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MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL EIGENVALUE PROBLEM WITH NONLOCAL CONDITIONS

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Abstract. The nonlinear fractional nonlocal boundary value problem

$$D_{0+}^{\alpha}u(t)+\lambda g(t)f(t,u(t))=0,\ \ t\in(0,1),\quad n-1<\alpha\leq n,$$
$$u(0)=0,\quad u^{(k)}(0)=0,\quad 1\leq k\leq n-2,\quad u''(1)=\theta[u],$$

is considered under some conditions concerning the principal characteristic value to the relevant linear operator, where $n-1<\alpha\leq n$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and $\theta[u]=\int_0^1 u(s)\,dA(s)$ is given by Riemann-Stieltjes integral with a signed measure. The existence of positive solutions is obtained by means of the fixed point index theory in cones.

1. Introduction

The purpose of this paper is to study the existence of a positive solutions for the following boundary value problem

$$D_{0+}^{\alpha}u(t)+\lambda g(t)f(t,u(t))=0,\ \ t\in(0,1),\quad n-1<\alpha\leq n,\tag{1}$$

with the nonlocal BCs

$$u(0)=0,\quad u^{(k)}(0)=0,\quad 1\leq k\leq n-2,\quad u''(1)=\theta[u],\tag{2}$$

where $\lambda>0$ is a parameter and $\theta[u]$ is given by a Riemann-Stieltjes integral

$$\theta[u]=\int_0^1 u(s)\,dA(s).\tag{3}$$

This type of BC includes, as particular cases, multi-point problems when $\theta[u]=\sum_{i=1}^{m-2}\alpha_i u(\zeta_i)$, (see [1,15,18,29]), and a continuously distributed case when $\theta[u]=\int_0^1\alpha(s)\,u(s)\,ds$,(see[4]).

The nonlocal BVPs have been studied extensively. The methods used therein mainly depend on the fixed-point theorems, degree theory, upper and lower solutions techniques, and monotone iteration. The existence results are available in

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the literature [2, 3, 5-10, 12, 16, 21, 26-28]. Recently, Wang et al. [19] studied the nonlocal BVP

$$D_{0+}^{\alpha}u(t)+q(t)f(t,u(t))=0,\quad t\in(0,1),\quad n-1<\alpha\leq n.$$
$$u(0)=u'(0)=...=u^{(n-2)}(0)=0,\quad u(1)=\int_0^1 u(s)\,dA(s),$$

where $\alpha\geq 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $q(t)$ may be singular at $t=0$ and/or $t=1$, $f(t,u)$ may also have singularity at $u=0$. $\int_0^1 u(s)\,dA(s)$ denotes the Riemann- Stieltjes integral with a signed measure. It is worth mentioning that the idea using a Riemann-Stieltjes integral with a signed measure is due to Webb and Infante in [23, 24]. The papers [13, 20-25] contain several new ideas, and give a unified approach to many BVPs.

In this paper, we obtain the results on the existence of one and two positive solution by utilizing the results of Webb and Lan [25] involving comparison with the principal characteristic value of a related linear problem to the fractional case. We then use the theory worked out by Webb and Infante in [22, 23] to study the general nonlocal BCs.

2. Preliminaries

In this section, we will present some definitions and lemmas that will be used in the proof of our main results.

Definition 2.1([14, 17]). The fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\rightarrow R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds,$$

provided that the integral on the right-hand side converges.

Definition 2.2([14, 17]). The standard Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:(0,\infty)\rightarrow R$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}y(s)\,ds,$$

where $n=[\alpha]+1$, provided that the integral on the right-hand side converges.

Definition 2.3([11]). Let E be a real Banach space. A nonempty closed convex set $K\subset E$ is called cone of E if it satisfies the following conditions

$$\begin{array}{ll}(1) & x\in K,\ \sigma\geq 0\quad \textit{implies}\quad \sigma x\in K;\\ (2) & x\in K,\ -x\in K\quad \textit{implies}\quad x=0.\end{array}$$

Definition 2.4([11]). An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Lemma 2.1([11, 30]). Suppose $T:K\rightarrow K$ is a completely continuous operator and has no fixed points on $\partial K_\rho\bigcap K$. Then the following are true:

(i) If $\|Tu\|\leq\|u\|$ for all $u\in\partial K_\rho\bigcap K$, then $i(T,K_\rho\bigcap K,K)=1$, where i is the fixed point index on K.

(ii) If $\|Tu\|\geq\|u\|$ for all $u\in\partial K_\rho\bigcap K$, then $i(T,K_\rho\bigcap K,K)=0$.

Lemma 2.2([11, 30]). Let K be a cone in Banach space E. Suppose that $T:\bar{K}_\rho\rightarrow K$ is a completely continuous operator. If there exists $u_0\in K\backslash\{0\}$ such that $u-Tu\neq\mu u_0$ for any $u\in\partial K_r$, and $\mu\geq 0$,$i(T,K_\rho,K)=0$.

Lemma 2.3([11, 30]). Let K be a cone in Banach space E. Suppose that $T:\bar{K}_\rho\to K$is a completely continuous operator. If $Tu\neq\mu u$ for any $u\in\partial K_r$ and $\mu\geq 1$, then $i\left(T,K_\rho,K\right)=1$.

