta,\alpha)M\int_0^\infty \frac{\Omega_r(s)}{1+|\omega|s^\alpha}ds
\end{aligned}$$

where, $\Omega_r(s) = \frac{1}{m(r,\rho)}\int_{-r}^{r}\|\phi(t-s)\|\rho(t)dt$. Using that the space $PAA_0(X,\rho)$ is translation invariant it follows that $t\to\phi(t-s)$ belongs to $PAA_0(X,\rho)$ for each $s\in R$ and hence $\Omega_r(s)\to 0$ as $r\to\infty$. Next, since Ω_r is bounded ($\|\Omega_r\| \le \|\phi\|_\infty$) and $\frac{1}{1+|\omega|s^\alpha}$ is integrable in $[0,\infty)$, using the Lebesgue dominated convergence theorem it follows that $\lim_{r\to\infty}\int_0^\infty \frac{\Omega_r(s)}{1+|\omega|s^\alpha}ds = 0$. The proof is now completed. □

The first existence and uniqueness result is based on Banach's contraction principle.

Theorem 3.1. *Let $\rho \in V_\infty$. Let also $f = g+\phi \in WPAA(\mathbb{R}\times X\times X,\rho)$ with $g \in AA(\mathbb{R}\times X\times X, X)$ and $\phi \in PAA_0(\mathbb{R}\times X\times X,\rho)$. Assume that (H1)-(H5) hold. Then (1.1) has a unique mild solution in $WPAA(X,\rho)$ provided*

$$L_f(1+\frac{c_k}{b}L_f^{'})C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} < 1.$$

Proof. Consider the operator $\Gamma : WPAA(X,\rho) \to WPAA(X,\rho)$ such that

$$(\Gamma x)(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s,x(s),Kx(s))ds,\ t\in\mathbb{R}.$$

In view of Lemma 3.2 and Lemma 3.3, the operator (Γx) is well-defined.

Now if $x, y \in WPAA(X,\rho)$, by inequality (2.3) we have

$$\begin{aligned}
&\|(\Gamma x)(t)-(\Gamma y)(t)\| \\
&= \Big\| \int_{-\infty}^{t} S_\alpha(t-s)\big[f(s,x(s),Kx(s)) - f(s,y(s),Ky(s))\big]ds \Big\| \\
&\le \int_{-\infty}^{t} \|S_\alpha(t-s)\|_{L(X)} \big\|f(s,x(s),Kx(s)) - f(s,y(s),Ky(s))\big\| ds \\
&\le \int_{-\infty}^{t} \frac{C(\theta,\alpha)M}{1+|\omega|(t-s)^\alpha}\big[L_f\big(\|x(s)-y(s)\| + \|Kx(s)-Ky(s)\|\big)\big]ds. \qquad (3.2)
\end{aligned}$$

Consider

$$\begin{aligned}
\|Kx(s)-Ky(s)\| &\le \int_{-\infty}^{t} |k(t-s)|\|h(s,x(s)) - h(s,y(s))\|ds \\
&\le \int_{-\infty}^{t} |k(t-s)|L_f^{'}\|x(s)-y(s)\|ds \\
&\le \sup_{t\in R}\|x(t)-y(t)\|L_f^{'}\Big(\int_{-\infty}^{t}|k(t-s)|ds\Big) \\
&\le \sup_{t\in R}\|x(t)-y(t)\|L_f^{'}\int_{0}^{\infty}|k(s)|ds \\
&\le \sup_{t\in R}\|x(t)-y(t)\|L_f^{'}\int_{0}^{\infty}c_k e^{-bs}ds \\
&\le \frac{c_k}{b}L_f^{'}\sup_{t\in R}\|x(t)-y(t)\|.
\end{aligned}$$

Using the above estimate, inequality (3.2) becomes,

$$\begin{aligned}
&\|(\Gamma x)(t)-(\Gamma y)(t)\| \\
&\le L_f(1+\frac{c_k}{b}L_f^{'})\sup_{t\in R}\|x(t)-y(t)\|\int_0^\infty \frac{C(\theta,\alpha)M}{1+|\omega|s^\alpha}ds \\
&\le L_f(1+\frac{c_k}{b}L_f^{'})C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\|x-y\|_{WPAA(X,\rho)},\ \forall\, t\in\mathbb{R}.
\end{aligned}$$

