Journal of Fractional Calculus and Applications,
Vol. 3. July 2012, No.11, pp. 1– 7.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

STATISTICAL PROPERTIES OF A TWO-ION SYSTEM WITH A LASER FIELD

M. ABDEL-ATY

Abstract. The behavior of the atomic occupation probabilities of the upper and lower levels gained by a two-ion system interacting with a laser field is investigated. At a fixed time the immediate relations of the populations are examined against several physical parameters. It is observed that for some intermediate ranges of the detuning parameters and the coupling strength, the presence of the off-resonance enhances the entanglement.

1. Introduction

The characterization of entangled states and entanglement is a challenging problem and considerable theoretical efforts have been invested in characterizing entanglement in a variety of physical situations [1, 2, 3]. Creation and characterization of entangled states of up to 8 trapped ions, the investigation of long-lived two-ion Bell-states and experiments toward entangling ions have recently been reported [4]. By coupling the electronic and vibrational states of a trapped ion with laser radiation, motional wavepackets of the ion can be manipulated and interesting states of motion, such as Fock states, squeezed states or Schrödinger-cat states, can be created which exhibit highly nonclassical behavior [5, 6, 7]. In order to observe these nonclassical features, methods for trapped ion quantum state tomography have been proposed [8] and experimental preparation of the motional state of a trapped ion, which has been initially laser cooled to the zero-point of motion, has been reported in [9]. Also, some schemes have been known for entangling two single atoms [10, 11, 12, 13, 14], different systems [15-25] as well as for entangling macroscopic atomic ensembles [26, 27]. In these schemes, feedback is typically applied to the system of interest based upon the outcome of certain measurements.

The main motivation for the present work is twofold. First, we demonstrate how a pair coherent state affect the entanglement for an ion-field interaction, and second to see how the time-dependent amplitude of the irradiating laser field affect this entanglement. In particular, we consider two three-level trapped ions (two qutrits) and discuss the roles played by the initial state setting and time-dependent amplitude of the laser field on the generation of entangled states. Most of the previous

2000 *Mathematics Subject Classification.* 81V80, 81T05, 81T70.
Key words and phrases. Ion-Field, Qubit.
Submitted May 1, 2012. Published July 1, 2012.

proposals for engineering the quantum state of a single trapped cold atom operate with two-level systems and use various extreme experimental conditions, such as the strong Raman excitation or the weak-coupling Lamb-Dicke approximations. In this paper we generalize the situation to the case, where the laser fields are applied to two three-level trapped atoms instead of two-level systems. For point like trapped atoms fixed at given positions, they can be mathematically related to each other by an amplitude shift transformation of the field mode operator. Even more intriguing effects can be expected in the quantum regime, when two time-dependent ion-field couplings exist [28-35].

The paper is presented as follows: In section 2, we present the quantized model and obtain an exact analytical solution of the system Schrödinger equation. In section 3, we analyze in detail how the pair cat state affect the general features of the negativity (a measure of the ion-field entanglement). This gives us the opportunity to stress the essential role played by pair cat states in this context and study the existence of a long living entanglement. We summarize our results at the end of the paper and make some conclusions.

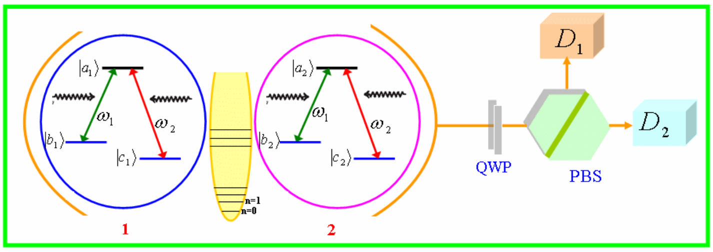

Figure 1. Experimental apparatus. Two trapped atoms are initially prepared in their excited states by laser pulses. In this figure, we denote photodetectors D_1 and D_2, quarter wave plates (QWP) and a polarizing beam splitter (PBS). We assume that the cavity is one sided so that the only leakage of photons occurs through the side facing QWP.

