Journal of Fractional Calculus and Applications,
Vol. 3. July 2012, No. 9, pp. 1– 14.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

EXISTENCE RESULTS FOR FIRST ORDER BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

SOTIRIS K. NTOUYAS

ABSTRACT. This paper studies a new class of boundary value problems of nonlinear differential equations and inclusions of fractional order with fractional integral boundary conditions. Some new existence and uniqueness results are obtained by using standard fixed point theorems. Some illustrative examples are also discussed.

1. INTRODUCTION

In this paper, we discuss the existence and uniquness of solutions for a boundary value problem of nonlinear fractional differential equations and inclusions of order $q \in (0,1]$ with fractional integral boundary conditions. More precisely, in Section 3, we consider the following boundary value problem of fractional differential equations

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ 0 < t < 1, \ 0 < q \leq 1, \\ x(0) = \alpha I^p x(\eta), \ 0 < \eta < 1, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\alpha \in \mathbb{R}$ is such that $\alpha \neq \Gamma(p+1)/\eta^p$, Γ is the Euler gamma function and $I^p, 0 < p < 1$ is the Riemann-Liouville fractional integral of order p.

Fractional differential equations have aroused great interest, which is caused by both the intensive development of the theory of fractional calculus and the application of physics, mechanics and chemistry engineering. For some recent development on the topic see [1]-[23] and the references cited therein.

In [26], the authors studied the following boundary value problem with fractional integral boundary conditions

$$\begin{cases} {}^cD^q_{0^+} x(t) = f(t, x(t), {}^c D^p_{0^+} x(t)), \ 0 < t < 1, \ 1 < q \leq 2, \ 0 < p < 1 \\ x(0) = 0, \quad x'(1) = \alpha I^p_{0^+} x(1). \end{cases} \tag{2}$$

2000 *Mathematics Subject Classification.* 34A08, 26A33, 34A60.

Key words and phrases. Fractional differential equations; nonlocal boundary conditions; fixed point theorems.

Submitted Apr. 23, 2012. Published July 1, 2012.

Existence and uniqueness results are proved via Banach's contraction principle and Leray-Schauder Nonlinear Alternative.

Ahmad *et al.* in [10] discussed existence results for nonlinear fractional differential equations with three-point integral boundary conditions

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)),\ 0 < t < 1,\ 1 < q \le 2, \\ x(0) = 0, \qquad x(1) = \alpha \displaystyle\int_0^\eta x(s)ds,\ 0 < \eta < 1, \end{cases} \tag{3}$$

where ${}^cD^q$ denotes the Caputo fractional derivative, and $\alpha \in \mathbb{R}$, $\alpha \neq 2/\eta^2$.

In Section 3, we prove new existence and uniqueness results for the problem (1). These results are based on Banach's fixed point theorem, Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type.

In Section 4, we continue our study for boundary value problems with fractional integral boundary conditions for multivalued maps (inclusion case) and consider the problem

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)),\ 0 < t < 1,\ 0 < q \le 1, \\ x(0) = \alpha I^p x(\eta),\ 0 < \eta < 1, \end{cases} \tag{4}$$

where $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

We establish existence results for the problem (4), when the right hand side is convex as well as non-convex valued. In the first result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the second result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

2. Linear Problem

Let us recall some basic definitions of fractional calculus [28, 31, 32].

Definition 2.1. *For at least n-times differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds,\ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds,\ q > 0,$$

provided the integral exists.

The following lemmas gives some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative [28].

Lemma 2.3. *Let $p, q \ge 0, f \in L_1[a, b]$. Then $I^p I^q f(t) = I^{p+q} f(t)$ and ${}^cD^q I^q f(t) = f(t)$, for all $t \in [a, b]$.*

Lemma 2.4. *Let $\beta > \alpha > 0, f \in L_1[a,b]$. Then ${}^cD^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t)$, for all $t \in [a,b]$.*

To define the solution of the boundary value problem (1) we need the following lemma, which deals with a linear variant of the problem (1).