Thus

$$\|\Gamma x-\Gamma y\|_{WPAA(X,\rho)} \le L_f(1+\frac{c_k}{b}L_f^{'})C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\|x-y\|_{WPAA(X,\rho)}.$$

This proves that Γ is a contraction, so by the Banach fixed point theorem there exist a unique $x \in WPAA(X,\rho)$ such that $\Gamma x = x$, that is $x(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s,x(s),Kx(s))ds$. □

We next study the existence of weighted pseudo almost automorphic mild solutions of equation (1.1) when the perturbation f is not necessarily Lipschitz continuous. For that, we require the following assumption:

(H6) There exists a continuous nondecreasing function $W : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,x,y)\| \le W(t)(\|x\|+\|y\|) \quad \text{for all} \quad t\in\mathbb{R} \text{ and } \quad x\in X.$$

The following existence result is based upon nonlinear Leray-Schauder alternative theorem.

Theorem 3.2. *Let $\rho \in V_\infty$. Assume that A is sectorial of type $\omega < 0$. Let (H2) be satisfied and $f \in WPAA(X,\rho)$ satisfying (H1) and (H6) and the following additional conditions:*

(i) *For each $C \geq 0$*
$$\lim_{|t|\to\infty} \frac{1}{h(t)} \int_{-\infty}^{t} \frac{W\big((1+k)Ch(s)\big)}{1+|\omega|(t-s)^\alpha} ds = 0,$$
where h is the function given in Lemma 2.6. We set
$$\beta(C) := C(\theta,\alpha)M \Big\| \int_{-\infty}^{t} \frac{W\big((1+k)Ch(s)\big)}{1+|\omega|(t-s)^\alpha} ds \Big\|,$$
where $C(\theta,\alpha)$ and M are constants given inequality (2.3).

(ii) *For each $\epsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $\|u-v\|_h \leq \delta$ implies that*
$$C(\theta,\alpha)M \int_{-\infty}^{t} \frac{\|f(s,u(s),Ku(s)) - f(s,v(s),Kv(s))\|}{1+|\omega|(t-s)^\alpha} ds \leq \epsilon, \text{ for all } t \in \mathbb{R}.$$

(iii) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(iv) *For all $a, b \in \mathbb{R}$, $a < b$ and $\Lambda > 0$, the set $\{f(s,x,Kx) : a \leq s \leq b,\ x \in C_h(X),\ \|x\|_h \leq \Lambda\}$ is relatively compact in X.*

Then equation (1.1) has a weighted pseudo almost automorphic mild solution.

Proof. We define the operator $\Gamma : C_h(X) \to C_h(X)$ by
$$(\Gamma x)(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s,x(s),Kx(s)) ds,\ t \in R.$$
We will show that Γ has a fixed point in $WPAA(X,\rho)$. For the sake of convenience, we divide the proof into several steps.

Step 1: For $x \in C_h(X)$, we have that
$$\begin{aligned}\|(\Gamma x)(t)\| &\leq C(\theta,\alpha)M \int_{-\infty}^{t} \frac{W(\|x(s)\| + K\|x(s)\|)}{1+|\omega|(t-s)^\alpha} ds \\ &\leq C(\theta,\alpha)M \int_{-\infty}^{t} \frac{W((1+\|K\|)\|x\|_h h(s))}{1+|\omega|(t-s)^\alpha} ds\end{aligned}$$
It follows from condition (i) that Γ is well defined.

Step 2: The operator Γ is continuous.

In fact, for any $\epsilon > 0$, we take $\delta > 0$ involved in condition (ii). If $x, y \in C_h(X)$ and $\|x-y\|_h \leq \delta$ then
$$\begin{aligned}\|(\Gamma x)(t) - (\Gamma y)(t)\| &\leq C(\theta,\alpha)M \int_{-\infty}^{t} \frac{\|f(s,x(s),Kx(s)) - f(s,y(s),Ky(s))\|}{1+|\omega|(t-s)^\alpha} ds \\ &\leq \epsilon,\end{aligned}$$
which shows the assertion.

Step 3: We will show that Γ is completely continuous.

