

ON EXISTENCE OF SOLUTION TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS FOR $0 < \alpha \leq 3$

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ABSTRACT. We investigate in this article the existence problem of a fractional nonlinear differential system with $\alpha \in (0, 3]$. We obtain the results by using Banach fixed-point theorem.

1. INTRODUCTION

In recent years, considerable interest in fractional calculus has been stimulated by the applications to numerical analysis and different areas of applied sciences like physics and engineering. One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot do. The fact that fractional differential equations are considered as alternative models to nonlinear differential equations which induced extensive researches in various fields including the theoretical part. The existence and uniqueness problems of fractional nonlinear differential equations as a basic theoretical part are investigated by many authors (see [1]-[10] and references therein). The Cauchy problems for some fractional abstract differential equations (in the case of $0 < \alpha \leq 1$) with nonlocal conditions are investigated by the authors in [2], [5] and [8] using the Banach and Krasnoselkii fixed point theorems. The Banach fixed point theorem is used in [4] to investigate the existence problem of fractional integrodifferential equations (in the case of $0 < \alpha \leq 1$) on Banach spaces. In [1] and [6], the authors obtained sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential equations (in the case of $1 < \alpha \leq 2$) involving the Caputo fractional derivative and nonlocal conditions using the Banach, Schaefer's, and Krasnoselkii fixed points theorems. In [10], the authors investigated the existence problem of a boundary value problem to fractional differential equation (in the case $2 < \alpha \leq 3$) by using Banach, and Schaefer's fixed points. Motivated by these works we study in this paper the existence of solution to fractional differential equations when $0 < \alpha \leq 3$ on Banach spaces which may be considered as a generalized survey to many articles. The problem is solved by using the contraction mapping theorem.

2000 *Mathematics Subject Classification.* 34A12, 34G20, 26A33.

Key words and phrases. Fractional-order differential equations, Banach fixed point, nonlocal condition, existence problem.

Submitted April 26, 2012. Published July 20, 2012.

2. PRELIMINARIES

We need some basic definitions and properties of fractional calculus (see [3], [7], and [9] which will be used in this paper.

Definition 1 A real function $f(t)$ is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2 A function $f \in C_\mu$, $\mu \geq -1$ is said to be fractional integrable of order $\alpha > 0$ if

$$(I^\alpha f)(t) = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds < \infty,$$

and if $\alpha_n = 0$, then $I^0 f(t) = f(t)$.

Next, we introduce the (Caputo) fractional derivative.

Definition 3 The fractional derivative in the Caputo sense is defined as

$$(D^\alpha f)(t) = D^\alpha f(t) = I^{n-\alpha} \left(\frac{d^n f}{dt^n} \right) (t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$.

Lemma 1 [1] If $n-1 < \alpha \leq n$, then

$$I^\alpha D^\alpha f(t) = f(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1},$$

where $t \in J = [0, T]$, c_0, c_1, \dots , and c_{n-1} are constants.

Let $Y = C(J, X)$ be a Banach space of all continuous functions $x(t)$ from a compact interval J into a Banach space X . Without loss of generality, we use the common norm $\|\bullet\|$ for all used normed spaces.

Consider the fractional nonlinear differential system

$$\begin{cases} D^{\alpha_n} x_n(t) = f_n(t, x_n(t)), \\ x_n(0) = z_n \in Y, x_2'(T) = x_3'(T) = x_3''(T) = 0, \end{cases} \quad (1)$$

where $n-1 < \alpha_n \leq n$; $n = 1, 2, 3$, $f_n : J \times Y \rightarrow Y$ is fractional integrable function of order α_n and satisfies the following hypothesis;

(H1) There exists a positive constant A_n such that

$$\|f_n(t, x_n) - f_n(t, y_n)\| \leq A_n \|x_n - y_n\|, n = 1, 2, 3,$$

for any $t \in J$, $x_n, y_n \in Y$. Moreover, let $B_n = \sup_{t \in J} \|f_n(t, 0)\|$.

Definition 4 A function $x_n \in C^n(J, X)$ with its α_n -derivative exists on J , is said to be a solution of (1) if x_n satisfies the equation $D^{\alpha_n} x_n(t) = f_n(t, x_n(t))$ on J , and conditions $x_n(0) = z_n \in Y$, $x_2'(T) = x_3'(T) = x_3''(T) = 0$.

