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EXISTENCE AND NONEXISTENCE RESULTS OF POSITIVE SOLUTION FOR NONLINEAR FRACTIONAL EIGENVALUE PROBLEM

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Abstract. In this work, we consider the following BVP

$$\begin{aligned} &{}^cD_0^\alpha u(t) = \lambda g(t) f(t, u(t)) \ ; \qquad t \in (0,1) \ , \ \alpha \in (2,3) \\ &u(0) + u^{'}(0) = 0 \\ &u(1) + u^{'}(1) = 0 \\ &au^{''}(0) + bu^{''}(1) = 0 \ ; \qquad a > 0 \ , b \leq 0 \ , a + b > 0 \end{aligned}$$

where ${}^cD_0^\alpha$ represents the fractional Caputo derivative of order α and λ is a positive parameter. Using a fixed point theorem for operators on a cone, we obtain sufficient conditions for the existence of positive solution of the above BVP. At the end, example is presented illustrate the main results.

1. Introduction

Recently, fractional differential equations(in short FDEs) have been studied extensively. The motivation for those works stems from both the development of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, economy, biology and so on. For an extensive collection of such results, we refer the readers to the monographs by Podlubny[7], Kilbas et al[5], Ross and Miller[6].
Some basic theory for initial value problems of FDE involving Caputo differential operators has been discussed by many researchers. Also there are some papers about the existence results of positive solution for nonlinear fractional boundary value problems by using techniques of fixed point theorem([1]-[4], [9]-[11]).
For example S.Zhang[8] considered the BVP of the following form:

$$\begin{aligned} &{}^cD_0^\alpha u(t) = f(t, u(t)) \ ; \qquad t \in (0,1) \ , \ \alpha \in (1,2] \\ &u(0) + u^{'}(0) = 0 \\ &u(1) + u^{'}(1) = 0. \end{aligned}$$

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In this paper we consider the following BVP

$$
\begin{aligned}
&{}^cD_0^{\alpha}u(t) = \lambda g(t)f(t,u(t)) \; ; \qquad t\in(0,1)\;,\;\alpha\in(2,3) && (1)\\
&u(0)+u^{'}(0)=0\\
&u(1)+u^{'}(1)=0\\
&au^{''}(0)+bu^{''}(1)=0\;; \qquad a>0\;,b\le 0\;,a+b>0 && (2)
\end{aligned}
$$

where ${}^cD_0^{\alpha}$ represents the fractional Caputo derivative of order α and λ is a positive parameter.
Assume that the following conditions hold:

(H_1) $f\in C((0,1)\times[0,\infty),[0,\infty))$ and f is nonzero, in particular $f(0,u)\neq 0$.
(H_2) $g\in C((0,1),[0,\infty))$and g does not vanish on any sub interval of $[0,1]$ and

$$0<\int_0^1 g(s)ds<\infty.$$

(H_3)

$$f_0=\lim_{u\mapsto 0^+}\frac{f(t,u)}{u}\;,\;f_{\infty}=\lim_{u\to\infty}\frac{f(t,u)}{u},\;t\in(0,1),\;0\le f_0,f_{\infty}\le\infty.$$

2. Applicable Preliminaries

Definition2.1. The fractional Riemann-Liouville integral of order $\alpha>0$ of the continuous function $u:\mathbb{R}^+\to\mathbb{R}$ is given by

$$I_0^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds \tag{3}$$

Definition2.2. The fractional Caputo derivative of order $\alpha>0$ of a continuous function $u:\mathbb{R}^+\to\mathbb{R}$ is defined by

$${}^cD_0^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}u^{(n)}(s)ds \tag{4}$$

where $n=[\alpha]+1$.
Lemma2.3.[7] Let $\alpha>0$ .If $u\in C^n[0,1]$, then

$$I_0^{\alpha\,c}D_0^{\alpha}u(t)=u(t)+c_1+c_2t+..+c_nt^{n-1}.$$

Moreover, fractional differential equation

$${}^cD_0^{\alpha}u(t)=0$$

has the unique solution

$$u(t)=c_1+c_2t+..+c_nt^{n-1}$$

where $n=[\alpha]+1$ and for every $i=1,2,..,n$; $c_i\in\mathbb{R}$ for details see[5].
Lemma2.4. If $y\in C[0,1]$ is given, then the unique solution of BVP

