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EFFICIENT SPECTRAL COLLOCATION METHOD FOR SOLVING MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS BASED ON THE GENERALIZED LAGUERRE POLYNOMIALS

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Abstract. In this paper, a new approximate formula of the fractional derivatives is derived. The proposed formula is based on the generalized Laguerre polynomials. Global approximations to functions defined on a semi-infinite interval are constructed. The fractional derivatives are presented in terms of Caputo sense. Special attention is given to study the convergence analysis of the method and estimate the upper bound of the error. The new spectral Laguerre collocation method is presented for solving multi-term fractional orders initial value problems (FIVPs). The properties of Laguerre polynomials are utilized to reduce FIVPs to a system of algebraic equations which can be solved using a suitable numerical method. Several numerical examples are provided to confirm the theoretical results and the efficiency of the proposed method.

1. Introduction

Ordinary and partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering [4]. Consequently, considerable attention has been given to the solutions of fractional differential equations and integral equations of physical interest. Most non-linear fractional differential equations do not have exact analytic solutions, so approximate and numerical techniques ([2], [5], [6], [28]) must be used. Several numerical methods to solve the fractional differential equations have been given such as variational iteration method [14], homotopy perturbation method ([26], [27]), Adomian's decomposition method [15], homotopy analysis method [13] and collocation method ([9], [18]-[20], [24], [29]).

Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and forms the basis of spectral methods of solution of differential equations ([7], [12]). In [17], Khader introduced an efficient numerical method for solving the fractional diffusion equation using the shifted Chebyshev polynomials. In [11] two Chebyshev spectral methods for solving multi-term fractional orders differential equations are introduced.

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In [16] the generalized Laguerre polynomials were used to compute a spectral solution of a non-linear boundary value problems. The classical generalized Laguerre polynomials constitute a complete orthogonal sets of functions on the semi-infinite interval $[0, \infty)$ [8]. Convolution structures of Laguerre polynomials were presented in [3]. The Laguerre polynomials arise in quantum mechanics, in the radial part of the solution of the Schrödinger equation for an one-electron atom. Physicists often use a definition for the Laguerre polynomials that is larger, by a factor of $n!$, than the definition used here. Furthermore, various physicists use somewhat different definitions of the so-called generalized Laguerre polynomials, for instance in modern quantum mechanics the definition is different than the one found below. A comparison of notations can be found in introductory quantum mechanics. Also, other spectral methods based on other orthogonal polynomials are used to obtain spectral solutions on unbounded intervals ([30], [31]). Spectral collocation methods are efficient and highly accurate techniques for numerical solution of nonlinear differential equations. The basic idea of the spectral collocation method is to assume that the unknown solution $u(x)$ can be approximated by a linear combination of some basis functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for a problem under consideration.

The main aim of the presented paper is concerned with the application of the proposed approach to obtain the numerical solution of multi-order fractional differential equations of the form

$$D^{\nu}u(x) = F\left(x; u(x), D^{\beta_1}u, ..., D^{\beta_n}u\right), \tag{1}$$

with the following initial conditions

$$u^{(k)}(0) = u_k, \qquad k = 0, 1, ..., m-1, \tag{2}$$

where $m < \nu \leq m+1$, $0 < \beta_1 < \beta_2 < ... < \beta_n < \nu$ and D^{ν} denotes Caputo fractional derivative of order ν. It should be noted that F can be non-linear in general.

The structure of this paper is arranged in the following way: In section 2, we introduce some basic definitions about Caputo fractional derivatives and properties of the classical generalized Laguerre polynomials. In section 3, we introduce the fundamental theorems for the fractional derivatives of the generalized Laguerre polynomials. In section 4, the convergence analysis of the derived approximate formula is given. In section 5, the procedure of solution for multi-order FIVPs is clarified. In section 6, numerical examples are given to solve the FIVPs and show the accuracy of the method. Finally, in section 7, the report ends with a brief conclusion and some remarks.

2. Preliminars and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1 The fractional derivative in the Caputo sense

Definition 1 The Caputo fractional derivative operator D^ν of order ν is defined in the following form

$$D^\nu f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}} dt, \quad \nu > 0, \quad x > 0,$$

where $m-1 < \nu \leq m,\ m \in \mathbb{N}$.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^\nu \left(\lambda\, f(x) + \mu\, g(x)\right) = \lambda\, D^\nu\, f(x) + \mu\, D^\nu\, g(x), \tag{3}$$

where λ and μ are constants. For the Caputo's derivative we have

$$D^\nu\, C = 0, \qquad \text{C is a constant,} \tag{4}$$

$$D^\nu\, x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \nu \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \nu \rceil. \end{cases} \tag{5}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν, and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\nu \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see ([10], [23], [25]).