Lemma 2.4([14]). Assume that $u\in C\left(0,1\right)\cap L\left(0,1\right)$ with a fractional derivative of order $\alpha>0$ that belongs to $C\left(0,1\right)\cap L\left(0,1\right)$. Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u\left(t\right)=u\left(t\right)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+...+c_nt^{\alpha-n}.$$

Lemma 2.5. Let $y\left(t\right)\in C\left[0,1\right]$ be a given function and $n-1<\alpha\leq n$, then $u\left(t\right)$ is a solution of BVP (1) − (2) if and only if $u\left(t\right)$ is a solution of the integral equation:

$$u\left(t\right)=\gamma\left(t\right)\theta\left[u\right]+\int_0^1 G_0\left(t,s\right)y\left(s\right)ds, \tag{4}$$

where

$$\gamma\left(t\right)=\frac{t^{\alpha-1}}{\left(\alpha-1\right)\left(\alpha-2\right)},\quad G_0\left(t,s\right)=\begin{cases}\frac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\leq s\leq t\leq 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}, & 0\leq t\leq s\leq 1.\end{cases} \tag{5}$$

Proof. Assume that $u\left(t\right)$ is a solution of BVP (1)-(2). Applying Lemma 2.4, (1) can be reduced to an equivalent integral equation

$$u\left(t\right)=-\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}y\left(s\right)ds+c_1t^{\alpha-1}+c_2\,t^{\alpha-2}+...+c_n\,t^{\alpha-n}. \tag{6}$$

By (2), we obtain
$c_n=...=c_2=0$, and $c_1=\frac{\theta[u]}{(\alpha-1)(\alpha-2)}+\frac{1}{\Gamma(\alpha)}\int_0^1\left(1-s\right)^{\alpha-3}y\left(s\right)ds.$
Therefore, we obtain

$$\begin{aligned}u\left(t\right)&=\tfrac{t^{\alpha-1}}{(\alpha-1)(\alpha-2)}\theta\left[u\right]+\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1\left(1-s\right)^{\alpha-3}y\left(s\right)ds-\tfrac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}y\left(s\right)ds\\&=\gamma\left(t\right)\theta\left[u\right]+\int_0^t\tfrac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y\left(s\right)ds+\int_t^1\tfrac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}y\left(s\right)ds\\&=\gamma\left(t\right)\theta\left[u\right]+\int_0^1G_0\left(t,s\right)y\left(s\right)ds.\end{aligned}$$

Conversely, if $u\left(t\right)$ is a solution of the integral equation (4), using the relation $D^\alpha t^{\alpha-m}=0$, where $m=1,2,...,n$, where n is the smallest integer greater than or equal to α, we have

$$\begin{aligned}D_{0+}^{\alpha}u\left(t\right)&=D_{0+}^{\alpha}t^{\alpha-1}\left(\tfrac{\theta[u]}{\alpha-1}\right)+D_{0+}^{\alpha}t^{\alpha-1}\left(\int_0^1\tfrac{(1-s)^{\alpha-3}}{\Gamma(\alpha)}y\left(s\right)ds\right)\\&\quad-D_{0+}^{\alpha}\left(\int_0^t\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y\left(s\right)ds\right)\\&=-D_{0+}^{\alpha}I^{\alpha}y\left(t\right)=-y\left(t\right).\end{aligned}$$

A simple computation showed $u\left(0\right)=0,\quad u^{(k)}\left(0\right)=0,\quad 1\leq k\leq n-2,\quad u''\left(1\right)=\theta\left[u\right]$.

Remark 2.1. $G_0\left(t,s\right)$is the Green's function for the local BVP

$$\begin{aligned}&D_{0+}^{\alpha}u\left(t\right)+\lambda g\left(t\right)f\left(t,u\left(t\right)\right)=0,\ \ t\in\left(0,1\right),\,n-1<\alpha\leq n,\\&u\left(0\right)=0,\,u^{(k)}\left(0\right)=0,\,1\leq k\leq n-2,\,u''\left(1\right)=0.\end{aligned} \tag{7}$$

Lemma 2.6. $G_0\left(t,s\right)$ has the following properties
(i) $G_0(t,s)\geq 0$ is continuous for all $t,s\in\left[0,1\right]$;
(ii) $c_0\left(t\right)\Phi_0\left(s\right)\leq G_0(t,s)\leq\Phi_0\left(s\right),\quad\forall t,s\in\left[0,1\right],$
where

$$\Phi_0\left(s\right)=G_0(1,s)=\frac{\left(1-s\right)^{\alpha-3}-\left(1-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)},c_0\left(t\right)=t^{\alpha-1}.$$

Proof. It is obvious that $G_0(t,s)$is nonnegative and continuous.
(i) For $s \leq t$, we have

$$\begin{aligned}\frac{\partial G_0(t,s)}{\partial t} &= \frac{(\alpha-1)}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-3}-(t-s)^{\alpha-2}\right]\\ &\geq \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-3}-(t-s)^{\alpha-2}\right]\\ &\geq \frac{(\alpha-1)t^{\alpha-2}s(1-s)^{\alpha-3}}{\Gamma(\alpha)} \geq 0,\end{aligned}$$

and

$$\frac{G_0(t,s)}{\Phi_0(s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{(1-s)^{\alpha-3}-(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