$$
\begin{aligned}
&{}^cD_0^{\alpha}u(t)=y(t)\;; \qquad t\in(0,1)\;,\;\alpha\in(2,3)\\
&u(0)+u^{'}(0)=0\\
&u(1)+u^{'}(1)=0\\
&au^{''}(0)+bu^{''}(1)=0\;; \qquad a>0\;,b\le 0\;,a+b>0
\end{aligned}
$$

is given by

$$u(t)=\int_0^1 G(t,s)y(s)ds \tag{5}$$

where $G(t,s)$ is called Green's function and

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(1-t)(1-s)^{\alpha-1}+(\alpha-1)(1-t)(1-s)^{\alpha-2}\\ +\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(-3+3t-t^2)(1-s)^{\alpha-3}+\\ (t-s)^{\alpha-1}\ ;\ 0\le s\le t\le 1\\ \\ (1-t)(1-s)^{\alpha-1}+(\alpha-1)(1-t)(1-s)^{\alpha-2}\\ +\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(-3+3t-t^2)(1-s)^{\alpha-3}\\ ;\ 0\le t\le s\le 1\end{cases} \tag{6}$$

Proof. Using Lemma 2.3 we have

$$\begin{aligned}{}^cD_0^\alpha u(t)=y(t)&\longrightarrow u(t)=-c_1-c_2t-c_3t^2+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ &\longrightarrow u^{'}(t)=-c_2-2c_3t+\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds\\ &\longrightarrow u^{''}(t)=-2c_3+\int_0^t\frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds\end{aligned} \tag{7}$$

Considering boundary conditions, we obtain

$$\begin{cases}u(0)=-c_1,\quad u^{'}(0)=-c_2,\quad u^{''}(0)=-2c_3\\ \\ u(1)=-c_1-c_2-c_3+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ \\ u^{'}(1)=-c_2-2c_3+\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds\\ \\ u^{''}(1)=-2c_3+\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds\end{cases}$$

Thus we compute $c_1,\ c_2,\ c_3$ as follows

$$\begin{aligned}c_1&=\frac{3b}{2(a+b)}\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ c_2&=\frac{-3b}{2(a+b)}\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds+\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ c_3&=\frac{b}{2(a+b)}\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds\end{aligned}$$

Substituting $c_1,\ c_2,\ c_3$ in (7) we obtain

$$\begin{aligned}
u(t) &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds - \frac{3b}{2(a+b)}\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds \\
&- t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - t\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds + \frac{3bt}{2(a+b)}\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds \\
&- \frac{bt^2}{2(a+b)}\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\
&= \int_0^t \frac{(1-t)(1-s)^{\alpha-1} + (\alpha-1)(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha)}y(s)ds \\
&+ \int_0^t \frac{\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(-3+3t-t^2)(1-s)^{\alpha-3} + (t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\
&+ \int_t^1 \frac{(1-t)(1-s)^{\alpha-1} + (\alpha-1)(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha)}y(s)ds \\
&+ \int_t^1 \frac{\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(-3+3t-t^2)(1-s)^{\alpha-3}}{\Gamma(\alpha)}y(s)ds \\
&= \int_0^1 G(t,s)y(s)ds.
\end{aligned}$$

The proof is complete.