By a solution of (1), we mean a continuous function $x(t)$ which satisfies the equation ${}^cD^q x(t) = f(t, x(t)),\ 0 < t < 1$, and the boundary condition $x(0) = \alpha I^p x(\eta), 0 < \eta < 1$.

Lemma 2.5. *Let $\alpha \neq \dfrac{\Gamma(p+1)}{\eta^p}$. Then for a given $g \in C([0,1],\mathbb{R})$, the solution of the fractional differential equation*

$$ {}^cD^q x(t) = g(t),\ 0 < q \le 1 \tag{5} $$

subject to the boundary condition

$$ x(0) = \alpha I^p x(\eta) \tag{6} $$

is given by

$$ \begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds \\ &+ \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} g(s)ds,\ t \in [0,1]. \end{aligned} \tag{7} $$

Proof. For some constant $c_0 \in \mathbb{R}$, we have [28]

$$ x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - c_0. \tag{8} $$

Using the Riemann-Liouville integral of order p for (8), we have

$$ \begin{aligned} I^p x(t) &= \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \left[\int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} g(r)dr - c_0 \right] ds \\ &= I^p I^q g(t) - c_0 \frac{t^p}{\Gamma(p+1)} = I^{p+q} g(t) - c_0 \frac{t^p}{\Gamma(p+1)}, \end{aligned} $$

where we have used Lemma 2.3. Using the condition (6) in the above expression, we get

$$ c_0 = -\frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} I^{p+q} g(\eta). $$

Substituting the value of c_0 in (8), we obtain (7). □

3. Existence results for single-valued case

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In view of Lemma 2.5, we define an operator $F : \mathcal{C} \to \mathcal{C}$ by

$$ \begin{aligned} (Fx)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\ &+ \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s))ds,\ t \in [0,1]. \end{aligned} \tag{9} $$

Observe that the problem (1) has solutions if and only if the operator equation $Fx = x$ has fixed points.

Our first existence result is obtained by the use of the well-known Banach's contraction principle.

Theorem 3.1. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and that there exists a constant $L > 0$ such that*

(A_1) $|f(t,x) - f(t,y)| \le L|x-y|,\ t \in [0,1],\ x, y \in \mathbb{R}$.

If $LA < 1$, where

$$A = \frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \alpha\eta^p|}, \tag{10}$$

then the boundary value problem (1) has a unique solution.

Proof. Let us set $\sup_{t\in[0,1]} |f(t,0)| = M$ and show that $FB_\rho \subset B_\rho$, where F is defined by (9), $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ and $\rho \ge \dfrac{MA}{1-LA}$, with A given by (10).

For $x \in B_\rho,\ t \in [0,1]$, we have

$$\begin{aligned}
& \|(Fx)(t)\| \\
\le\ & \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \right. \\
& \left. + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s))| ds \right\} \\
\le\ & \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (|f(s,x(s)) - f(s,0)| + |f(s,0)|) ds \right. \\
& \left. + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} (|f(s,x(s)) - f(s,0)| + |f(s,0)|) ds \right\} \\
=\ & [L\rho + M] \left\{ \frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \alpha\eta^p|} \right\} \\
\le\ & [L\rho + M]\, A \le \rho.
\end{aligned}$$

This shows that $FB_\rho \subset B_\rho$.

Now, for $x, y \in C([0,1],\mathbb{R})$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
& \|(Fx) - (Fy)\| \\
\le\ & \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \right. \\
& \left. + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s)) - f(s,y(s))| ds \right\} \\
\le\ & L\|x-y\| \left\{ \frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \alpha\eta^p|} \right\} \\
=\ & LA\|x-y\|.
\end{aligned}$$

As $LA \in (0,1)$ by assumption, therefore F is a contraction. Hence Banach's contraction principle applies and the problem (1) has a unique solution. □

Now, we prove the existence of solutions of (1) by applying Krasnoselskii's fixed point theorem [30].