For $s \geq t$, we have

$$\frac{\partial G_0(t,s)}{\partial t} = \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-3}}{\Gamma(\alpha)} \geq 0,$$

and

$$\frac{G_0(t,s)}{\Phi_0(s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{(1-s)^{\alpha-3}-(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

Thus, (i) holds.
Defining $\mathrm{G}_A(s) = \int_0^1 G_0(t,s)\,dA(t)$, it is shown in [21] that the Green's function for nonlocal BVP (1)-(2) is given by

$$G(t,s) = \frac{\gamma(t)}{[1-\theta[\gamma]]}\mathrm{G}_A(s) + G_0(t,s). \tag{8}$$

By similar arguments to [23], we obtain the following Lemma.
Lemma 2.7. If G_0 satisfies (i), (ii), then G satisfies (i), (ii) for a function Φ, with the same interval $[a,b]$and the same constant c, where

$$\Phi(s) = \Phi_0(s) + \frac{\|\gamma\|}{[1-\theta[\gamma]]}\mathrm{G}_A(s),$$

$\Phi_0(s)$ defined in Lemma 2.6, and $c = \min\{c_0(t), \quad t \in [a,b]\}$
Proof. We have

$$\begin{aligned}G(t,s) &= \frac{\gamma(t)}{[1-\theta[\gamma]]}\mathrm{G}_A(s) + G_0(t,s)\\ &\leq \frac{\|\gamma\|}{[1-\theta[\gamma]]}\mathrm{G}_A(s) + \Phi_0(s) =: \Phi(s),\end{aligned}$$

and for $t \in [a,b]$

$$G(t,s) \geq \frac{c\|\gamma\|}{[1-\theta[\gamma]]}\mathrm{G}_A(s) + c\Phi_0(s) = c\Phi(s).$$

Note that $g\Phi \in L^\infty$ because A has finite variation and $\mathrm{G}_A(s) \leq \Phi(s)\,var(A)$.
Thus , the Green's function $G(t,s)$ satisfies (i) , (ii) for a function Φ and the constant c. Throughout the paper we assume that:
(iii) A is a function of bounded variation, and $\mathrm{G}_A(s) = \int_0^1 G_0(t,s)\,dA(t)$ satisfies $\mathrm{G}_A(s) \geq 0$ for a. e. $s \in [0,1]$. Note that $\mathrm{G}_A(s)$ exists for a. e. $s \in [0,1]$by (i).
(iv) The functions g, Φ satisfy $g \geq 0$ almost everywhere, $g\Phi \in L^1[0,1]$, and

$$\int_a^b \Phi(s)\,g(s)\,ds > 0.$$

(v) $f : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Caratheodory conditions, that is, $f(\cdot,u)$ is measurable for each fixed $u \in [0,\infty)$and $f(t,\cdot)$ is continuous for almost every

$t \in [0,1]$, and for each $r > 0$, there exists $\phi_r \in L^\infty[0,1]$ such that $0 \le f(t,u) \le \phi_r$ for all $u \in [0,r]$and almost all $t \in [0,1]$.
(vi) $\gamma \in C[0,1]$, $\quad \gamma(t) \ge 0, \quad 0 \le \theta[\gamma] < 1.$

3. Main Result

Set $E = C[0,1]$is a Banach space with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Let $P = \{u \in E : u \ge 0\}$ denote the standard cone of non-negative functions. Define

$$K = \left\{ u \in P, \min_{a \le t \le b} u(t) \ge c\|u\| \right\}, \tag{9}$$

where $[a,b]$ is some subset of $[0,1]$ and $c = \min\{c_0(t) : t \in [a,b]\}$.
Note that $\gamma \in K$ so $K \ne \{0\}$. For any $0 < r < R < +\infty$, let $K_r = \{u \in K : \|u\| < r\}$, $\partial K_r = \{u \in K : \|u\| = r\}$, $\bar{K}_r = \{u \in K : \|u\| \le r\}$, $\bar{K}_R \backslash K_r = \{u \in K : r \le \|u\| \le R\}$ and $V_r = \left\{ u \in K : \min_{t\in[a,b]} u(t) < r \right\}$and V_r is bounded for K. Recall that a cone K in Banach space E is said to be reproducing if $E = K - K$, and is a total cone if $E = \overline{K - K}$. Define a nonlinear operator $T : P \to K$ and a linear operator $L : P \to K$ by

$$Tu(t) = \lambda \int_0^1 G(t,s)\, g(s)\, f(s, u(s))\, ds. \tag{10}$$

and

$$Lu(t) := \int_0^1 G(t,s)\, g(s)\, u(s)\, ds. \tag{11}$$

Lemma 3.1([24]). Under the hypotheses (i) −(vi) the maps $T : P \to E$ defined in (10) is compact.
Theorem 3.1. Under the hypotheses (i) −(vi) the map $T : P \to K$.
Proof.
For $u \in P$ and $t \in [0,1]$ we have :

$$Tu(t) \le \lambda \int_0^1 \Phi(s)\, g(s)\, f(s, u(s))\, ds.$$

Hence,

$$\|Tu\| \le \lambda \int_0^1 \Phi(s)\, g(s)\, f(s, u(s))\, ds.$$

Also , for $t \in [a,b]$, we have :

$$\begin{aligned} Tu(t) &\ge c\lambda \textstyle\int_0^1 \Phi(s)\, g(s)\, f(s, u(s))\, ds \\ &\ge c\|Tu\|. \end{aligned}$$