Lemma 2 The fractional differential system

$$\begin{cases} D^{\alpha_n} x_n(t) = f_n(t), \\ x_n(0) = z_n \in Y, x_2'(T) = x_3'(T) = x_3''(T) = 0, \end{cases} \quad (2)$$

is equivalent to

$$x_n(t) = z_n + \int_0^t G_n(t, s) f_n(s) ds, t \in J, n-1 < \alpha_n \leq n, n = 1, 2, 3 \quad (3)$$

where

$$\begin{aligned}
 G_1(t, s) &= \begin{cases} \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & \text{for } 0 \leq s \leq t \\ 0, & \text{for } t \leq s < T \end{cases} \\
 G_2(t, s) &= \begin{cases} \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} - \frac{t(T-s)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}, & \text{for } 0 \leq s \leq t \\ -\frac{t(T-s)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}, & \text{for } t \leq s < T \end{cases} \\
 G_3(t, s) &= \begin{cases} \frac{(t-s)^{\alpha_3-1}}{\Gamma(\alpha_3)} - \frac{t(T-s)^{\alpha_3-2}}{\Gamma(\alpha_3-1)} + \frac{t(T-\frac{t}{2})(T-s)^{\alpha_3-3}}{\Gamma(\alpha_3-2)}, & \text{for } 0 \leq s \leq t \\ \frac{t(T-\frac{t}{2})(T-s)^{\alpha_3-3}}{\Gamma(\alpha_3-2)} - \frac{t(T-s)^{\alpha_3-2}}{\Gamma(\alpha_3-1)}, & \text{for } t \leq s < T \end{cases}
 \end{aligned}$$

Proof. The case $0 < \alpha_1 \leq 1$ is trivial. Let $1 < \alpha_2 \leq 2$, then by Lemma 1 and conditions in (2), we have

$$I^{\alpha_2} f_2(t) = I^{\alpha_2} D^{\alpha_2} x_2(t) = x_2(t) + c_0 + c_1 t.$$

For $t = 0$, we get $0 = z_2 + c_0$ which implies that $c_0 = -z_2$. On the other hand, for $t = T$, we get $I^{\alpha_2-1} f_2(T) = x_2'(T) + c_1$ which implies that

$$c_1 = I^{\alpha_2-1} f_2(T) = \frac{1}{\Gamma(\alpha_2 - 1)} \int_0^T (T - s)^{\alpha_2-2} f_2(s) ds.$$

Hence

$$x_2(t) = z_2 - \frac{t}{\Gamma(\alpha_2 - 1)} \int_0^T (T - s)^{\alpha_2-2} f_2(s) ds + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t - s)^{\alpha_2-1} f_2(s) ds.$$

Finally, let $2 < \alpha_3 \leq 3$, then again by Lemma 1 and conditions in (2), we have

$$I^{\alpha_3} f_3(t) = I^{\alpha_3} D^{\alpha_3} x_3(t) = x_3(t) + c_0 + c_1 t + c_2 t^2.$$

Hence, for $t = 0$, we get $0 = z_3 + c_0$ which implies that $c_0 = -z_3$. For $t = T$, we get $I^{\alpha_3-1} f_3(T) = x_3'(T) + c_1 + 2c_2 T$, and $I^{\alpha_3-2} f_3(T) = x_3''(T) + 2c_2$, which imply that $c_2 = \frac{1}{2} I^{\alpha_3-2} f_3(T)$, and $c_1 = I^{\alpha_3-1} f_3(T) - T I^{\alpha_3-2} f_3(T)$. Therefore

$$\begin{aligned}
 x_3(t) &= z_3 + \frac{t}{\Gamma(\alpha_3 - 2)} \left(T - \frac{t}{2} \right) \int_0^T (T - s)^{\alpha_3-3} f_3(s) ds \\
 &\quad - \frac{t}{\Gamma(\alpha_3 - 1)} \int_0^T (T - s)^{\alpha_3-2} f_3(s) ds + \frac{1}{\Gamma(\alpha_3)} \int_0^t (t - s)^{\alpha_3-1} f_3(s) ds.
 \end{aligned}$$

Conversely, if we apply the fractional differential operator D^{α_n} , $n - 1 < \alpha_n \leq n$, to the integral equations in (3), one can easily get the system (2).