2.2 The definition and properties of the generalized Laguerre polynomials

The generalized Laguerre polynomials $[L_n^{(\alpha)}(x)]_{n=0}^\infty$, $\alpha > -1$ are defined on the semi-infinite interval $[0, \infty)$ and can be determined with the aid of the following recurrence formula

$$(n+1)L_{n+1}^{(\alpha)}(x) + (x - 2n - \alpha - 1)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0, \quad n = 1, 2, 3, ..., \tag{6}$$

where, $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$. The explicit formula of these polynomials of degree n is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k = \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}, \tag{7}$$

with $(a)_0 := 1$ and $(a)_k := a(a+1)(a+2)...(a+k-1),\ k = 1, 2, 3, ...$ and $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$. These polynomials are orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = \frac{1}{\Gamma(1+\alpha)} x^\alpha e^{-x}$. The orthogonality relation is

$$\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \binom{n+\alpha}{n} \delta_{mn}. \tag{8}$$

Also, they satisfy the differentiation formula

$$D^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x), \quad k = 0, 1, ..., n. \tag{9}$$

Any function $u(x)$ belongs to the space $L^2_w[0,\infty)$ of all square integrable functions on $[0,\infty)$ with weight function $w(x)$, can be expanded in the following Laguerre series

$$u(x)=\sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(x), \tag{10}$$

where the coefficients c_i are given by

$$c_i=\frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)}\int_0^{\infty} x^{\alpha}e^{-x}L_i^{(\alpha)}(x)u(x)dx, \quad i=0,1,2,\dots. \tag{11}$$

Consider only the first $(m+1)$ terms of classical Laguerre polynomials, so we can write

$$u_m(x)=\sum_{i=0}^{m} c_i L_i^{(\alpha)}(x). \tag{12}$$

A new approximate formula for the fractional derivative of the approximate function using Laguerre polynomials is discussed in the following section.

3. An approximate formula for fractional derivatives of $L_n^{(\alpha)}(x)$

The main goal of this section is to present the following theorems for the approximated fractional derivatives of the classical Laguerre polynomials.

Lemma 1 Let $L_n^{(\alpha)}(x)$ be a generalized Laguerre polynomial then

$$D^{\nu}L_n^{(\alpha)}(x)=0, \quad n=0,1,...,\lceil\nu\rceil-1, \quad \nu>0. \tag{13}$$

Proof. This lemma can be proved directly using a combination of Eqs.(3)-(5).

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

Theorem 1 Let $u(x)$ be approximated by the generalized Laguerre polynomials as (12) and also suppose $\nu>0$ then, its approximated fractional derivative can be written in the following form

$$D^{\nu}(u_m(x))\cong\sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,\,k}^{(\nu)}\, x^{k-\nu}, \tag{14}$$

where $w_{i,\,k}^{(\nu)}$ is given by

$$w_{i,\,k}^{(\nu)}=\frac{(-1)^k}{\Gamma(k+1-\nu)}\begin{pmatrix} i+\alpha \\ i-k\end{pmatrix}. \tag{15}$$

Proof. Since the Caputo's fractional differentiation is a linear operation we have

$$D^{\nu}(u_m(x))=\sum_{i=0}^{m} c_i\, D^{\nu}(L_i^{(\alpha)}(x)). \tag{16}$$

Employing Eqs.(4)-(5) in Eq.(7) we have

$$D^{\nu}\, L_i^{(\alpha)}(x)=0, \qquad i=0,1,...,\lceil\nu\rceil-1, \quad \nu>0. \tag{17}$$

Therefore, for $i = \lceil \nu \rceil, ..., m$, by using Eqs.(4)-(5) in Eq.(7), we get

$$D^{\nu}\, L_i^{(\alpha)}(x) = \sum_{k=0}^{i} \frac{(-1)^k}{k!} \left(\begin{array}{c} i+\alpha \\ i-k \end{array} \right) D^{\nu}\, x^k = \sum_{k=\lceil \nu \rceil}^{i} \frac{(-1)^k}{\Gamma(k+1-\nu)} \left(\begin{array}{c} i+\alpha \\ i-k \end{array} \right) x^{k-\nu}. \tag{18}$$

A combination of Eqs.(16) and (18) leads to the desired result.

Test example

Consider the function $u(x) = x^3$ with $m = 3$, $\nu = 1.5$ and $\alpha = -0.5$, the generalized Laguerre series of x^3 is

$$x^3 = 1.875\, L_0^{(\alpha)}(x) - 11.25\, L_1^{(\alpha)}(x) + 15\, L_2^{(\alpha)}(x) - 6\, L_3^{(\alpha)}(x).$$

Now, by using Eq.(14), we obtain

$$D^{1.5}\ x^3 = \sum_{i=2}^{3} \sum_{k=2}^{i} c_i\, w_{i,\,k}^{(1.5)}\, x^{k-1.5},$$

where, $w_{2,\,2}^{(1.5)} = 1.12838$, $w_{3,\,2}^{(1.5)} = 2.82095$, $w_{3,\,3}^{(1.5)} = -0.752253$, therefore,

$$D^{1.5}\ x^3 = c_2\, w_{2,\,2}^{(1.5)}\, x^{0.5} + c_3\, w_{3,\,2}^{(1.5)}\, x^{0.5} + c_3\, w_{3,\,3}^{(1.5)}\, x^{1.5} = \frac{\Gamma(4)}{\Gamma(2.5)}\ x^{1.5},$$

which agree with the exact derivative (5).