Lemma2.5. The Green's function $G(t,s)$ given by (6) satisfies the following conditions :

- (P_1) For all $t,s \in (0,1)$, $G(t,s) > 0$ and $G(t,s) \in C([0,1]\times[0,1])$.
- (P_2) There exist $\gamma(s) \in C(0,1)$:

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s) \ge \gamma(s) \max_{0\le t\le 1} G(t,s) \tag{8}$$

where for all $s \in (0,1)$

$$M(s) = \frac{(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2} - \frac{3(\alpha-1)(\alpha-2)b}{2(a+b)}(1-s)^{\alpha-3}}{\Gamma(\alpha)}, \tag{9}$$

$$m(s) = \frac{8(1-s)^{\alpha-1} + 8(\alpha-1)(1-s)^{\alpha-2} - 21\frac{(\alpha-1)(\alpha-2)b}{a+b}(1-s)^{\alpha-3}}{32\Gamma(\alpha)} \tag{10}$$

and $\gamma(s) = \frac{m(s)}{M(s)}$.

Proof. (P_1) is clear by definition of $G(t,s)$. Note that for all $s \in (0,1)$ $G(t,s)$ is decreasing for $t \le s$. Now let

$$\begin{aligned}
g_1(t,s) &= \frac{(1-t)(1-s)^{\alpha-1} + (\alpha-1)(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha)} \\
&+ \frac{\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(-3+3t-t^2)(1-s)^{\alpha-3} + (t-s)^{\alpha-1}}{\Gamma(\alpha)}\ ; \qquad s \le t
\end{aligned}$$

,

$$g_2(t,s)=\frac{(1-t)(1-s)^{\alpha-1}+(\alpha-1)(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha)}$$
$$+\frac{\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(-3+3t-t^2)(1-s)^{\alpha-3}}{\Gamma(\alpha)}\ ; \qquad t\le s$$

where $g_1(t,s)$ for $1/4\le t\le 3/4$ and $g_2(t,s)$ with respect to t are decreasing and continuous. Thus we have

$$\min_{\frac14\le t\le\frac34} g_1(t,s)\ge\frac{8(1-s)^{\alpha-1}+8(\alpha-1)(1-s)^{\alpha-2}-21\frac{(\alpha-1)(\alpha-2)b}{a+b}(1-s)^{\alpha-3}}{32\Gamma(\alpha)},$$
$$\max_{0\le t\le 1} g_1(t,s)\le\frac{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}-3\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(1-s)^{\alpha-3}}{\Gamma(\alpha)}.$$

Also we have

$$\min_{\frac14\le t\le\frac34} g_2(t,s)\ge\frac{8(1-s)^{\alpha-1}+8(\alpha-1)(1-s)^{\alpha-2}-21\frac{(\alpha-1)(\alpha-2)b}{a+b}(1-s)^{\alpha-3}}{32\Gamma(\alpha)},$$
$$\max_{0\le t\le 1} g_2(t,s)\le\frac{(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}-3\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(1-s)^{\alpha-3}}{\Gamma(\alpha)}$$
$$\le\frac{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}-3\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(1-s)^{\alpha-3}}{\Gamma(\alpha)}.$$

So there for we conclude that for $s\in[0,1)$

$$\min_{\frac14\le t\le\frac34} G(t,s)\ge m(s)=\frac{8(1-s)^{\alpha-1}+8(\alpha-1)(1-s)^{\alpha-2}-21\frac{(\alpha-1)(\alpha-2)b}{a+b}(1-s)^{\alpha-3}}{32\Gamma(\alpha)} \tag{11}$$

,

$$\max_{0\le t\le 1} G(t,s)\le M(s)=\frac{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}-3\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(1-s)^{\alpha-3}}{\Gamma(\alpha)}. \tag{12}$$

Now for $s\in(0,1)$ we set

$$\gamma(s)=\frac{1}{32}\frac{8(1-s)^{\alpha-1}+8(\alpha-1)(1-s)^{\alpha-2}-21\frac{(\alpha-1)(\alpha-2)b}{a+b}(1-s)^{\alpha-3}}{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}-3\frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(1-s)^{\alpha-3}} \tag{13}$$

hence $\gamma\in C((0,1),(0,+\infty))$. This complete the proof.