We set $B_\Lambda(X)$ for the closed ball with center at 0 and radius Λ in the space X. Let $V'(t) = \Gamma(B_\Lambda(C_h(X)))$ and $v' = \Gamma(x)$ for $x \in B_\Lambda(C_h(X))$. First, we will prove that $V^{'}(t)$ is a relatively compact subset of X for each $t \in \mathbb{R}$. It follows form condition (i) that the function $s \to \frac{W((1+K)\Lambda h(t-s))}{1+|\omega|s^{\alpha}}$ is integrable on $[0,\infty)$. Hence, for $\epsilon > 0$, we can choose $a \geq 0$ such that $C(\theta,\alpha)M\int_a^{\infty}\frac{W((1+K)\Lambda h(t-s))}{1+|\omega|s^{\alpha}}ds \leq \epsilon$. Since,

$$\begin{aligned} v^{'}(t) &= \int_0^a S_{\alpha}(s)f(t-s,x(t-s),Kx(t-s))ds \\ &\quad + \int_a^{\infty} S_{\alpha}(s)f(t-s,x(t-s),Kx(t-s))ds \end{aligned}$$

and

$$\begin{aligned} &\Big\|\int_a^{\infty} S_{\alpha}(s)f(t-s,x(t-s),Kx(t-s))ds\Big\| \\ &\leq C(\theta,\alpha)M\int_a^{\infty}\frac{W((1+K)\Lambda h(t-s))}{1+|\omega|s^{\alpha}}ds \\ &\leq \epsilon \end{aligned}$$

we get $v^{'}(t) \in a\overline{c_0(N)} + B_{\epsilon}(X)$ where $c_0(N)$ denotes the convex hull of N and $N = \{S_{\alpha}(s)f(\xi,x,Kx) : 0 \leq s \leq a, t-a \leq \xi \leq t, \|x\|_h \leq \Lambda\}$. Using the strong continuity of $S_{\alpha}(\cdot)$ and property (iv) of f, we can infer that N is a relatively compact set and $V^{'}(t) \subset a\overline{c_0(N)} + B_{\epsilon}(X)$, which establishes our assertion.
Next, we show that the set $V^{'}$ is equicontinuous. In fact, we can decompose

$$\begin{aligned} v^{'}(t+s) - v^{'}(t) &= \int_0^s S_{\alpha}(\sigma)f(t+s-\sigma,x(t+s-\sigma),Kx(t+s-\sigma))d\sigma \\ &\quad + \int_0^a [S_{\alpha}(\sigma+s) - S_{\alpha}(\sigma)]f(t-\sigma,x(t-\sigma),Kx(t-\sigma))d\sigma \\ &\quad + \int_a^{\infty} [S_{\alpha}(\sigma+s) - S_{\alpha}(\sigma)]f(t-\sigma,x(t-\sigma),Kx(t-\sigma))d\sigma. \end{aligned}$$

For each $\epsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} &\Big\|\int_0^s S_{\alpha}(\sigma)f(t+s-\sigma,x(t+s-\sigma),Kx(t+s-\sigma))d\sigma \\ &\quad + \int_a^{\infty} [S_{\alpha}(\sigma+s) - S_{\alpha}(\sigma)]f(t-\sigma,x(t-\sigma),Kx(t-\sigma))d\sigma\Big\| \\ &\leq C(\theta,\alpha)M\Big[\int_0^s \frac{W((1+K)\Lambda h(t+s-\sigma))}{1+|\omega|\sigma^{\alpha}}d\sigma \\ &\quad + 2\int_a^{\infty}\frac{W((1+K)\Lambda h(t-\sigma))}{1+|\omega|\sigma^{\alpha}}d\sigma\Big] \\ &\leq \frac{\epsilon}{2} \end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\sigma,x(t-\sigma),Kx(t-\sigma)) : 0 \leq \sigma \leq a, x \in B_{\Lambda}(C_h(X))\}$ is a relatively compact set and $S_{\alpha}(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[S_{\alpha}(\sigma+s) - S_{\alpha}(\sigma)]f(t-\sigma,x(t-\sigma),Kx(t-\sigma))\| \leq \frac{\epsilon}{2a}$ for $s \leq \delta_2$. Combining these estimates, we get $\|v^{'}(t+s) - v^{'}(t)\| \leq \epsilon$ for s small enough and independent of $x \in B_{\Lambda}(C_h(X))$.