Theorem 2 The Caputo fractional derivative of order ν for the generalized classical Laguerre polynomials can be expressed in terms of the generalized Laguerre polynomials themselves in the following form

$$D^{\nu} L_i^{(\alpha)}(x) = \sum_{k=\lceil \nu \rceil}^{i} \sum_{j=0}^{k-\lceil \nu \rceil} \Omega_{i\,j\,k}\, L_j^{(\alpha)}(x), \qquad i = \lceil \nu \rceil, \lceil \nu \rceil + 1, ..., m\,, \tag{19}$$

where

$$\Omega_{i\,j\,k} = \frac{(-1)^{j+k}\,(\alpha+i)!\,(k-\nu+\alpha)!}{(i-k)!\,(\alpha+k)!\,(k-\nu-j)!\,(\alpha+j)!}\,.$$

Proof. From the properties of the generalized Laguerre polynomials [1] and expand $x^{k-\nu}$ in Eq.(18) in the following form

$$x^{k-\nu} = \sum_{j=0}^{k-\lceil \nu \rceil} c_{kj} L_j^{(\alpha)}(x), \tag{20}$$

where c_{kj} can be obtained using (11) where $u(x) = x^{k-\nu}$ then,

$$c_{kj} = \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} \int_0^{\infty} x^{k+\alpha-\nu} e^{-x} L_j^{(\alpha)}(x) dx, \quad j = 0, 1, 2, \dots. \tag{21}$$

then,

$$x^{k-\nu} = \sum_{j=0}^{k-\lceil \nu \rceil} \frac{(-1)^j\,(k-\nu)!\,(k-\nu+\alpha+1)!}{(k-\nu-j)!\,(\alpha+j)!}\, L_j^{(\alpha)}(x). \tag{22}$$

Therefore, the Caputo fractional derivative $D^{\nu}L_i^{(\alpha)}(x)$ in Eq.(18) can be written in the following form

$$D^{\nu}L_i^{(\alpha)}(x)=\sum_{k=\lceil\nu\rceil}^{i}\sum_{j=0}^{k-\lceil\nu\rceil}\frac{(-1)^{j+k}\,(\alpha+i)!\,(k-\nu+\alpha)!}{(i-k)!\,(\alpha+k)!\,(k-\nu-j)!\,(\alpha+j)!}L_j^{(\alpha)}(x), \qquad (23)$$

for $i=\lceil\nu\rceil,\lceil\nu\rceil+1,...,m$.
Eq.(23) leads to the desired result and completes the proof of the theorem.

4. Convergence analysis of the proposed approximate formula

In this section, special attention is given to estimate the upper bound of the error of the proposed approximate formula.

Theorem 3 For the Laguerre polynomials $L_n^{(\alpha)}(x)$, we have the following global uniform bounds estimates

$$|L_n^{(\alpha)}(x)|\leq\begin{cases}\frac{(\alpha+1)_n}{n!}\,e^{x/2}, & \text{for}\quad \alpha\geq 0,\ \ x\geq 0,\ \ n=0,1,2,...;\\ \left(2-\frac{(\alpha+1)_n}{n!}\right)e^{x/2}, & \text{for}\quad -1<\alpha\leq 0,\ \ x\geq 0,\ \ n=0,1,2,....\end{cases} \qquad (24)$$

Proof. These estimates were presented in [21] and [22], Szego proved them in [1].

Theorem 4 The error in approximating $D^{\nu}u(x)$ by $D^{\nu}u_m(x)$ is bounded by

$$|E_T(m)|\leq\sum_{i=m+1}^{\infty}c_i\Pi_{\nu}(i,j)\ \frac{(\alpha+1)_j}{j!}\ e^{x/2},\ \ \alpha\geq 0,\ \ x\geq 0,\ \ j=0,1,2,..., \qquad (25)$$

$$|E_T(m)|\leq\sum_{i=m+1}^{\infty}c_i\Pi_{\nu}(i,j)\ \left(2-\frac{(\alpha+1)_j}{j!}\right)e^{x/2},\ \ -1<\alpha\leq 0,\ \ x\geq 0,\ \ j=0,1,2,.... \qquad (26)$$

such that, $\Pi_{\nu}(i,j)=\sum_{k=\lceil\nu\rceil}^{i}\sum_{j=0}^{k-\nu}\Omega_{i\,j\,k}$, where, $\Omega_{i\,j\,k}$ is defined in Eq.(19).
Proof. A combination of Eqs.(10), (12) and (19) leads to

$$|E_T(m)|=|D^{\nu}u(x)-D^{\nu}u_m(x)|\leq\sum_{i=m+1}^{\infty}c_i\ \Pi_{\nu}(i,j)\,|L_j^{(\alpha)}(x)|, \qquad (27)$$

using Eqs.(24), (27) and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem.