Remark. We notice that $\lim_{\alpha_n \rightarrow n^-} G_n = \lim_{\alpha_{n+1} \rightarrow n^+} G_{n+1}$ for $n = 1, 2$. Moreover, $\lim_{s \rightarrow T^-} G_n(t, s) = 0$, for $n = 1, 2, 3$, and $t \leq s$.

Now, using Lemma 2, the corresponding integral form to nonlinear system (1) can be written in the form

$$x_n(t) = z_n + \int_0^T G_n(t, s) f_n(s, x_n(s)) ds, t \in J, n-1 < \alpha_n \leq n, n = 1, 2, 3. \quad (4)$$

3. EXISTENCE OF THE SOLUTION

The existence problem to the given fractional nonlinear differential system is investigated in this section by using the well-known Banach fixed point theorem. The first result is the existence of solution for the system (1). The following condition is essential to get the contraction property.

(H2) Let, for $n = 1, 2, 3$,

$$\begin{cases} C_n = \max\{A_n, B_n\} T^{\alpha_n} \left(\frac{1}{\Gamma(\alpha_n+1)} + \frac{n(n-1)}{n! \Gamma(\alpha_n)} + \frac{1}{2} \left| \frac{(n-2)(n-1)}{\Gamma(\alpha_n-1)} \right| \right) < 1 \\ r_n \geq \frac{\|z_n\| + C_n}{1 - C_n} \end{cases}$$

and $n-1 < \alpha_n \leq n$. Moreover, let $B_{r_n} = \{x \in Y : \|x\| \leq r_n\}$.

Theorem 1 If the hypotheses (H1), and (H2) are satisfied, then the fractional differential system (1) has a solution on J .

Proof. We prove, by using the Banach fixed point, the operator $\Lambda_n : Y \rightarrow Y, n = 1, 2, 3$, given by (see (4))

$$\Lambda_n x_n(t) = z_n + \int_0^T G_n(t, s) f_n(s, x_n(s)) ds, t \in J$$

has a fixed point on $B_{r_n}; n = 1, 2, 3$. This fixed point is then a solution of the system (1). Firstly, we show that $\Lambda_n B_{r_n} \subset B_{r_n}$. Let $n = 1$, and $0 < \alpha_1 \leq 1$, then

$$\begin{aligned} \|\Lambda_1 x_1(t)\| &\leq \|z_1\| + \int_0^T \|G_1(t, s)\| \|f_1(s, x_1(s)) - f_1(s, 0) + f_1(s, 0)\| ds \\ &\leq \|z_1\| + \frac{A_1 \|x_1\|}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} ds + \frac{B_1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} ds \\ &\leq \|z_1\| + (A_1 \|x_1\| + B_1) \frac{t^{\alpha_1}}{\Gamma(\alpha_1 + 1)}. \end{aligned}$$

Next, for $n = 2$, and $1 < \alpha_2 \leq 2$, we have

$$\|\Lambda_2 x_2(t)\| \leq \|z_2\| + (A_2 \|x_2\| + B_2) \left(\frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \frac{t T^{\alpha_2-1}}{\Gamma(\alpha_2)} \right).$$

The last case, $n = 3$, and $2 < \alpha_3 \leq 3$, we have

$$\|\Lambda_3 x_3(t)\| \leq \|z_3\| + (A_3 \|x_3\| + B_3) \left(\frac{t^{\alpha_3}}{\Gamma(\alpha_3 + 1)} + \frac{t T^{\alpha_3-1}}{\Gamma(\alpha_3)} + \frac{t(T - \frac{t}{2}) T^{\alpha_3-2}}{\Gamma(\alpha_3 - 1)} \right).$$