Similar to the proofs of Lemma 3.1 and Theorem 3.1, $Lu(t)$ is compact and maps P into K. We shall use the Krein-Rutman theorem. We recall that λ is an eigenvalue of L with corresponding eigenfunction ϕif $\phi \ne 0$ and $\lambda\phi = L\phi$. The reciprocals of eigenvalues are called characteristic values of L. The radius of the spectrum of L,denoted $r(L)$, is given by the well-known spectral radius formula $r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}$.

Theorem 3.2. **[25]** Let K be a total cone in a real Banach space E and let $\hat{L}: E \to E$ be a compact linear operator with $\hat{L}(K) \subseteq K$. If $r\left(\hat{L}\right) > 0$ then there is $\phi_1 \in K \backslash \{0\}$ such that $\hat{L}\phi_1 = r\left(\hat{L}\right)\phi_1$.

Thus $\lambda_1 := r\left(\hat{L}\right)$ is an eigenvalue of $\hat{L}$, the largest possible real eigenvalue and $\mu_1 = \frac{1}{\lambda_1}$ is the smallest positive characteristic value.

Lemma 3.2. **[25]**
Assume that (i) −(iv) hold and let L be as defined in (11). Then $r(L) > 0$.

Theorem 3.3.
When (i) −(iv) hold, $r(L)$ is an eigenvalue of L with eigenfunction ϕ_1 in K.

Proof. $r(L)$ is an eigenvalue of L with eigenfunction in P, by Theorem 3.2. As L maps P into K, the eigenfunction belongs to K.

Theorem 3.4 ([25]). Let $\mu_1 = 1/r(L)$ and $\phi_1(t)$ be a corresponding eigenfunction in P of norm 1. Then $m \le \mu_1 \le M$, where

$$m = \left(\sup_{t\in[0,1]} \int_0^1 G(t,s)\, g(s)\, ds\right)^{-1}, \quad M = \left(\inf_{t\in[a,b]} \int_a^b G(t,s)\, g(s)\, ds\right)^{-1}. \quad (12)$$

If $g(t) > 0$ for $t \in [0,1]$ and $G(t,s) > 0$ for $t,s \in [0,1]$, the first inequality is strict unless $\phi_1(t)$ is constant for $t \in [0,1]$. If $g(t)\,\phi(t) > 0$for $t \in [a,b]$, the second inequality is strict unless $\phi_1(t)$ is constant for $t \in [a,b]$.

For the local BVP (7) **if** $g(t) \equiv 1$**:**
We now compute the constant m and the optimal value of $M(a,b)$, that is, we determine a, b so that $M(a,b)$ is minimal.
For $s \le t$, we have by direct integration

$$\begin{aligned} \textstyle\int_0^t G_0(t,s)\, ds &= \textstyle\int_0^t \left[\frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right] ds \\ &= -\tfrac{t^{\alpha-1}(1-t)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)} + \tfrac{t^{\alpha-1}}{(\alpha-2)\Gamma(\alpha)} - \tfrac{t^{\alpha}}{\alpha\Gamma(\alpha)}. \end{aligned}$$

For $s \ge t$,

$$\int_t^1 G_0(t,s)\, ds = \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} ds = \frac{t^{\alpha-1}(1-t)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)}.$$

Then we have :

$$\int_0^1 G_0(t,s)\, ds = \frac{t^{\alpha-1}}{(\alpha-2)\Gamma(\alpha)} - \frac{t^{\alpha}}{\alpha\Gamma(\alpha)}.$$

And the maximum of this expression occurs when $t = 1$, hence

$$\sup_{t\in[0,1]} \int_0^1 G_0(t,s)\, ds = \frac{1}{(\alpha-2)\Gamma(\alpha)} - \frac{1}{\alpha\Gamma(\alpha)} = \frac{2}{(\alpha-2)\Gamma(\alpha+1)}.$$

Then $m = \frac{(\alpha-2)\Gamma(\alpha+1)}{2}$.
For $a < b$, we have by direct integration

$$\begin{aligned} \textstyle\int_a^t G_0(t,s)\, ds &= -\tfrac{t^{\alpha-1}(1-t)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)} + \tfrac{t^{\alpha-1}(1-a)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)} - \tfrac{(t-a)^{\alpha}}{\alpha\Gamma(\alpha)}, \\ \textstyle\int_t^b G_0(t,s)\, ds &= -\tfrac{t^{\alpha-1}(1-b)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)} + \tfrac{t^{\alpha-1}(1-t)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)}. \end{aligned}$$