5. Procedure of solution for the multi-order FIVPs

Consider the multi-order fractional differential equation of type given in Eq.(1). Let $w(x)=\frac{1}{\alpha!}x^{\alpha}e^{-x}$ be a positive weight function on the interval $I=[0,\infty)$ and $L_w^2(I)$ is the weighted L^2 space with inner product

$$(u,v)_w=\int_0^{\infty}w(x)u(x)v(x)dx,$$

and the associated norm $\|u\|_w=(u,u)_w^{\frac{1}{2}}$. It is well known that $[L_n^{(\alpha)}(x):n\geq 0]$ forms a complete orthogonal system in $L_w^2(I)$, so if we define

$$S_m(I)=\mathrm{Span}[L_0^{(\alpha)}(x),L_1^{(\alpha)}(x),...,L_m^{(\alpha)}(x)],$$

then, the Laguerre spectral solution to Eq.(1) is to find $u_m \in S_m(I)$ such that

$$u_m(x) \cong \sum_{i=0}^{m} c_i \, L_i^{(\alpha)}(x). \tag{28}$$

From Eqs.(1), (28) and Theorem 1 we have

$$\sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} c_i \, w_{i,\,k}^{(\nu)} \, x^{k-\nu} = F\left(x; \sum_{i=0}^{m} c_i \, L_i^{(\alpha)}(x), \sum_{i=\lceil\beta_1\rceil}^{m} \sum_{k=\lceil\beta_1\rceil}^{i} c_i \, w_{i,\,k}^{(\beta_1)} \, x^{k-\beta_1}, ..., \sum_{i=\lceil\beta_n\rceil}^{m} \sum_{k=\lceil\beta_n\rceil}^{i} c_i \, w_{i,\,k}^{(\beta_n)} \, x^{k-\beta_n} \right). \tag{29}$$

We now collocate Eq.(29) at $(m+1-\lceil\nu\rceil)$ points x_p, $p = 0, 1, ..., m-\lceil\nu\rceil$ as

$$\sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} c_i \, w_{i,\,k}^{(\nu)} \, x_p^{k-\nu} = F\left(x_p; \sum_{i=0}^{m} c_i \, L_i^{(\alpha)}(x_p), \sum_{i=\lceil\beta_1\rceil}^{N} \sum_{k=\lceil\beta_1\rceil}^{i} b_i \, w_{i,\,k}^{(\beta_1)} \, x_p^{k-\beta_1}, ..., \sum_{i=\lceil\beta_n\rceil}^{N} \sum_{k=\lceil\beta_n\rceil}^{i} b_i \, w_{i,\,k}^{(\beta_n)} \, x_p^{k-\beta_n} \right). \tag{30}$$

For suitable collocation points we use roots of the generalized Laguerre polynomial $L_{m+1-\lceil\nu\rceil}^{(\alpha)}(x)$.
Also, by substituting Eqs.(9) and (28) in the initial conditions (2) and using the property $L_i^{(\alpha)}(0) = \begin{pmatrix} \alpha+i \\ i \end{pmatrix}$ we can obtain $\lceil\nu\rceil$ equations

$$\sum_{i=0}^{m} c_i \, (-1)^k \begin{pmatrix} \alpha+i \\ i-k \end{pmatrix} = u_k, \qquad k = 0, 1, 2, ..., m-1. \tag{31}$$

Equation (30), together with $\lceil\nu\rceil$ equations of the initial conditions (31), give $(m+1)$ algebraic equations which can be solved, for the unknowns c_i, $i = 0, 1, ..., m$, using a suitable numerical method, as described in the following section.

6. Numerical simulation and comparison

In order to illustrate the effectiveness of the proposed method, we implement it to solve the multi-term fractional orders FIVPs with different numerical examples.

Example 1:

Consider the following fractional Cauchy problem with variable coefficients of the linear form on $[0, 1]$

$$D^{1.5}u(x)+2\,Du(x)+3\sqrt{x}D^{0.5}u(x)+(1-x)\,u(x) = \frac{2}{\Gamma(1.5)}\,x^{0.5}+4\,x+\frac{4}{\Gamma(1.5)}\,x^2+(1-x)x^2, \tag{32}$$

with the following initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{33}$$

The exact solution to this problem is $u(x) = x^2$.