The three cases for $n = 1, 2, 3$ can be written as

$$\begin{aligned} & \|\Lambda_n x_n(t)\| \\ \leq & \|z_n\| \\ & + T^{\alpha_n} \left(\frac{1}{\Gamma(\alpha_n + 1)} + \frac{n(n-1)}{n! \Gamma(\alpha_n)} + \frac{1}{2} \left| \frac{(n-2)(n-1)}{\Gamma(\alpha_n - 1)} \right| \right) (A_n \|x_n\| + B_n), \end{aligned}$$

where $n-1 < \alpha_n \leq n$. Therefore, if $x_n \in B_{r_n}$, we get $\|\Lambda_n x_n(t)\| \leq (1 - C_n)r_n + C_n r_n = r_n$. Hence, the operator Λ_n maps B_{r_n} into itself. Next, we prove that Λ_n is a contraction mapping on B_{r_n} . Let $x_n, y_n \in B_{r_n}$, then for $n = 1, 2, 3$, we have

$$\begin{aligned} & \|\Lambda_n x_n(t) - \Lambda_n y_n(t)\| \\ \leq & \int_0^T |G_n(t, s)| \|f_n(s, x_n(s)) - f_n(s, y_n(s))\| ds \\ \leq & A_n T^{\alpha_n} \left(\frac{1}{\Gamma(\alpha_n + 1)} + \frac{n(n-1)}{n! \Gamma(\alpha_n)} + \frac{1}{2} \left| \frac{(n-2)(n-1)}{\Gamma(\alpha_n - 1)} \right| \right) \|x_n - y_n\| \\ \leq & C_n \|x_n - y_n\|, \end{aligned}$$

where $n-1 < \alpha_n \leq n$. Hence, the operator Λ_n has a unique fixed point x_n which is a solution to the system (1) for each $n-1 < \alpha_n \leq n, n = 1, 2, 3$.

Next result in this section is the existence of solution to the system (1) in the vector form.

Let $x = (x_1, x_2, x_3) \in Z = Y \times Y \times Y$ such that $\|x\| = \|x_1\| + \|x_2\| + \|x_3\|$. Then $(Z, \|\bullet\|)$ becomes a complete normed space. Also, assume that $D^\alpha x = (D^{\alpha_1} x_1, D^{\alpha_2} x_2, D^{\alpha_3} x_3); n-1 < \alpha_n \leq n, n = 1, 2, 3$, and $f(t, x) = (f_1(t, x_1), f_2(t, x_2), f_3(t, x_3))$, $G(t, s) = (G_1(t, s), G_2(t, s), G_3(t, s))$. Moreover,

we shall use the following logical notation

$$\begin{aligned} & \int_0^T G(t, s) * f(s, x(s)) ds \\ = & \left(\int_0^T G_1(t, s) f_1(s, x_1(s)) ds, \int_0^T G_2(t, s) f_2(s, x_2(s)) ds, \int_0^T G_3(t, s) f_3(s, x_3(s)) ds \right). \end{aligned}$$

Therefore, the system (1) can be rewritten in the form

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), t \in J, x \in Z \\ z(0) = (x_1(0), x_2(0), x_3(0)) = z_0 \in Z, x'_2(T) = x'_3(T) = x''_3(T) = 0 \end{cases} \quad (5)$$

which is equivalent to (see eq.(4))

$$x(t) = z_0 + \int_0^T G(t, s) * f(s, x(s)) ds. \quad (6)$$

(H3) Let D, C , and r be positive constants such that

$$\begin{cases} D = \max_{1 \leq n \leq 3} \{A_n, B_n\} \\ C = D \left(\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{T^{\alpha_2}}{\Gamma(\alpha_2)} + \frac{T^{\alpha_3}}{\Gamma(\alpha_3+1)} + \frac{T^{\alpha_3}}{\Gamma(\alpha_3)} + \frac{T^{\alpha_3}}{\Gamma(\alpha_3-1)} \right) < 1 \\ r \geq \frac{\|z_0\|+C}{1-C} \end{cases}$$

Moreover, let $B_r = \{z \in Z : \|x\| \leq r\}$.

Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in Z$, in view of (H1), it easy to get

$$\begin{aligned} \|f(t, x) - f(t, y)\| &= \|(f_1(t, x_1) - f_1(t, y_1), f_2(t, x_2) - f_2(t, y_2), f_3(t, x_3) - f_3(t, y_3))\| \\ &\leq A_1\|x_1 - y_1\| + A_2\|x_2 - y_2\| + A_3\|x_3 - y_3\| \leq D\|x - y\|, \end{aligned}$$

Theorem 2 If the hypotheses (H1), and (H3) are satisfied, then the fractional differential system (5) has a solution on J .