Remark2.6. We consider the Banach space $B=C[0,1]$ such that equipped with the norm

$$\| u \|=\max_{0\le t\le 1}\mid u(t)\mid$$

also we define the cone $P\subset B$ with the following form

$$P=\left\{u\in B\ \middle|\ u(t)\ge 0\right\}.$$

Finally, we define the integral Hammerstein operator as below

$$T:P\longrightarrow B\ ;\ Tu(t)=\lambda\int_0^1 G(t,s)g(t)f(s,u(s))ds. \tag{14}$$

Obviously from definition (14) we conclude that $TP \subset P$.
Lemma2.7. Assume that the conditions $(H_1),(H_2)$ hold. Then the operator $T : P \longrightarrow P$ defined by (14) is a completely continuous operator.
Proof. By conditions $(H_1),(H_2)$ and Lemma 2.5, clearly T, is a continuous operator. Now let $\Omega \subset P$ is bounded. Thus

$$\exists M \in \mathbb{R}^+,\ \forall u \in \Omega\ ; \qquad \| u \| \leq M.$$

Let

$$L_1 = \max_{t\in[0,1],u\in[0,M]} \mid f(t,u(t)) \mid +1\ ,\ \forall t \in [0,1]\ ;\ a(t) \leq L_2\ ,\ L = L_1L_2 + 1,$$

hence for every $u \in \Omega$,we have

$$\mid Tu(t) \mid = \lambda \int_0^1 G(t,s)g(s)f(s,u(s))ds \leq \lambda L \int_0^1 G(s,s)ds < +\infty.$$

So $T\Omega$ is bounded.
At last we prove that operator T,is equicontinuous. Let $u \in \Omega$ and $t_2 > t_1$ for every $s \in (0,1),\ t_1,t_2 \in [0,s]$. Thus indeed

$$\begin{aligned} \mid Tu(t_2) - Tu(t_1) \mid &\leq \lambda L \int_0^1 \mid G(t_2,s) - G(t_1,s) \mid ds \\ &\leq \lambda L \int_0^{t_1} \mid G(t_2,s) - G(t_1,s) \mid ds \\ &\quad + \lambda L \int_{t_1}^{t_2} \mid G(t_2,s) - G(t_1,s) \mid ds \\ &\quad + \lambda L \int_{t_2}^{1} \mid G(t_2,s) - G(t_1,s) \mid ds \end{aligned}$$

$$\begin{aligned} &= \int_0^{t_1} \frac{(t_2-t_1)\left[(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2} + \frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(3-(t_1+t_2))(1-s)^{\alpha-3}\right]}{\Gamma(\alpha)}ds \\ &+ \int_0^{t_1} \frac{\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]}{\Gamma(\alpha)}ds \\ &+ \int_{t_1}^{t_2} \frac{2(t_2-t_1)\left[(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2} + \frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(3-(t_1+t_2))(1-s)^{\alpha-3}\right]}{\Gamma(\alpha)}ds \\ &+ \int_{t_1}^{t_2} \frac{\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]}{\Gamma(\alpha)}ds \\ &+ \int_{t_2}^{1} \frac{(t_2-t_1)\left[(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2} + \frac{(\alpha-1)(\alpha-2)b}{2(a+b)}(3-(t_1+t_2))(1-s)^{\alpha-3}\right]}{\Gamma(\alpha)}ds \end{aligned}$$

Thus, when $t_1 \to t_2$ we conclude that

$$\mid Tu(t_2) - Tu(t_1) \mid \longrightarrow 0.$$

Hence $T\Omega$ is equicontinuous on [0,1]. So using $Arzela - Ascoli$ theorem we attain that, operator $T : P \longrightarrow P$ is completely continuous. The proof is complete.
Theorem2.8.[4] Let X be a real Banach space and $P \subset X$ be a cone in X .

Assume Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1$,$\overline{\Omega}_1 \subset \Omega_2$ and $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow P$ be a completely continuous operator such that

(i) $\| Tu \| \leq \| u \|,\ u \in P \cap \partial\Omega_1$ and $\| Tu \| \geq \| u \|,\ u \in P \cap \partial\Omega_2$, or

(ii) $\| Tu \| \leq \| u \|,\ u \in P \cap \partial\Omega_2$ and $\| Tu \| \geq \| u \|,\ u \in P \cap \partial\Omega_1$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main Results

The following theorems relies on Theorem 2.8 which has two possibilities (i) and (ii). We have to prove any case may occur. That is why we stated the results in two different Theorems 3.1 and 3.2, separately.