To solve this problem, by applying the proposed technique described in section 5 with $m=3$, we approximate the solution as

$$u_3(x) \cong c_0 L_0^{(\alpha)}(x) + c_1 L_1^{(\alpha)}(x) + c_2 L_2^{(\alpha)}(x) + c_3 L_3^{(\alpha)}(x).$$

Using Eq.(29), with $\nu = 1.5$, $\beta_1 = 1$, $\beta_2 = 0.5$ and $\alpha = -0.5$ we have

$$\sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,k}^{(\nu)}\, x^{k-\nu} + 2\sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=\lceil\beta_1\rceil}^{i} c_i\, w_{i,k}^{(\beta_1)}\, x^{k-\beta_1} + g(x)\sum_{i=\lceil\beta_2\rceil}^{m}\sum_{k=\lceil\beta_2\rceil}^{i} c_i\, w_{i,k}^{(\beta_2)}\, x^{k-\beta_2} + h(x)\left(\sum_{i=0}^{m} c_i\, L_i^{(\alpha)}(x)\right) = r(x), \tag{34}$$

where $g(x) = 3\sqrt{x}$, $h(x) = 1-x$ and $r(x) = \frac{2}{\Gamma(1.5)}\, x^{0.5} + 4\,x + \frac{4}{\Gamma(1.5)}\, x^2 + (1-x)x^2$.
Now, we collocate Eq.(34) at the roots x_p as

$$\sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,k}^{(\nu)}\, x_p^{k-\nu} + 2\sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=\lceil\beta_1\rceil}^{i} c_i\, w_{i,k}^{(\beta_1)}\, x_p^{k-\beta_1} + g(x_p)\sum_{i=\lceil\beta_2\rceil}^{m}\sum_{k=\lceil\beta_2\rceil}^{i} c_i\, w_{i,k}^{(\beta_2)}\, x_p^{k-\beta_2} + h(x_p)\left(\sum_{i=0}^{m} c_i\, L_i^{(\alpha)}(x)\right) = r(x_p). \tag{35}$$

where x_p are roots of the generalized Laguerre polynomial $L_2^{(\alpha)}(x)$, i.e.,

$$x_0 = 0.275255, \quad x_1 = 2.72474\,.$$

Using Eq.(35) at the collocation points x_0, x_1 and using Eqs.(31) and (33), we obtain the following system of algebraic equations

$$\sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,k}^{(\nu)}\, x_0^{k-\nu} + 2\sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=\lceil\beta_1\rceil}^{i} c_i\, w_{i,k}^{(\beta_1)}\, x_0^{k-\beta_1} + g(x_0)\sum_{i=\lceil\beta_2\rceil}^{m}\sum_{k=\lceil\beta_2\rceil}^{i} c_i\, w_{i,k}^{(\beta_2)}\, x_0^{k-\beta_2} + h(x_0)\left(\sum_{i=0}^{m} c_i\, L_i^{(\alpha)}(x_0)\right) = r(x_0), \tag{36}$$

$$\sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,k}^{(\nu)}\, x_1^{k-\nu} + 2\sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=\lceil\beta_1\rceil}^{i} c_i\, w_{i,k}^{(\beta_1)}\, x_1^{k-\beta_1} + g(x_p)\sum_{i=\lceil\beta_2\rceil}^{m}\sum_{k=\lceil\beta_2\rceil}^{i} c_i\, w_{i,k}^{(\beta_2)}\, x_1^{k-\beta_2} + h(x_1)\left(\sum_{i=0}^{m} c_i\, L_i^{(\alpha)}(x_1)\right) = r(x_1), \tag{37}$$

$$c_0 r_0 + c_1 r_1 + c_2 r_2 + c_3 r_3 = 0, \tag{38}$$

$$c_0 s_0 + c_1 s_1 + c_2 s_2 + c_3 s_3 = 0, \tag{39}$$

where, $r_i = \begin{pmatrix} \alpha + i \\ i \end{pmatrix}, \quad s_i = \begin{pmatrix} \alpha + i \\ i-1 \end{pmatrix}, \quad i = 0, 1, 2, 3.$

By solving Eqs.(36)-(39) using conjugate gradient method, we can obtain the coefficients c_i.

Therefore, the approximate solution is given by

$$u_3(x) \cong 0.75\ L_0^{(\alpha)}(x) - 3\ L_1^{(\alpha)}(x) + 2\ L_2^{(\alpha)}(x) + 0\ L_3^{(\alpha)}(x) = x^2,$$

which is the exact solution of the problem.