2. The model

In absence of the rotating wave approximation, the two-qutrit system under consideration can be specified by the Hamiltonian

$$\hat{H} = \hat{H}_{cm} + \hat{H}_a + \hat{H}_{int}, \tag{1}$$

where

$$\begin{aligned} \hat{H}_{cm} &= \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2, \\ \hat{H}_a &= \sum_{i=1}^{2} \sum_{j=a,b,c} \hbar\Omega_i S_{jj}^{(i)}, \\ \hat{H}_{int} &= \hbar\sum_{i=1}^{2} \left(\varepsilon_1(t)\zeta_1^{(i)} e^{-i(k_1\hat{x}_i+\phi_1)} S_{bc}^{(i)} + \varepsilon_2(t)\zeta_2^{(i)} e^{-i(k_2\hat{y}_i+\phi_2)} S_{ac}^{(i)} + H.c. \right). \end{aligned} \tag{2}$$

We denote by $\zeta_j^{(i)} = \frac{\overrightarrow{\mu}^{(i)}.\overrightarrow{E}_{ij}}{\hbar}$ the Rabi frequency characterizing the coupling strength, where $\overrightarrow{\mu}^{(j)}$ is the dipole moment operator. $\varepsilon_i(t)$ is a modulated amplitude of the irradiating laser field, and $\overrightarrow{E}_{ij}$ is the polarization unit direction. The operator $\hat{k}_i$ is the wave vector of the driving laser field, $\hat{x}_i$ and $\hat{y}_i$ are the center of mass position of the atoms. We denote by $\phi_i(t)$ the fluctuations in the laser phase.

The center-of-mass motion can be described in terms of annihilation and creation operators of vibrational quanta in the usual way $\hat{x}_i = \Delta x_i(\hat{a}_1^\dagger + \hat{a}_1)$, and $\hat{y}_i = \Delta y_i(\hat{a}_2^\dagger + \hat{a}_2)$, where the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the annihilation and creation operators. The quantities Δx_i and Δy_i correspond to the fluctuations in the center of mass motion in the ground state of the system. As usual, to describe this system we use the operators $S_{lm}^{(i)} = |l^{(i)}\rangle\langle m^{(i)}|, (l, m = a, b, c$ and $i = 1, 2$), where $|l^{(i)}\rangle$ denotes the eigenstate of the i^{th} atomic level with the energy $\hbar\Omega_i$. For the sake of simplicity (but without loss of generality), we shall deal with the case in which $\phi_1 = 0$, $\phi_2 = \phi$, and $\Delta = \omega_{ab} - \omega_1 = \omega_{ac} - \omega_2$.

If we express the center-of-mass position in terms of the creation and annihilation operators and apply the rotating wave approximation in the Lamb-Dicke regime we can expand the interaction Hamiltonian as

$$\begin{aligned}\hat{H}_{int} \quad = \quad & -\hbar\Delta\sum_{i=1}^{2}\left(S_{bb}^{(i)} + S_{aa}^{(i)}\right) + \hbar\gamma_1^{(1)}(t)\left(\hat{a}_1 S_{ab}^{(1)} + \hat{a}_1^\dagger S_{ba}^{(1)}\right) \\ & +\hbar\gamma_2^{(1)}(t)\left(\hat{a}_2 S_{ac}^{(1)} + \hat{a}_2^\dagger S_{ca}^{(1)}\right) + \hbar\gamma_1^{(2)}(t)\left(\hat{a}_1 S_{ab}^{(2)} e^{-i\phi} + \hat{a}_1^\dagger S_{ba}^{(2)} e^{i\phi}\right) \\ & +\hbar\gamma_2^{(2)}(t)\left(\hat{a}_2 S_{ac}^{(2)} e^{-i\phi} + \hat{a}_2^\dagger S_{ca}^{(2)} e^{i\phi}\right) \end{aligned} \tag{3}$$

with new coupling parameter $\gamma_i^{(j)}(t) = -i\eta_i\epsilon_i(t)\zeta_i^{(j)}$ including the Lamb-Dicke parameter η_i in its definition.

The wavefunction under the Hamiltonian $\hat{H}$ at any time $t > 0$ can be written as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=1}^{9}\hat{\mathcal{U}}_j(n, m, t)|\xi_j\rangle, \tag{4}$$

where $\hat{\mathcal{U}}_j(n, m, t)$ are probability amplitudes of finding the system in state $|\xi_j\rangle$. In this case, $|\xi_1\rangle = |a_1, a_2\rangle\otimes|n, m\rangle$, $|\xi_2\rangle = |b_1, a_2\rangle\otimes|n+1, m\rangle$, $|\xi_3\rangle = |a_1, b_2\rangle\otimes|n+1, m\rangle$, $|\xi_4\rangle = |c_1, a_2\rangle\otimes|n, m+1\rangle$, $|\xi_5\rangle = |a_1, c_2\rangle\otimes|n, m+1\rangle$, $|\xi_6\rangle = |b_1, c_2\rangle\otimes|n+1, m+1\rangle$, $|\xi_7\rangle = |c_1, c_2\rangle\otimes|n, m+2\rangle$, $|\xi_8\rangle = |b_1, b_2\rangle\otimes|n+2, m\rangle$, $|\xi_9\rangle = |c_1, b_2\rangle\otimes|n+1, m+1\rangle$.