Theorem3.1. Let conditions $(H_1), (H_2), (H_3)$ are satisfied. Then for every λ satisfying

$$\frac{1}{\int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds f_\infty} < \lambda < \frac{1}{\int_0^1 M(s)g(s)ds f_0} \tag{15}$$

boundary value problem (1),(2) has at least one positive solution in P.

Proof. Let λ be given as in (15) . Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{\int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds(f_\infty - \epsilon)} < \lambda < \frac{1}{\int_0^1 M(s)g(s)ds(f_0 + \epsilon)} \tag{16}$$

By Lemmas 2.4 and 2.8, we know that $T : P \longrightarrow P$ is completely continuous and boundary value problem (1),(2) has a solution u if and only if u solves the operator equation $u = Tu$.

Now turning to f_0,there exist $r_1 > 0$ such that $f(t, u) \leq (f_0 + \epsilon)u$ for every $0 < u \leq r_1$.

Let $c_1 = r_1,\ \Omega_1 = \{u \in P| \parallel u \parallel < c_1\}$. For $u \in \partial\Omega_1$, we have $0 \leq u(t) \leq c_1$ for $t \in [0, 1]$. It follows from Lemma 2.5 that for $t \in [0, 1]$:

$$\begin{aligned}
\| Tu \| &= \max_{0\leq t\leq 1} \lambda \int_0^1 G(t, s)g(s)f(s, u(s))ds \\
&\leq \lambda \int_0^1 M(s)g(s)(f_0 + \epsilon)u(s)ds \\
&\leq \lambda \int_0^1 M(s)g(s)ds(f_0 + \epsilon)c_1 \leq c_1 = \| u \| .
\end{aligned}$$

Hence,

$$\forall u \in P \cap \partial\Omega_1 \ ; \qquad \| Tu \| \leq \| u \| . \tag{17}$$

Next, considering f_∞,there exist $r_2 > 0$ such that $f(t, u) > (f_\infty - \epsilon)u$, for all $u \geq r_2$.

Let $c_2 = max\{1 + c_1, r_2\}$, and $\Omega_2 = \{u \in P| \parallel u \parallel \leq c_2\}$. For every $u \in \partial\Omega_2$,and for every $t \in [0, 1]$, we have $0 \leq u(t) \leq c_2$.

If $u \in P$ with $u(t) \geq c_2$ and from Lemma 2.5 and

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)g(s)f(s,u(s))ds \\
&\geq \lambda \int_{1/4}^{3/4} G(t,s)g(s)f(s,u(s))ds \\
&\geq \lambda \int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds(f_\infty - \epsilon)c_2 \\
&\geq c_2,
\end{aligned}$$

thus,$\| Tu \| \geq c_2$. Hence,

$$\forall u \in P \cap \partial\Omega_2 \ ; \qquad \| Tu \| \geq \| u \| \ . \tag{18}$$

Applying (17)and (18)and first part of Theorem 2.8 we conclude that boundary value problem (1),(2) has at least one positive solution in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This complete the proof.

Theorem3.2. Assume that conditions $(H_1), (H_2), (H_3)$ are satisfied. Then, there for each λ satisfying

$$\frac{1}{\displaystyle\int_{1/4}^{3/4} \gamma(s)M(s)g(s)dsf_0} < \lambda < \frac{1}{\displaystyle\int_0^1 M(s)g(s)dsf_\infty} \tag{19}$$

there exist at least one positive solution of boundary value problem (1),(1) in P.

Proof. Let λ be given as in (19). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{\displaystyle\int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds(f_0 - \epsilon)} < \lambda < \frac{1}{\displaystyle\int_0^1 M(s)g(s)ds(f_\infty + \epsilon)}. \tag{20}$$

By Lemmas 2.4 and 2.8, we know that $T : P \longrightarrow P$ is completely continuous and boundary value problem (1),(2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$.