Example 2:

Consider the following non-linear initial value problem

$$D^3 u(x) + D^{2.5} u(x) + u^2(x) = x^4, \tag{40}$$

with the following initial conditions

$$u(0) = u'(0) = 0, \quad u''(0) = 2. \tag{41}$$

We apply the suggested method with $m = 3$, and approximate the solution $u(x)$ as follows

$$u_3(x) \cong \sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x). \tag{42}$$

Using Eq.(29), with $\nu = 3,\ \beta_1 = 2.5,\ \alpha = -0.5,$ we have

$$\sum_{i=3}^{3}\sum_{k=3}^{i} c_i\, w_{i,\,k}^{(3)}\, x^{k-3} + \sum_{i=3}^{3}\sum_{k=3}^{i} c_i\, w_{i,\,k}^{(2.5)}\, x^{k-2.5} + \left(\sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x)\right)^2 = x^4. \tag{43}$$

Now, we collocate Eq.(43) at the roots x_p as

$$\sum_{i=3}^{3}\sum_{k=3}^{i} c_i\, w_{i,\,k}^{(3)}\, x_p^{k-3} + \sum_{i=3}^{3}\sum_{k=3}^{i} c_i\, w_{i,\,k}^{(2.5)}\, x_p^{k-2.5} + \left(\sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x_p)\right)^2 = x_p^4, \tag{44}$$

where x_p are roots of the classical Laguerre polynomial $L_1^{(\alpha)}(x)$, i.e., $x_0 = 1 + \alpha = 0.5$.

By using Eqs.(31) and (44) we obtain the following non-linear system of algebraic equations

$$c_3(w_{3,3}^{(3)} + w_{3,3}^{(2.5)}\, x_0^{0.5}) + (k_0 c_0 + k_1 c_1 + k_2 c_2 + k_3 c_3)^2 = x_0^4, \tag{45}$$

$$c_0 r_0 + c_1 r_1 + c_2 r_2 + c_3 r_3 = 0, \tag{46}$$

$$c_0 s_0 + c_1 s_1 + c_2 s_2 + c_3 s_3 = 0, \tag{47}$$

$$c_0 t_0 + c_1 t_1 + c_2 t_2 + c_3 t_3 = 2, \tag{48}$$

where, $k_i = L_i^{(\alpha)}(x_0)$, $r_i = \begin{pmatrix} \alpha + i \\ i \end{pmatrix}$, $s_i = \begin{pmatrix} \alpha + i \\ i - 1 \end{pmatrix}$, $t_i = \begin{pmatrix} \alpha + i \\ i - 2 \end{pmatrix}$, $i = 0, 1, 2, 3$. By solving Eqs.(45)-(48) we obtain

$$c_0 = 0.75, \qquad c_1 = -3, \qquad c_2 = 2, \qquad c_3 = 0.$$

Therefore

$$u(x) = \begin{pmatrix} 0.75\,, & -3\,, & 2\,, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -x + 0.5 \\ 0.5x^2 - 1.5x + 0.375 \\ -0.16667x^3 + 1.25x^2 - 1.875x + 0.3125 \end{pmatrix} = x^2,$$

which is the exact solution of this problem.

It is clear that in this example the presented method can be considered as an efficient method.

Example 3:

In this example, we consider the following non-linear fractional differential equation

$$D^4u(x) + D^{3.5}u(x) + u^3(x) = x^9, \tag{49}$$

subject to the initial conditions

$$u(0) = u^{(1)}(0) = u^{(2)}(0) = 0, \quad u^{(3)}(0) = 6\,. \tag{50}$$

To solve the above problem, by applying the proposed technique with $m = 4$, we approximate the solution as

$$u_4(x) \cong c_0L_0^{(\alpha)}(x) + c_1L_1^{(\alpha)}(x) + c_2L_2^{(\alpha)}(x) + c_3L_3^{(\alpha)}(x) + c_4L_4^{(\alpha)}(x).$$

Using Eq.(29), with $\nu = 4,\ \beta_1 = 3.5,\ \alpha = -0.5$ we have

$$\sum_{i=4}^{4}\sum_{k=4}^{i} c_i\, w_{i,\,k}^{(4)}\, x^{k-4} + \sum_{i=4}^{4}\sum_{k=4}^{i} c_i\, w_{i,\,k}^{(3.5)}\, x^{k-3.5} + \left(\sum_{i=0}^{4} c_i\, L_i^{(\alpha)}(x)\right)^3 = x^9. \tag{51}$$

Now, we collocate Eq.(51) at the roots x_p as

$$\sum_{i=4}^{4}\sum_{k=4}^{i} c_i\, w_{i,\,k}^{(4)}\, x_p^{k-4} + \sum_{i=4}^{4}\sum_{k=4}^{i} c_i\, w_{i,\,k}^{(3.5)}\, x_p^{k-3.5} + \left(\sum_{i=0}^{4} c_i\, L_i^{(\alpha)}(x_p)\right)^3 = x_p^9, \tag{52}$$

where x_p are roots of the generalized Laguerre polynomial $L_1^{(\alpha)}(x)$, i.e., $x_0 = 0.5\,$. By using Eqs.(31) and (52) we obtain the following non-linear system of algebraic equations

$$c_4\left(w_{4,4}^{(4)} + w_{4,4}^{(3.5)}\, x_0^{0.5}\right) + (k_0c_0 + k_1c_1 + k_2c_2 + k_3c_3 + k_4c_4)^3 = x_0^9, \tag{53}$$

$$c_0r_0 + c_1r_1 + c_2r_2 + c_3r_3 + c_4r_4 = 0, \tag{54}$$

$$c_0s_0 + c_1s_1 + c_2s_2 + c_3s_3 + c_4s_4 = 0, \tag{55}$$

$$c_0t_0 + c_1t_1 + c_2t_2 + c_3t_3 + c_4t_4 = 0, \tag{56}$$

$$c_0z_0 + c_1z_1 + c_2z_2 + c_3z_3 + c_4z_4 = 6, \tag{57}$$

where, (for $i = 0, 1, 2, 3, 4$)