Proof. Let $x = (x_1, x_2, x_3) \in Z$. Define the operator Λ on Z given by

$$\Lambda x(t) = (\Lambda_1 x_1(t), \Lambda_2 x_2(t), \Lambda_3 x_3(t)) = z_0 + \int_0^T G(t, s) * f(s, x(s)) ds,$$

where $\|\Lambda x\| = \|\Lambda_1 x_1\| + \|\Lambda_2 x_2\| + \|\Lambda_3 x_3\|$. Hence, following the proof of Theorem 1, we have

$$\begin{aligned} \|\Lambda x(t)\| &\leq \|z_0\| + D(\|x_1\| + 1) \frac{T^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + DT^{\alpha_2}(\|x_2\| + 1) \left(\frac{1}{\Gamma(\alpha_2 + 1)} + \frac{1}{\Gamma(\alpha_2)} \right) \\ &\quad + DT^{\alpha_3}(\|x_3\| + 1) \left(\frac{1}{\Gamma(\alpha_3 + 1)} + \frac{1}{\Gamma(\alpha_3)} + \frac{1}{\Gamma(\alpha_3 - 1)} \right) \\ &\leq \|z_0\| + C\|x\| + C. \end{aligned}$$

Therefore, if $x \in B_r$, so does Λx . On the other hand, if $x, y \in B_r$, then

$$\begin{aligned} \|\Lambda x(t) - \Lambda y(t)\| &= \left\| \int_0^T G_1(t, s) (f_1(s, x_1(s)) - f_1(s, y_1(s))) ds \right\| \\ &\quad + \left\| \int_0^T G_2(t, s) (f_2(s, x_2(s)) - f_2(s, y_2(s))) ds \right\| \\ &\quad + \left\| \int_0^T G_3(t, s) (f_3(s, x_3(s)) - f_3(s, y_3(s))) ds \right\| \\ &\leq \frac{DT^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \|x_1 - y_1\| + DT^{\alpha_2} \left(\frac{1}{\Gamma(\alpha_2 + 1)} + \frac{1}{\Gamma(\alpha_2)} \right) \|x_2 - y_2\| \\ &\quad + DT^{\alpha_3} \left(\frac{1}{\Gamma(\alpha_3 + 1)} + \frac{1}{\Gamma(\alpha_3)} + \frac{1}{\Gamma(\alpha_3 - 1)} \right) \|x_3 - y_3\| \\ &\leq C\|x - y\|, \end{aligned}$$

which ends the proof.

We close the article by considering the following nonlocal fractional differential system

$$\begin{cases} D^{\alpha_n} x_n(t) = f_n(t, x_n(t)), \\ x_n(0) = g_n(x_n), x'_2(T) = a, x'_3(T) = b, x''_3(T) = c \end{cases} \quad (7)$$

where $f_n, n = 1, 2, 3$, as before, is fractional integrable of order $\alpha_n; n - 1 < \alpha_n \leq n$ and satisfies the hypothesis (H1), a, b , and c are constants, and the nonlinear function g_n satisfies the following assumption.

(H4) The function g_n is defined on the Banach space X , satisfying the Lipschitz condition, i.e., there exists a positive constant C_n such that

$$\|g_n(x_n) - g_n(y_n)\| \leq C_n \|x_n - y_n\|, n = 1, 2, 3$$

for any $x_n, y_n \in Y$.

The system (7) is equivalent to

$$x(t) = g(x) + At + Bt^2 + \int_0^T G(t, s) * f(s, x(s)) ds \in Z$$

where $g(x) = g(x_1, x_2, x_3) = (g_1(x_1), g_2(x_2), g_3(x_3)), A = (0, a, b - cT), B = (0, 0, \frac{c}{2})$, and the last term is defined as previous.

Before going on to the next result in the sequel, we need to modify the hypothesis (H3) to be suitable for the case.