Beginning with f_0, there exist $r_1 > 0$ such that $f(t,u) > (f_0 - \epsilon)u$, for every $0 < u \leq r_1$.

Let $c_1 = r_1$, $\Omega_1 = \{u \in P | \| u \| < c_1\}$. For $u \in P \cap \partial\Omega_1$, we have $0 \leq u(t) \leq c_1$ for all $t \in [0,1]$.

If $u \in P$, $u(t) \geq c_1$, from Lemma 2.5 we have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)g(s)f(s,u(s))ds \\
&\geq \lambda \int_{1/4}^{3/4} G(t,s)g(s)f(s,u(s))ds \\
&\geq \lambda \int_{1/4}^{3/4} \gamma(s)M(s)g(s)(f_0 - \epsilon)u(s)ds \\
&\geq \lambda \int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds(f_0 - \epsilon)c_1 \\
&\geq c_1.
\end{aligned}$$

Thus,$\| Tu \| \geq c_1$. Hence

$$\forall u \in P \cap \partial\Omega_1 \; ; \qquad \| Tu \| \geq \| u \| \; . \tag{21}$$

It remain to consider f_∞. There exist $r_2 > 0$ such that $f(t,u) < (f_\infty + \epsilon)u$, for all $u \geq r_2$.
There are the two cases ;

(a) f is bounded. In this case suppose $N > 0$ is such that $f(t,u) \leq N$, for all $0 < u < \infty$.
Let $c_2 = max\{1 + c_1, N\lambda \int_0^1 g_1(0,s)g(s)ds\}$.
Then, for $u \in P$ with $\| u \| = c_2$, by Lemma 2.5 we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)g(s)f(s,u(s))ds \\ &\leq N\lambda \int_0^1 M(s)g(s)ds \\ &\leq c_2 = \| u \|, \end{aligned}$$

so there for $\| Tu \| \leq \| u \|$. So, if $\Omega_2 = \{u \in P | \; \| u \| < c_2\}$, then

$$\forall u \in P \cap \partial\Omega_2 \; ; \qquad \| Tu \| \leq \| u \| \; . \tag{22}$$

(b) f is unbounded. In this case , let $c_2 = max\{1 + c_1, r_2\}$, be such that for $0 < u \leq c_2, \; f(t,u) \leq f(t,c_2)$.
Now choosing $u \in P$ with $\| u \| = c_2$, and from Lemma 2.5, we have

$$\begin{aligned} Tu(t) &\leq \lambda \int_0^1 M(s)g(s)f(s,u(s))ds \\ &\leq \lambda \int_0^1 M(s)g(s)ds(f_\infty + \epsilon)c_2 \\ &\leq c_2 = \| u \|, \end{aligned}$$

so $\| Tu \| \leq \| u \|$. For this case if we let

$$\Omega_2 = \{u \in P | \; \| u \| < c_2\}$$

then

$$\forall u \in P \cap \partial\Omega_2 \; ; \qquad \| Tu \| \leq \| u \| \; . \tag{23}$$

Applying (21)-(23) and second part of Theorem 2.8 we conclude that boundary value problem (1),(2) has at least one positive solution in P .The proof is complete.
Example3.3. Consider the boundary value problem

$$\begin{aligned} &{}^cD_0^{\frac{5}{2}} u(t) = \lambda g(t) f(t,u(t)) \; ; \; t \in (0,1) \qquad (24) \\ &u(0) + u^{'}(0) = 0 \\ &u(1) + u^{'}(1) = 0 \\ &\frac{3}{2} u^{''}(0) - u^{''}(1) = 0 \qquad (25) \end{aligned}$$

where

$$\lambda = g(t) = 1,$$

$$f(t,u(t)) = \begin{cases} 1 + 14u^2 & ;\ 0 \le u \le 1 \\ 14 + u & ;\ u > 1 \end{cases}$$

A direct computation showed

$$\int_0^1 M(s)g(s)ds = 4.739 \quad , \quad \int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds = 1.004 \ ,$$

$$f_0 = \infty \quad , \quad f_\infty = 1.$$

So for each $0 < \lambda < 0.211$, according to Theorem 3.2, boundary value problem (24),(25) has at least one positive solution u in P .