To that end, differentiate both sides of equation (4) with respect to time to get [28]

$$\hat{H}|\psi(t)\rangle = i\hbar\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=1}^{9}\frac{\partial\mathcal{U}_j(n, m, t)}{\partial t}|\xi_j\rangle. \tag{5}$$

Invoking the interaction Hamiltonian $\hat{H}_{int}$ from equation (3) and $|\psi(t)\rangle$ from equation (4) into equation (5) and comparing the coefficients of $|\xi_j\rangle$ we get a closed set of nine differential equations. For an arbitrary configuration of two three-level systems we can write the general set of differential equations in the following form

$$i\hbar\frac{\partial\mathcal{U}_j(n, m, t)}{\partial t} = \Re_{j1}\mathcal{U}_1(n, m, t) + \Re_{j2}\mathcal{U}_2(n, m, t) + ... + \Re_{j9}\mathcal{U}_9(n, m, t), \tag{6}$$

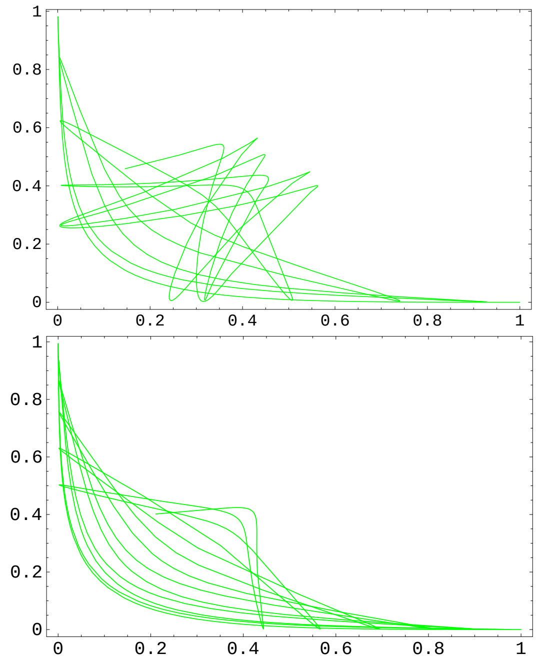

FIGURE 2. Plot of the relation between the upper state population and the lower state population. The parameters are $\Delta = 0, \phi = \pi/3$, the initial state of the system is assumed to be $|a_1, a_2, 0, 0\rangle$ and for different values of $\eta/\gamma(\gamma = \gamma_i^{(i)}$ where (a) $\eta/\gamma = 0.2$ and (b) $\eta/\gamma = 0.12$.

where $i = 1, 2, 3.., 9$ and $\Re_{ij} = \langle \xi_i | \hat{H} | \xi_j \rangle$. In order to consider the most general case, we solve equation (6), by assuming a new variable $\mathcal{G}(n, m, t)$ as [29]

$$\mathcal{G}_i(n,m,t) = \mathcal{U}_1(n,m,t) + x_1\mathcal{U}_2(n,m,t) + x_2A_3(n,m,t) + ... + x_9\mathcal{U}_9(n,m,t), \quad (7)$$

which means that

$$i\frac{d\mathcal{G}(n,t)}{dt} = z\gamma(t)\mathcal{G}(n,t), \quad (8)$$

where $\gamma_i(t) = \gamma_i\gamma(t)$.

Using the above equations, we can write

$$\mathcal{U}_j(n,m,t) = \sum_{l=1}^{9} \mathcal{U}_j(n,m,0)Q_{jl}(n,m,t)\exp\left(-iz_l\int_0^t \gamma(\tau)d\tau\right), \quad (9)$$

where $Q_{jl}(n,m,t) = O_{kj}^{-1}\mathcal{G}_j(n,m,0)$, and O_{kj}^{-1} is the inverse of the matrix O_{kj}, a 9×9 matrix whose columns are $(1,1,...,1)^T$, $(x_{11}, x_{12}, ..., x_{19})^T$,...., $(x_{81}, x_{82}, ..., x_{89})^T$ and x_{ij} are the roots for x_i, statisfing the transformation from equation (6) to equation (8).The eigenvalues of x_i in turn determine eigenvalues for z_i.

We have thus completely determined the dynamics of the two three-level system in presence of time-dependent couplings, a detuning and phase shift.