Then

$$\begin{aligned}\int_a^b G_0(t,s)\,ds &= \tfrac{t^{\alpha-1}(1-a)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)} - \tfrac{(t-a)^\alpha}{\alpha\Gamma(\alpha)} - \tfrac{t^{\alpha-1}(1-b)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)} \\ &= \tfrac{1}{(\alpha-2)\Gamma(\alpha)}\left[t^{\alpha-1}\left((1-a)^{\alpha-1}-(1-b)^{\alpha-1}\right) - \tfrac{(\alpha-2)}{\alpha}(t-a)^\alpha\right] \\ &= R(t,a,b),\end{aligned}$$

$$\frac{\partial R(t,a,b)}{\partial t} = \frac{1}{(\alpha-2)\Gamma(\alpha)}\left[(\alpha-1)t^{\alpha-2}\left((1-a)^{\alpha-2}-(1-b)^{\alpha-2}\right)-(\alpha-2)(t-a)^{\alpha-1}\right].$$

The sign of derivative $\frac{\partial R}{\partial t}$ shows that this is an increasing function of t so the minimum occurs at $t=a$. Let

$$R(a,b) = \frac{a^{\alpha-1}}{(\alpha-2)\Gamma(\alpha)}\left((1-a)^{\alpha-2}-(1-b)^{\alpha-2}\right).$$

The minimal value of $M(a,b)$corresponds to the maximal value of $R(a,b)$.

$$\frac{\partial R(a,b)}{\partial b} = \frac{a^{\alpha-1}(1-b)^{\alpha-3}}{\Gamma(\alpha)} > 0.$$

The quantity $R(a,b)$is an increasing function of b so its maximum is when $b=1$. Let

$$R(a) = \frac{a^{\alpha-1}(1-a)^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)}.$$

Then the maximal of $R(a)$occurs when $a=\frac{\alpha-1}{2\alpha-3}$

$$\min_{t\in[a,b]}\int_a^b G_0(t,s)\,ds = R\left(\frac{\alpha-1}{2\alpha-3},1\right) = \frac{(\alpha-1)^{\alpha-1}}{(2\alpha-3)\Gamma(\alpha)}.$$

Hence the minimal value of $M(a,b)$is :

$$M\left(\frac{\alpha-1}{2\alpha-3},1\right) = \frac{(2\alpha-3)\Gamma(\alpha)}{(\alpha-1)^{\alpha-1}}.$$

4. The existence of at least one positive solution

For convenience, we introduce the following notations

$$\begin{aligned}&\bar f(u) := \sup_{t\in[0,1]} f(t,u), \qquad \underline{f}(u) := \inf_{t\in[0,1]} f(t,u);\\ &f^0 := \limsup_{u\to0^+}\bar f(u)/u, \quad f_0 := \liminf_{u\to0^+}\underline{f}(u)/u;\\ &f^\infty := \limsup_{u\to\infty}\bar f(u)/u, \quad f_\infty := \liminf_{u\to\infty}\underline{f}(u)/u,\end{aligned}$$

$$f^{0,r} := \sup_{\{0\le t\le1,0\le u\le r\}} f(t,u)/r,\, f_{r,r/c} := \inf_{\{a\le t\le b,r\le u\le r/c\}} f(t,u)/r.$$

Under the hypotheses (i)-(iv) let $\tilde L$ be defined by :

$$\tilde L u(t) = \int_a^b G(t,s)\,g(s)\,u(s)\,ds.$$

Then $\tilde L$ is a compact linear operator and $\tilde L(P)\subseteq K$.
Hence $r\left(\tilde L\right)$ is an eigenvalue of $\tilde L$ with an eigenfunction $\tilde\phi_1$ in K. Let $\tilde\mu_1 := \frac{1}{r(\tilde L)}$.
Note that $\tilde\mu_1\ge\mu_1$, hence the condition in the following theorem is more stringent than if could use$r(L)$.

Theorem 4.1. Assume that

(A1) $0 \leq \lambda f^0 < \mu_1$,
(A2) $\tilde{\mu}_1 < \lambda f_\infty \leq \infty$.
Then (1)-(2) has at least one positive solution.
Proof. (A1) Let $\varepsilon > 0$ be such that $f^0 \leq \frac{1}{\lambda}(\mu_1 - \varepsilon)$. Then there exists $\rho_0 > 0$ such that $f(t,u) \leq \frac{1}{\lambda}(\mu_1 - \varepsilon) u$ for all $u \in [0, \rho_0]$ and almost all $t \in [0,1]$. Let $\rho \in (0, \rho_0]$, we prove that

$$Tu \neq \beta u \,\text{for}\, u \in \partial K_\rho \quad \text{and} \quad \beta \geq 1, \tag{13}$$

which implies the result. In fact , if (13) doesn't hold, then there exist $u \in \partial K_\rho$ and $\beta \geq 1$ such that $Tu = \beta u$. This implies

$$\begin{aligned}\beta u(t) &= \lambda \textstyle\int_0^1 G(t,s)\, g(s)\, f(s, u(s))\, ds \\ &\leq (\mu_1 - \varepsilon) \textstyle\int_0^1 G(t,s)\, g(s)\, u(s)\, ds = (\mu_1 - \varepsilon) Lu(t).\end{aligned}$$