Theorem3.4. Let conditions $(H_1) - (H_3)$ hold. If $f_0, f_\infty < \infty$, then there exist a positive constant λ_0, such that for every $0 < \lambda < \lambda_0$, the boundary value problem (1),(2) has no positive solution.

Proof. Since $f^0, f^\infty < \infty$,thus

$$\begin{aligned} &\exists c_1, c_2, r_1, r_2 > 0 \ :\ r_1 < r_2,\ t \in [0,1]\ ; \\ &f(t,u) < c_1 u\ ;\ u \in [0, r_1] \\ &f(t,u) < c_2 u\ ;\ u \in [r_2, +\infty). \end{aligned}$$

Let

$$C = max\left\{c_1, c_2, \sup_{r_1 \le u \le r_2} \frac{f(t,u)}{u}\right\}$$

Thus we have

$$f(t,u) \le Cu\ ;\ u \in [0,+\infty), t \in [0,1].$$

Assume $w(t)$ is a positive solution of the boundary value problem (1),(2). We will show that this leads to a contradiction for every $0 < \lambda < \lambda_0$ with

$$\lambda_0 = \frac{A}{C}, \qquad A = \left(\lambda \int_0^1 M(s)g(s)ds\right)^{-1}.$$

In this case we have

$$\begin{aligned} w(t) = Tw(t) &= \lambda \int_0^1 G(t,s)g(s)f(s,w(s))ds \\ &\le \lambda C \parallel w \parallel \int_0^1 G(t,s)g(s)ds. \end{aligned}$$

Thus

$$\begin{aligned}\| w \| &\leq \lambda C \| w \| \int_0^1 \sup_{t\in[0,1]} G(t,s)g(s)ds \\ &= \frac{\lambda C}{A} \| w \| \\ &< \| w \|,\end{aligned}$$

which is a contradiction. So therefore the boundary value problem (1),(2) has no positive solution.The proof is complete.

Theorem3.5. Let conditions $(H_1) - (H_3)$ hold. If $f_0, f_\infty > 0$, then there exist a positive constant λ_0, such that for every $\lambda > \lambda_0$, the boundary value problem (1),(2) has no positive solution.

Proof. Since $f_0, f_\infty > 0$, thus we conclude that

$$\begin{aligned}&\exists m_1, m_2, r_1, r_2 > 0 \ ; \ r_1 < r_2, \ t \in [1/4, 3/4] \ ; \\ &\qquad f(t,u) \geq m_1 u \ ; \ u \in [0, r_1] \\ &\qquad f(t,u) \geq m_2 u \ ; \ u \in [r_2, +\infty).\end{aligned}$$

Assume that

$$m = min\left\{ m_1, m_2, \min_{r_1 \leq u \leq r_2} \frac{f(t,u)}{u} \right\}.$$

Hence we have

$$f(t,u) \geq mu \ , u \in [0, +\infty), t \in [1/4, 3/4].$$

Let $w(t)$ is a positive solution of the boundary value problem (1),(2). We will show that this leads to a contradiction for every

$$\lambda > \lambda_0, \quad \lambda_0 = \frac{B}{m}, \qquad B = \left(\lambda \int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds \right)^{-1}.$$

So we have

$$\begin{aligned}w(t) = Tw(t) &= \lambda \int_0^1 G(t,s)g(s)f(s,w(s))ds \\ &\geq m\lambda w \int_0^1 G(t,s)g(s)ds.\end{aligned}$$

Hence

$$\begin{aligned}\| w \| &\geq m\lambda \| w \| \int_{1/4}^{3/4} \gamma(s)M(s)g(s)ds \\ &= \frac{\lambda m}{B} \| w \| \\ &> \| w \|,\end{aligned}$$

which is a contradiction. So therefore the boundary value problem (1),(2)has no positive solution. This complete the proof.

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