$k_i = L_i^{(\alpha)}(x_0),\ r_i = \begin{pmatrix} \alpha+i \\ i \end{pmatrix},\ s_i = \begin{pmatrix} \alpha+i \\ i-1 \end{pmatrix},\ t_i = \begin{pmatrix} \alpha+i \\ i-2 \end{pmatrix},\ z_i = \begin{pmatrix} \alpha+i \\ i-3 \end{pmatrix}.$

By solving Eqs.(53)-(57) we obtain

$$c_0 = 1.875, \qquad c_1 = -11.25, \qquad c_2 = 15, \qquad c_3 = -6, \qquad c_4 = 0.$$

Therefore, $u(x)$ has the form

$$u(x) = \begin{pmatrix} 1.875, & -11.25, & 15, & -6, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -x + 0.5 \\ 0.5x^2 - 1.5x + 0.375 \\ -0.1667x^3 + 1.25x^2 - 1.875x + 0.3125 \\ 0.042x^4 - 0.583x^3 + 2.188x^2 - 2.188x + 0.273 \end{pmatrix} = x^3,$$

which is the exact solution of this problem. It is clear that in this example the presented method can be considered as an efficient method.

Example 4:

In this example, we consider the following non-linear differential equation

$$D^2u(x)+g(x)D^{1.5}u(x)+u^2(x)=2+2x+x^4, \tag{58}$$

where $g(x)=\Gamma(1.5)x^{0.5}$ and subject to the initial conditions

$$u(0)=u'(0)=0. \tag{59}$$

The exact solution of this example is $u(x)=x^2$. To solve this problem, by applying the proposed technique described in section 5 with $m=3$, we approximate the solution as

$$u_3(x)\cong c_0L_0^{(\alpha)}(x)+c_1L_1^{(\alpha)}(x)+c_2L_2^{(\alpha)}(x)+c_3L_3^{(\alpha)}(x).$$

Using Eq.(29), with $\nu=2$, $\beta_1=1.5$ and $\alpha=-0.25$ we have

$$\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(2)}\,x^{k-2}+g(x)\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(1.5)}\,x^{k-1.5}+\left(\sum_{i=0}^{3}c_i\,L_i^{(\alpha)}(x)\right)^2=2+2x+x^4. \tag{60}$$

Now, we collocate Eqs.(60) at the roots x_p as

$$\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(2)}\,x_p^{k-2}+g(x_p)\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(1.5)}\,x_p^{k-1.5}+\left(\sum_{i=0}^{3}c_i\,L_i^{(\alpha)}(x_p)\right)^2=2+2x_p+x_p^4, \tag{61}$$

where x_p are roots of the generalized Laguerre polynomial $L_2^{(\alpha)}(x)$, i.e.,

$$x_0=2+\alpha-\sqrt{2+\alpha}=0.427124,\quad x_1=2+\alpha+\sqrt{2+\alpha}=3.07288\,.$$

By using Eqs.(31) and (61) we obtain the following non-linear system of algebraic equations

$$\begin{aligned}&c_2w_{2,2}^{(2)}+c_3\left(w_{3,2}^{(2)}+w_{3,3}^{(2)}x_0\right)+g(x_0)c_2w_{2,2}^{(1.5)}x_0^{0.5}+c_3g(x_0)\left(w_{3,2}^{(1.5)}x_0^{0.5}+w_{3,3}^{(1.5)}x_0^{1.5}\right)\\&\quad+(k_0c_0+k_1c_1+k_2c_2+k_3c_3)^2=2+2x_0+x_0^4,\end{aligned} \tag{62}$$

$$\begin{aligned}&c_2w_{2,2}^{(2)}+c_3\left(w_{3,2}^{(2)}+w_{3,3}^{(2)}x_1\right)+g(x_1)c_2w_{2,2}^{(1.5)}x_1^{0.5}+c_3g(x_1)\left(w_{3,2}^{(1.5)}x_1^{0.5}+w_{3,3}^{(1.5)}x_1^{1.5}\right)\\&\quad+(l_0c_0+l_1c_1+l_2c_2+l_3c_3)^2=2+2x_1+x_1^4,\end{aligned} \tag{63}$$

$$c_0r_0+c_1r_1+c_2r_2+c_3r_3=0, \tag{64}$$

$$c_0s_0+c_1s_1+c_2s_2+c_3s_3=0, \tag{65}$$

where, $k_i=L_i^{(\alpha)}(x_0)$, $l_i=L_i^{(\alpha)}(x_1)$, $r_i=\left(\begin{array}{c}\alpha+i\\ i\end{array}\right)$, $s_i=\left(\begin{array}{c}\alpha+i\\ i-1\end{array}\right)$, $i=0,1,2,3$. By solving Eqs.(62)-(65) using Newton iteration method, we can obtain the coefficients c_i.