3. Occupation probabilities

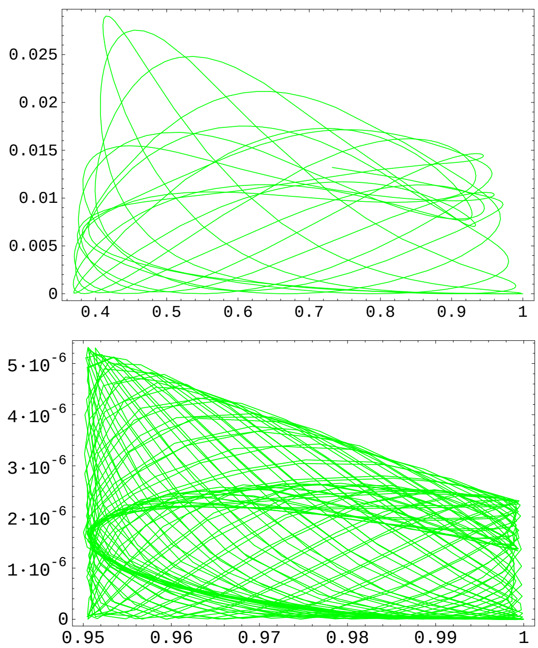

Figure 3. Plot of the relation between the upper state population and the lower state population. The parameters are $\eta/\gamma = 0.2\phi = \pi/3$, the initial state of the system is assumed to be $|a_1, a_2, 0, 0\rangle$ and for different values of Δ/γ where (a) $\Delta/\gamma = 3$ and (b) $\Delta/\gamma = 15$.

In this paper we consider the relation between the upper-state population and the lower-state population for a fixed value of the interaction time as an indicator of the entanglement between the two atoms. In Fig. 2, we plot this relation for different values of the Lamb-Dicke parameter η_i. It is shown that for small values of the Lamb-Dicke parameter symmetric behavior is observed while for larger values of this parameter, the relation between the upper-state population and the lower-state population has some sudden changes especially around the points, $x = y$.

On the other hand, the detuning effect shown in Fig. 3, gives an indication for the pure state observation. Since in Fig. 3a, where $\Delta/\gamma = 3$, we see that the upper-state population starts from lower values while the lower-state population takes the normal value (the maximum 1). This means that increasing the detuning further, the system will be completely in its lower state i.e. we obtained a complete pure state case (see Fig. 3b). Also, in order to generate the maximum entangled state we may prepare the system, in level $|b_1, b_2\rangle$ and let the atoms pass through two cavities successively, which are prepared initially in vacuum state. The transition from level $|b_1, b_2\rangle$ to other levels is in resonance with the fields. We adjust the interaction time of the atoms with first cavity field such that it sees a $\pi/(4\sqrt{2})$ pulse. Hence, there occurs equal probability of finding the atoms in state $|b_1, c_2\rangle$ and $|c_1, b_2\rangle$, leaving the cavity mode in vacuum state i.e. with a proper choice for the interaction time ($\sqrt{2}t = \pi/4$), the final state will project into a maximally entangled state.

Acknowledgments

I would like to acknowledge the support from the project 2011/17 of Deanship for Scientific Research, University of Bahrain.

4. Conclusion

In summary, we have found an exact solution for the transition probabilities in two qutrits interacting with laser fields taking into account the presence of time-dependent modulated functions and an instantaneous phase shift experienced by one of the atoms that can be easily interpreted physically, and thus provides insight into the behavior of more complicated multi-qutrit systems. The relation between the occupation probabilities are discussed as an indicator of the entanglement between the two qutrits has been discussed. It is shown that the detuning plays an important roles in this system for obtaining the pure state.

5. Acknowledgments

We would like to acknowledge the support from the project 2011/17 of Deanship for Scientific Research, University of Bahrain.

References

[1] C. H. Bennett, H. J. Bernstein, S. Popesu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[2] V. Vedral, M. B. Plenio, K. Jacobs, and P. L. Knight, Phys. Rev. A **56**, 4452 (1997).

[3] S. J. D. Phoenix and P. L. Knight, Ann. Phys. (N. Y) **186**, 381 (1988); Phys. Rev. A **44**, 6023 (1991); Phys. Rev. Lett. **66**, 2833 (1991).

[4] C. Becher, et. al., Laser Spectroscopy, eds. E. A. Hinds, A. Ferguson, E. Riis, pp. 381-392, World Scientific, 2005.

[5] For an overview of theoretical and experimental work on quantum dynamics of single trapped ions, see, for example, D. Leibfried, R. Blatt, C. Monroe and D. Wineland, Rev. Mod. Phys., **75**, 281 (2003).