Thus , we have shown $u(t) \leq (\mu_1 - \varepsilon) Lu(t)$. This gives

$$u(t) \leq (\mu_1 - \varepsilon) L[(\mu_1 - \varepsilon) Lu(t)] = (\mu_1 - \varepsilon)^2 L^2 u(t),$$

and iterating $u(t) \leq (\mu_1 - \varepsilon)^n L^n u(t)$ for $n \in N$. Therefore

$$\begin{aligned}&\|u\| \leq (\mu_1 - \varepsilon)^n \|L^n\| \|u\| \\ &1 \leq (\mu_1 - \varepsilon)^n \|L^n\|,\end{aligned}$$

and we have

$$1 \leq (\mu_1 - \varepsilon) \lim_{n \to 0} \|L^n\|^{1/n} = (\mu_1 - \varepsilon) \frac{1}{\mu_1} < 1,$$

a contradiction. It follows that

$$i_k(T, K_\rho) = 1, \text{ for each}\, \rho \in (0, \rho_0]. \tag{14}$$

(A2) Let $\rho_1 > 0$, $\rho_1 > \rho$ be chosen so that $f(t,u) > \frac{\tilde{\mu}_1}{\lambda} u$ for all $u \geq c\rho_1$, c as in (ii) and almost all $t \in [0,1]$.
We claim that $u \neq Tu + \beta \tilde{\phi}_1$ for all $\beta > 0$ and $u \in \partial K_{\rho^*}$ when $\rho^* > \rho_1$. Note that for $u \in K$with $\|u\| = \rho^* \geq \rho_1$.
We have $u(t) \geq c\rho_1$ for all $t \in [a,b]$.
Now, if our claim is false , then we have

$$u(t) = \lambda \int_0^1 G(t,s)\, g(s)\, f(s, u(s))\, ds + \beta \tilde{\phi}_1(t).$$

Therefore,

$$\begin{aligned}u(t) &\geq \tilde{\mu}_1 \textstyle\int_a^b G(t,s)\, g(s)\, u(s)\, ds + \beta \tilde{\phi}_1(t) \\ &= \tilde{\mu}_1 \tilde{L} u(t) + \beta \tilde{\phi}_1(t).\end{aligned} \tag{15}$$

From (15) we firstly deduce that $u(t) \geq \beta \tilde{\phi}_1(t)$ on $[a,b]$. Then we have

$$\tilde{\mu}_1 \tilde{L} u(t) \geq \tilde{\mu}_1 \tilde{L}\left(\beta \tilde{\phi}_1(t)\right) = \beta \tilde{\phi}_1(t).$$

Inserting this into (15) we obtain $u(t) \geq 2\beta \tilde{\phi}_1(t)$ for $t \in [a,b]$. Repeating this process gives $u(t) \geq n\beta \tilde{\phi}_1(t)$ for $t \in [a,b]$, $n \in N$. Since $\tilde{\phi}_1(t)$is strictly positive on $[a,b]$ this is a contradiction, then

$$i_K(T, K_{\rho^*}) = 0, \text{for}\, u \in \partial K_{\rho^*}. \tag{16}$$

By (14) and (16), one has

$$i_K\left(T, K_{\rho^*} \backslash \bar{K}_\rho\right) = i_K(T, K_{\rho^*}) - i_K(T, K_\rho) = -1.$$

Therefore, T has at least one fixed point $u_0 \in K_{\rho^*}\backslash\bar{K}_\rho$, and u_0 is a positive solution of BVP (1)-(2).

Theorem 4.2. Assume that

(A3) $\mu_1 < \lambda f_0 \leq \infty$,

(A4) $0 \leq \lambda f^\infty < \mu_1$.

Then (1)-(2) has at least one positive solution.

Proof. (A3) Let $\varepsilon > 0$ satisfy $f_0 > \frac{1}{\lambda}(\mu_1 + \varepsilon)$. Then there exists $R_1 > 0$ such that

$$f(t,u) \geq \frac{1}{\lambda}(\mu_1 + \varepsilon)\, u\, forallt \in [0,1],\ u \in [0,R_1]. \tag{17}$$

For any $u \in \partial K_{R_1}$we have by (17) that

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)\, g(s)\, f(s,u(s))\, ds \\ &\geq (\mu_1 + \varepsilon) \int_0^1 G(t,s)\, g(s)\, u(s)\, ds \\ &\geq \mu_1 Lu(t), \quad \forall t \in [0,1]. \end{aligned} \tag{18}$$

Let $\tilde{u}_1$ be the positive eigenfunction of L corresponding to μ_1, that $\tilde{u}_1 = \mu_1 L\tilde{u}_1$. We may suppose that T has no fixed point on ∂K_{R_1}, otherwise, the proof is finished. In the following we will show that

$$u - Tu \neq \beta\tilde{u}_1 \text{ for all } u \in \partial K_{R_1},\ \beta \geq 0. \tag{19}$$