(H5) Let D, C , and r be positive constants such that

$$\begin{cases} D = \max_{1 \leq n \leq 3} \{A_n, B_n \cdot C_n\} \\ C = D \left(1 + \frac{T^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{T^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \frac{T^{\alpha_2}}{\Gamma(\alpha_2)} + \frac{T^{\alpha_3}}{\Gamma(\alpha_3 + 1)} + \frac{T^{\alpha_3}}{\Gamma(\alpha_3)} + \frac{T^{\alpha_3}}{\Gamma(\alpha_3 - 1)} \right) < 1 \\ r \geq \frac{\|g(0)\| + (|a| + |b - cT|)T + \frac{1}{2}|c|T^2 + C}{1 - C} \end{cases}$$

Moreover, let $B_r = \{x \in Z : \|x\| \leq r\}$.

Now, we can state the next theorem whose proof is similar to that of Theorem 2 with some modifications.

Theorem 3 If the hypotheses (H1), (H4), and (H5) are satisfied, then the fractional differential system (7) has a solution on J .

Example. Consider the following fractional differential system

$$\begin{bmatrix} D^{\alpha_1} x_1(t) \\ D^{\alpha_2} x_2(t) \\ D^{\alpha_3} x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{x_1(t)}{(2+t)^3} \\ \frac{1}{7+e^t} \frac{x_2(t)}{1+x_2(t)} \\ \sin \frac{tx_3(t)}{8} \end{bmatrix}, t \in [0, 1]$$

where $x_n \in C([0, 1], \mathbb{R}), x'_2(1) = a, x'_3(1) = b, x''_3(1) = c, x_n(0) = g_n(x_n) = \sum_{k=1}^m c_k x_n(t_k), 0 < t_1 < t_2 < \dots < t_m < 1$, and c_k 's are positive real numbers

such that $\sum_{k=1}^m c_k = \frac{1}{8}$. To apply Theorem 3, we verify that the hypotheses (H1),

(H4), and (H5) are satisfied. The following constants can be easily specified as $A_n < \frac{1}{8}, n = 1, 2, 3, B_2 = \frac{1}{8}, B_1 = B_3 = 0, C_1 = C_2 = C_3 = \frac{1}{8}$,

which implies that $D < \frac{1}{8}$. On the other hand, it is clear that

$$\frac{1}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(\alpha_2 + 1)} + \frac{1}{\Gamma(\alpha_2)} + \frac{1}{\Gamma(\alpha_3 + 1)} + \frac{1}{\Gamma(\alpha_3)} + \frac{1}{\Gamma(\alpha_3 - 1)} < 7.$$

Hence, by Theorem 3, the system has a solution on $[0, 1]$.

REFERENCES

- [1] M. Benchohra, S. Hamani, S.K. Ntouyas, "Boundary value problems for differential equations with fractional order and nonlocal conditions" , *Nonlinear Analysis*, 71 (2009) 2391- 2396.
- [2] K. Balachandran, J.Y. Park, "Nonlocal Cauchy problem for abstract fractional semilinear evolution equations", *Nonlinear Analysis*, 71 (2009) 4471-4475.
- [3] A.A. Kilbas, H.M.Srivastava, J.J.Trujillo, "Theory and applications of fractional differential equations", Elsevier, Amsterdam (2006).
- [4] M. Matar, "On existence and uniqueness of the mild solution for fractional semilinear integro-differential equations", *Journal of Integral Equations and Applications*, 23.(2011),457-466.
- [5] M. Matar, "Existence and uniqueness of solutions to fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions", *Electronic Journal of Differential Equations*, 155 (2009) 1-7.
- [6] M. Matar, "Boundary value problem for fractional integro-differential equations with nonlocal conditions", *Int. J. Open Problems Compt. Math.*, 3 (2010) 481-489.
- [7] K.S. Miller, B. Ross, "An introduction to the fractional calculus and fractional differential equations", Wiley, New York, (1993).
- [8] G.M. N'Guérékata, "A Cauchy problem for some fractional abstract differential equation with nonlocal condition", *Nonlinear Analysis*, 70 (2009) 1873-187.
- [9] I. Podlubny, "Fractional differential equations", Academic Press, New York, (1999).
- [10] R. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for fractional differential equations, *Georgian Mathematical Journal*,16(3) (2009), 401-411.

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