[6] C. Monroe, D. M. Meekhof, B. E. King, S. R. Jeffers, W. M Itano, D. J. Wineland, and P. Gould, Phys. Rev. Lett. **75**, 4011 (1995).

[7] C. Monroe, D. M. Meekhof, B. E. King, D. Leibfried, W. M Itano and D. J. Wineland, Acc. Chem. Res. **29**, 585 (1996).

[8] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano and D. J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996); J. F. Poyatos, R. Walser, J. I. Cirac, R. Blatt, and P. Zoller, Phys. Rev. A **53**, R1966 (1996).

[9] D. Leibfried, D.M. Meekhof, B.E. King, C. Monroe, W.M. Itano, and D.J. Wineland, J. Mod. Opt. **44**, 2485 (1997).

[10] C. Cabrillo, J. I. Cirac, P. G-Fernandez, P. Zoller, Phys. Rev. A 59, 1025 (1999).

[11] S. Bose, P. L. Knight, M. B. Plenio, V. Vedral, Phys. Rev. Lett. 83, 5158 (1999).

[12] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A 59, 2468 (1999).

[13] J. Hong and H.-W. Lee, Phys. Rev. Lett. **89**, 237901 (2002).

[14] A. S. Sorensen, K. Molmer, quant-ph/0206142

[15] M. Sebawe Abdalla and L. Thabet, Appl. Math. Inf. Sci. **5**, 570 (2011).

[16] A. El-Barakaty, M. Darwish and A.-S. F. Obada, Appl. Math. Inf. Sci. **5**, 122 (2011).

[17] Z. Ficek, Appl. Math. Inf. Sci. **3**, 375 (2009).

[18] Li-Hui Sun, Gao-Xiang Li and Z. Ficek, Appl. Math. Inf. Sci. **4**, 315 (2010).

[19] H. Eleuch, Appl. Math. Inf. Sci. **3**, 185 (2009).

[20] F. N. M. Al-Showaikh, Appl. Math. Inf. Sci. **2**, 21 (2008).

[21] N. Metwally and M. R. B. Wahiddin, Appl. Math. Inf. Sci. **1**, 23 (2007).

[22] E. M. Khalil and G. M. Abd Al-Kader, Appl. Math. Inf. Sci. **1**, 35 (2007).

[23] S. Abdel-Khalek, Appl. Math. Inf. Sci. **1** 53 (2007).

[24] A. Becir, A. Messikh, and M. R. B. Wahiddin, Appl. Math. Inf. Sci. **1**, 95 (2007).
[25] N. H. Abdel-Wahab, Appl. Math. Inf. Sci. **1**, 263 (2007).
[26] L. M. Duan, M. D. Lukin, J. I. Cirac, P,. Zoller, Nature 414, 413 (2001); Y.-C. Ou, C.-H. Yuan and Z.-M. Zhang, J. Phys. B: At. Mol. Opt. Phys. **39**, 7 (2006)
[27] L. M. Duan, Phys. Rev. Lett. 88, 170402 (2002).
[28] R. R. Puri, J. Mod. Opt. **46**, 1465 (1999)
[29] B. W. Shore and P. L. Knight, J. Mod. Opt. **40**, 1195 (1993); A. M. Abdel-Hafez, A. M. M. Abu-Sitta, A.-S. F. Obada: Physica A, **156**, 689 (1989); J. H. Mc-Guire, K. K. Shakov and K. Y. Rakhimov, J. Phys. B: At. Mol. Opt. Phys. **36**, 3145 (2003).
[30] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[31] A. Peres, Phys. Rev. Lett. 77, 1413 (1996); W. K. Wootters, Phys. Rev. Lett. **80** , 2245 (1998).
[32] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996)
[33] J. Lee and M. S. Kim, Phys. Rev. Lett. 84, 4236 (2000); J. Lee, M. S. Kim, Y. J. Park and S. Lee, J. Mod. Opt. **47**, 2151 (2000).
[34] G. S. Agarwal, J. Opt. Soc. Am. B **5**, 1940 (1988); G. S. Agarwal, Phys. Rev. Lett. **57**, 827 (1986).
[35] P. Lougovski, E. Solano, and H. Walther, Phys. Rev. A **71**, 013811 (2005)

M. Abdel-Aty
Scientific Publishing Center, University of Bahrain 32038 Kingdom of Bahrain, Mathematics Department, Faculty of Science, Sohag University, Sohag, Egypt
E-mail address: `mabdelaty@uob.edu.bh`