If (19) is not true, then there is $\tilde{u}_0 \in \partial K_{R_1}$ and $\beta_0 \geq 0$ such that $\tilde{u}_0 - T\tilde{u}_0 = \beta_0\tilde{u}_1$. It is clear that $\beta_0 > 0$ and $\tilde{u}_0 = T\tilde{u}_0 + \beta_0\tilde{u}_1 \geq \beta_0\tilde{u}_1$. Set

$$\beta^* = \sup\{\beta : \quad \tilde{u}_0 \geq \beta\tilde{u}_1\}. \tag{20}$$

Obviously, $\beta^* \geq \beta_0 > 0$. It follows from $L(P) \subset P$ that

$$\mu_1 L\tilde{u}_0 \geq \mu_1 L\beta^*\tilde{u}_1 = \beta^*\mu_1 L\tilde{u}_1 = \beta^*\tilde{u}_1.$$

Using this and (18), we have

$$\tilde{u}_0 = T\tilde{u}_0 + \beta_0\tilde{u}_1 \geq \mu_1 L\tilde{u}_0 + \beta_0\tilde{u}_1 \geq \beta^*\tilde{u}_1 + \beta_0\tilde{u}_1,$$

which contradicts (24). Thus, (19) holds.

By Lemma 2.2, we have

$$i_K(T, K_{R_1}) = 0. \tag{21}$$

On the other hand, Let $\varepsilon > 0$ satisfy $f^\infty < \frac{1}{\lambda}(\mu_1 - \varepsilon)$. Then there exists $R_2 > R_1$ such that:

$$f(t,u) \leq \frac{1}{\lambda}(\mu_1 - \varepsilon)\, u. \quad \forall t \in [0,1], u \geq R_2. \tag{22}$$

By (v) there exists an L^∞function φ_1 such that $f(t,u) \leq \frac{1}{\lambda}\varphi_1(t)$ for all $u \in [0, R_2]$ and $t \in [0,1]$. Hence, we have

$$f(t,u) \leq \frac{1}{\lambda}[(\mu_1 - \varepsilon)\, u + \varphi_1(t)] \text{ for all} u \in R^+, t \in [0,1]. \tag{23}$$

Since $1/\mu_1$ is the radius of the spectrum of L, $(I/(\mu_1 - \varepsilon) - L)^{-1}$ exists. Let: $C = \int_0^1 \varphi_1(s)\Phi(s)\, g(s)\, ds$ and $R_0 = (I/(\mu_1 - \varepsilon) - L)^{-1}(c/(\mu_1 - \varepsilon))$. We prove that for each $R > R_0$,

$$Tu \neq \beta u \text{ for all} u \in \partial K_R \text{ and} \beta \geq 1. \tag{24}$$

In fact , if not, there exist $u \in \partial K_R$ and $\beta \geq 1$ such that $Tu = \beta u$.

This together with (23) , implies

$$\begin{aligned} u(t) &\leq \int_0^1 G(t,s)\, g(s)\left((\mu_1-\varepsilon)\, u(s)+\varphi_1(s)\right) ds \\ &= (\mu_1-\varepsilon)\int_0^1 G(t,s)\, g(s)\, u(s)\, ds + \int_0^1 G(t,s)\, g(s)\, \varphi_1(s)\, ds \\ &= (\mu_1-\varepsilon)\, Lu(t) + C. \end{aligned}$$

This implies

$\left(\frac{I}{\mu_1-\varepsilon}-L\right) u(t) \leq \frac{C}{\mu_1-\varepsilon}$ and $u(t) \leq \left(\frac{I}{\mu_1-\varepsilon}-L\right)^{-1}\left(\frac{C}{\mu_1-\varepsilon}\right) = R_0.$

Therefore, we have $\|u\| \leq R_0 < R$, a contradiction. Take $R > R_2$, it follows from (24) and properties of index that

$$i_K(T, K_R) = 1, \quad \forall R > R_0. \tag{25}$$

Now (21) and (25) combined imply

$$i_K\left(T, K_R \backslash \bar{K}_{R_1}\right) = i_K(T, K_R) - i_K\left(T, \bar{K}_{R_1}\right) = 1.$$

Therefore, T has at least one fixed point $u_0 \in K_R / \bar{K}_{R_1}$, and u_0 is a positive solution of BVP (1)-(2).

5. The existence of two positive solution

Theorem 5.1. Suppose that (A2), (A3) and
(A5) $\lambda f^{0,\rho'} \leq m$ for some $\rho' > 0$.
Then (1)-(2) has at least two positive solutions.
Proof. By (A5), we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)\, g(s)\, f(s, u(s))\, ds \\ &\leq \int_0^1 G(t,s)\, g(s)\, \rho' m ds, \end{aligned}$$

so that $\|Tu\| \leq \rho' = \|u\|$, for all $u \in \partial V_{\rho'}$. Now Lemma 2.1, yields

$$i_k(T, V_{\rho'}) = 1. \tag{26}$$

On the other hand, in view of (A2), we may take $\rho^* > \rho'$so that (16) holds (see the proof of Theorem 4.1). From (A3), We may take $R_1 \in (0, \rho')$so that (21) holds (see the proof Theorem 4.2).
Combining (26), (16) and (21), we arrive at