Therefore, the approximate solution is given by

$$u_3(x)\cong 1.38658\;L_0^{(\alpha)}(x)-3.324026\;L_1^{(\alpha)}(x)+1.2591210\;L_2^{(\alpha)}(x)+0.465689\;L_3^{(\alpha)}(x).$$

The obtained numerical results by means of the proposed method and exact solution are shown in figure 1, with $m=3$ and $\nu=2$, $\beta_1=1.5$ at $\alpha=-0.25$.

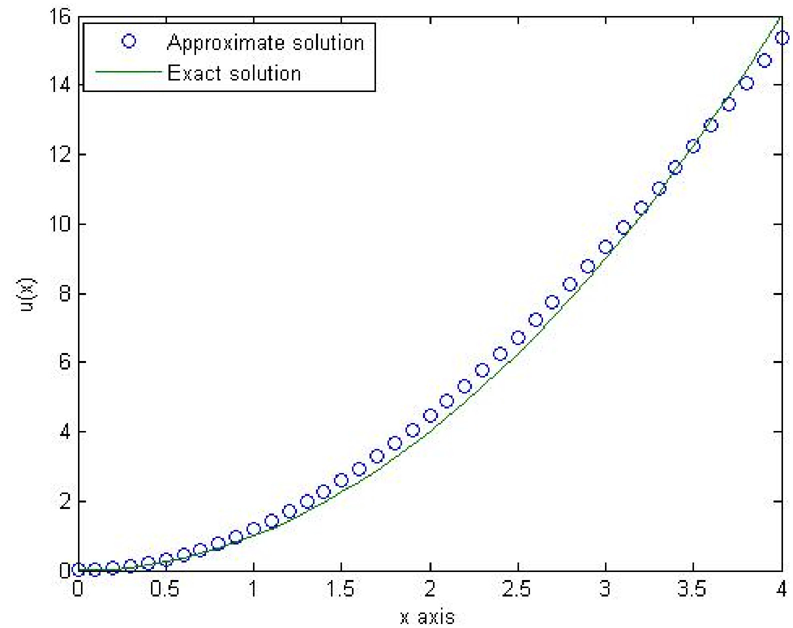

Figure 1. Comparison between the exact solution and the approximate solution.

From this figure we can conclude that the numerical results are excellent agreement with the exact solution. Also, the obtained numerical results by means of the proposed method are shown in table 1. In this table, the absolute errors between the exact solution u_{ex} and the approximate solution u_{approx}, at $m = 3$ and $m = 5$ are given. From table 1, it is evident that the overall errors can be made smaller by adding new terms from the series (28).

Table 1: The absolute error between the exact solution and the approximate solution at $m = 3$ and $m = 5$.

x	$\lvert u_{ex} - u_{approx.}\rvert$ at $m = 3$	$\lvert u_{ex} - u_{approx.}\rvert$ at $m = 5$
0.0	0.170849 e-03	0.274260 e-04
0.1	0.021094 e-03	0.420794 e-04
0.2	0.176609 e-03	0.376716 e-04
0.3	0.301420 e-03	0.844125 e-04
0.4	0.404138 e-03	0.327010 e-04
0.5	0.489044 e-03	0.361133 e-04
0.6	0.563305 e-03	0.194954 e-04
0.7	0.633367 e-03	0.295780 e-04
0.8	0.705677 e-03	0.492488 e-04
0.9	0.786679 e-03	0.283224 e-04
1.0	0.882821 e-03	0.773238 e-04

7. Conclusion and remarks

We have presented an approximate formula for the Caputo fractional derivative of the generalized Laguerre polynomials in terms of classical Laguerre polynomials themselves. We used it to solve numerically the multi-term FIVPs. Special attention is given to present study of the error analysis in which we estimate the upper bound of the error for the fractional derivatives of the approximated functions. The results are useful in the construction of a new spectral Laguerre method to obtain the solutions of FIVPs on a semi-infinite interval. Also, from the proposed examples we can find that, when we used the introduced formula. We obtained the exact solution of some examples with small m and an excellent agreement numerical solution with the exact solution in other examples. This show that the proposed method is an efficient method. All numerical results are obtained by building fast algorithms using Matlab 7.1.

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