Finally, applying condition (i), we can see that

$$\begin{aligned}\frac{\|v^{'}(t)\|}{h(t)} &\leq \frac{C(\theta,\alpha)M}{h(t)}\int_{-\infty}^{t}\frac{W((1+K)\Lambda h(s)}{1+|\omega|(t-s)^{\alpha}}ds \\ &\to 0, \quad |t|\to\infty,\end{aligned}$$

and this convergence is independent of $x \in B_{\Lambda}(C_h(X))$. Hence by Lemma 2.6, $V^{'}$ is a relatively compact set in $C_h(X)$.

Step 4:

Let us assume that $x^{\lambda}(\cdot)$ is a solution of equation $x^{\lambda} = \lambda\Gamma(x^{\lambda})$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned}\|x^{\lambda}(t)\| &= \lambda\Big\|\int_{-\infty}^{t} S_{\alpha}(t-s)f(s,x^{\lambda}(s),Kx^{\lambda}(s))ds\Big\| \\ &\leq C(\theta,\alpha)M\int_{-\infty}^{t}\frac{W((1+K)\|x^{\lambda}\|_h h(s))}{1+|\omega|(t-s)^{\alpha}}ds \\ &\leq \beta(\|x^{\lambda}\|_h)h(t).\end{aligned}$$

Hence we get

$$\frac{\|x^{\lambda}\|_h}{\beta(\|x^{\lambda}\|_h)} \leq 1$$

and combining with condition (iii), we conclude that the set $\{x^{\lambda} : x^{\lambda} = \lambda\Gamma(x^{\lambda}), \lambda \in (0,1)\}$ is bounded.

Step 5: It follows from hypothesis (H1)-(H2) and Lemma 3.2 that the function $t \to f(t,x(t),Kx(t))$ belongs to $WPAA(X,\rho)$ whenever $x \in WPAA(X,\rho)$. Hence using Lemma 3.3, we get $\Gamma(WPAA(X,\rho)) \subset WPAA(X,\rho)$ and noting that $WPAA(X,\rho)$ is a closed subspace of $C_h(X)$, consequently we can consider, $\Gamma : WPAA(X,\rho) \to WPAA(X,\rho)$. Using Steps 1-3, we deduce that this map is completely continuous. Applying Leray-Schauder alternative theorem, we infer that Γ has a fixed point $x \in WPAA(X,\rho)$, which completes the proof. $\square$

Corollary 3.2. *Let $\rho \in V_{\infty}$. Assume that A is sectorial of type $\omega < 0$. Let (H2) satisfying and $f \in WPAA(X,\rho)$ satisfying (H1) and the inequality (2.3) and the following Holder type condition:*

$$\|f(t,x_1,x_2) - f(t,y_1,y_2)\| \leq \gamma[\|x_1-y_1\|^{\beta} + \|x_2-y_2\|^{\beta}],\ 0<\beta<1$$

for all $t \in \mathbb{R}$ and $x_i, y_i \in X$ for $i = 1,2$, where $\gamma > 0$ is a constant. Moreover, assume the following conditions:

(a) $f(t,0,0) = q$

(b) $\sup_{t\in R} C(\theta,\alpha)M\int_{-\infty}^{t}\frac{(1+K)h(s)^{\beta}}{1+|\omega|(t-s)^{\alpha}}ds = \gamma_2 < \infty.$

(c) *For all $a,b \in \mathbb{R}$, $a < b$ and $p > 0$, the set $\{f(s,x,Kx) : a \leq s \leq b,\ x \in C_h(X),\ \|x\|_h \leq p\}$ is relatively compact in X.*

Then equation (1.1) has a weighted pseudo almost automorphic mild solution.