$$i_k\left(T, K_{\rho^*} \backslash \bar{V}_{\rho'}\right) = 0 - 1 = -1,$$

and

$$i_k\left(T, V_{\rho'} \backslash \bar{K}_{R_1}\right) = 1 - 0 = 1.$$

Consequently, Thas at least two fixed points, with one on $K_{\rho^*} \backslash \bar{V}_{\rho'}$ and the other on $V_{\rho'} \backslash \bar{K}_{R_1}$. Therefore, (1)-(2) has at least two positive solutions.
Theorem 5.2. Suppose that (A1),(A4) and
(A6) $\lambda f_{\rho', \rho'/c} \geq M$ for some $\rho' > 0$.
Then (1)-(2) has at least two positive solutions.
Proof. By (A6), we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)\, g(s)\, f(s, u(s))\, ds \\ &\geq \lambda \int_a^b G(t,s)\, g(s)\, f(s, u(s))\, ds \\ &\geq \int_a^b G(t,s)\, g(s)\, M \rho' ds, \end{aligned}$$

so that $\|Tu\| \geq \rho' = \|u\|$, for all $u \in \partial V_{\rho'}$, and by Lemma 2.1, yields

$$i_k(T, V_{\rho'}) = 0. \tag{27}$$

On the other hand, in view of (A1), We may take $\rho \in (0, \rho')$so that (14) holds (see the proof Theorem 4.1). In addition, from (A4), we may take $R > \rho'$ so that (25) holds (see the proof of Theorem 4.2).
Combining (27), (14) and (25), we arrive at

$$i_k \left(T, K_R \backslash \bar{V}_{\rho'}\right) = 1 - 0 = 1,$$

and

$$i_k \left(T, V_{\rho'} \backslash \bar{K}_{\rho}\right) = 0 - 1 = -1.$$

Hence, T has at least two fixed points, with one on $V_{\rho'} \backslash \bar{K}_{\rho}$ and the other on $K_R \backslash \bar{V}_{\rho'}$. Therefore, (1)-(2) has at least two positive solutions.

6. Nonexistence results

We now give a nonexistence result which shows that the above result on existence of one solution is sharp.
Definition 6.1. We say that a bounded linear operator L is $u_0 - positive$on the cone P, if there exists $u_0 \in P\backslash \{0\}$, such that for every $u \in P\backslash \{0\}$ there are positive constants $k_1 (u), k_2 (u)$ such that $k_1 (u) u_0 (t) \leq Lu (t) \leq k_2 (u) u_0 (t)$, for every $t \in [0, 1]$.
Theorem 6.1([7,19]). Suppose that L is $u_0 - positive$for some $u_0 \in P\backslash \{0\}$. Let $\mu_1 = 1/r (L)$be the principal characteristic value of L. Suppose that one of the following conditions hold.
(i) $f (t, u) < \mu_1 u$, for all $u > 0$ and almost all $t \in [0, 1]$.
(ii) $f (t, u) > \mu_1 u$, for all $u > 0$ and almost all $t \in [0, 1]$.
If (i) holds, then 0 is the unique fixed point of T in P. If (ii) holds, then 0 is the only possible fixed point of T in P.
Theorem 6.2. If g and $g\mathrm{G}_A (s)$ are integrable functions, then $G (t, s) \leq W (s) c_0 (t)$ for a function $W$with $Wg \in L^1 (0, 1)$,so L_0 is $c_0 - positive$on P.
Proof. We have

$$\begin{aligned} G (t, s) &= \tfrac{\gamma(t)\mathrm{G}_A(s)}{1-\theta[\gamma]} - \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} H (t - s) + \tfrac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} H (1 - s) \\ &\leq c_0 (t) \left[\tfrac{\mathrm{G}_A(s)}{1-\theta[\gamma]} + \tfrac{(1-s)^{\alpha-3}}{\Gamma(\alpha)} H (1 - s)\right] \\ &= c_0 (t) W (s). \end{aligned}$$

We illustrate the applicability of these results with some examples.
Example 6.1. Consider the problem

$$\begin{aligned} &D^{(6.5)} u (t) + \lambda (5t + 3) \left(\tfrac{6u^2+u}{u+1}\right) (3 + \sin u) = 0, \ t \in (0, 1), \\ &u (0) = 0, \quad u^{(k)} (0) = 0, \quad 1 \leq k \leq 5, \quad u'' (1) = 0. \end{aligned} \tag{28}$$

Here we have $g (t) = 5t + 3, \quad f (u) = (3 + \sin u) \frac{6u^2+u}{u+1}$ and $6 < \alpha \leq 7$.
It is readily shown that $f^0 = f_0 = 3, \quad f^\infty = 24, \quad f_\infty = 12$. Also, $3u \leq f (u) \leq 24u$for $u \geq 0$. By calculation, we find $m = 945.1744$, the smallest M calculated is $M (a, b) \approx M (0.5661, 1) \approx 203765.1892$. We find $\mu_1 \approx 107683$. Hence, by Theorem 4.1, there is at least one positive solution if $3\lambda < \mu_1$ and $12\lambda > \mu_1$; that is, there is a positive solution if $\lambda \in (8973.5833, 35894.3333)$. By Theorem 6.1, there does not exist a positive solution if either $3\lambda > \mu_1$ or $24\lambda < \mu_1$; that is, if $\lambda < 4486.7917$ or $\lambda > 35894.3333$ no positive solution exists. For $g (t) \equiv 1$ the corresponding constants are

$$m = 4210.3222, \quad M = 1261771.943, \quad \mu_1 \approx 105890.$$

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