Proof. Let $\gamma_0 = \|q\|, \gamma_1 = \gamma$. We take $W(\xi_1+\xi_2) = \gamma_0 + \gamma_1[\xi_1^{\beta}+\xi_2^{\beta}]$. Then condition (H6) is satisfied. It follows from (b), we can see that function f satisfies (i) in Theorem 3.2. Note that for each $\epsilon > 0$ there is $0 < \delta^{\beta} < \frac{\epsilon}{\gamma_1\gamma_2}$ such that for every $x, y \in C_h(X)$, $\|x-y\| \leq \delta$ implies that

$$C(\theta,\alpha)M\int_{-\infty}^{t}\frac{\|f(s,x(s),Kx(s))-f(s,y(s),Ky(s))\|}{1+|\omega|(t-s)^{\alpha}}ds \leq \epsilon$$

for all $t \in \mathbb{R}$. The hypothesis (iii) in the statement of the Theorem 3.2 can be easily verified using the definition of W. So by Theorem 3.2, we can prove equation (1.1) has a weighted pseudo almost automorphic mild solution. $\square$

4. Example

To illustrate Theorem 3.1, we consider the following fractional integro-differential equation:

$$D_t^{\alpha}w(t,x) = \frac{\partial^2}{\partial x^2}w(t) - aw(t,x) + D_t^{\alpha-1}f(w(t,x),Kw(t,x)), \quad t \in \mathbb{R}, \quad x \in [0,\pi],$$

$$Kw(t,x) = \int_{-\infty}^{t} k(t-s)h(w(s,x))ds, \tag{4.1}$$

$$w(t,0) = w(t,\pi) = 0, \tag{4.2}$$

where $1 < \alpha < 2$, k is a real valued function satisfying $|k(t)| \leq c_k e^{-bt}$ for $t \geq 0$ and c_k, b are positive constants, K is bounded and $K = \gamma I_d$, $f(w(t,x),Kw(t,x)) = \left(\sin\frac{1}{2+\cos t+\cos\sqrt{2}t} + e^{-(t+m^2)^2}\right)(sin(w(t,x)+\gamma w(t,x)))$ for each $t \in \mathbb{R}$ and $a, \gamma > 0$.

Set $(X, \|\cdot\|_X) = (L^2([0,\pi]), \|\cdot\|_2)$ and define

$$D(A) = \{w \in L^2([0,\pi]) : w^{''} \in L^2([0,\pi]), w(0) = w(\pi) = 0\}$$
$$Aw = \Delta w = w^{''}, \quad \text{for all } w \in D(A).$$

It is well known that A is the infinitisimal generator of an analytic semigroup on $L^2([0,\pi]$. Thus A is of sectorial type $\omega = -a < 0$. Set $\rho(t) = (t+m^2)^2$ for $t \in \mathbb{R}$ then $PAA_0(X,\rho)$ is tranlation invarient. We have

$$\begin{aligned}\|f(t,w(t,\cdot),\gamma w(t,\cdot)) - f(t,w_1(t,\cdot),\gamma w_1(t,\cdot))\|_2 &\leq \|w(t,\cdot)-w_1(t,\cdot)\|_2 + \gamma\\ &\leq \|w(t,\cdot)-w_1(t,\cdot)\|_2 \leq (1+\gamma)\\ &\leq \|w(t,\cdot)-w_1(t,\cdot)\|_2\end{aligned}$$

for all $w(t,\cdot), w_1(t,\cdot) \in L^2([0,\pi]), t \in \mathbb{R}$. Furthermore, one can easily check that $t \to \sin\frac{1}{2+\cos t+\cos\sqrt{2}t} + e^{-(t+m^2)^2}$ belongs to $WPAA(X, e^{(t+m^2)^2})$ with $e^{-(t+m^2)^2}$ as ergodic component and $\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}$ as its almost automorphic component. Consequently, f is weighted pseudo almost automorphic function with weight $\rho(t) = (t+m^2)^2$ for $t \in \mathbb{R}$. Hence choosing γ and a such that

$$(1+\gamma)a^{1/\alpha} < \frac{\alpha\sin(\pi/\alpha)}{C(\theta,\alpha)M}$$

assumption of Theorem 3.1 is satisfied and (4.1)-(4.2) has a unique solution in $WPAA(X,\rho)$.

5. Conclusion

In this paper, existence results for weighted pseudo almost automorphic integrodifferential equation of fractional order with $1 < \alpha < 2$ was proved. These results constitute an extension of the pseudo almost automorphic conditions for some nonlinear integrodifferential equations given by Abbas [1] and neutral fractional differential equations given by [55] to weighted pseudo almost automorphic fractional integrodifferential equations of order $1 < \alpha < 2$. As a possible application of the theoretical results, an example was presented.

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