Theorem 3.2. *(Krasnoselskii's fixed point theorem). Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1). In addition we assume that the following assumption holds:*

(A_2) $|f(t,x)| \leq \mu(t)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$, *and* $\mu \in C([0,1], \mathbb{R}^+)$.

Then the boundary value problem (1) has at least one solution on $[0,1]$, *provided that*

$$\frac{L|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \alpha\eta^p|} < 1. \tag{11}$$

Proof. Letting $\sup_{t\in[0,1]} |\mu(t)| = \|\mu\|$, we fix

$$\overline{r} \geq \|\mu\| \left\{ \frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \alpha\eta^p|} \right\}$$

and consider $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \leq \overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$\begin{aligned}
(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds, \ t \in [0,1], \\
(\mathcal{Q}x)(t) &= \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s))ds, \ t \in [0,1].
\end{aligned}$$

For $x, y \in B_{\overline{r}}$, we find that

$$\begin{aligned}
\|\mathcal{P}x + \mathcal{Q}y\| &\leq \|\mu\| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds + \frac{\|\mu\||\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} ds \\
&\leq \|\mu\| \left\{ \frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \alpha\eta^p|} \right\} \\
&\leq \overline{r}.
\end{aligned}$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_1) together with (11) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\| \leq \frac{\|\mu\|}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}}|f(t,x)| = \overline{f}$, and consequently we have

$$\begin{aligned}
|(\mathcal{P}x)(t_1)-(\mathcal{P}x)(t_2)| &= \Bigg|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s))ds \\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\Bigg| \\
&\le \frac{\overline{f}}{\Gamma(q+1)}(t_2^q-t_1^q),
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem 3.2 are satisfied. So the conclusion of Theorem 3.2 implies that the boundary value problem (1) has at least one solution on $[0, 1]$. □

The next existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.4. *(Nonlinear alternative for single valued maps)*[25]. *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

Theorem 3.5. *Assume that:*

(A_3) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $\phi \in L^1([0,1],\mathbb{R}^+)$ such that*

$$|f(t,x) \le \phi(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,1]\times\mathbb{R};$$

(A_4) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\left[I^q\phi(1)+\dfrac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^{p+q}\phi(\eta)\right]} > 1.$$

Then the boundary value problem (1) has at least one solution on $[0, 1]$.

Proof. Observe that the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by (9) is continuous. Next we show that *F maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, we have

$$\begin{aligned}|(Fx)(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\ &\quad+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\\ &\le \psi(\|x\|)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\phi(s)ds\\ &\quad+\frac{\psi(\|x\|)|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}\phi(s)ds\\ &\le \psi(\|x\|)\left[I^q p(1)+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^{p+q}\phi(\eta)\right].\end{aligned}$$

Thus,

$$\|Fx\|\le\psi(\rho)\left[I^q\phi(1)+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^{p+q}\phi(\eta)\right].$$

Now we show that F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t', t''\in[0,1]$ with $t'<t''$ and $x\in B_\rho$. Then

$$\begin{aligned}&|(Fx)(t'')-(Fx)(t')|\\ \le&\left|\psi(\rho)\int_0^{t'}\left[\frac{(t''-s)^{q-1}-(t'-s)^{q-1}}{\Gamma(q)}\right]\phi(s)ds+\psi(\rho)\int_{t'}^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}\phi(s)ds\right|.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t''-t'\to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $F: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous.

Now let $\lambda\in(0,1)$ and let $x=\lambda Fx$. Then for $t\in[0,1]$ we have

$$x(t)=\lambda\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\lambda\frac{\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}\int_0^\eta\frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds.$$

Then, using the computations by the first step, we have

$$|x(t)|\le\psi(\|x\|)\left[I^q\phi(1)+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^{p+q}\phi(\eta)\right].$$

Consequently,

$$\frac{\|x\|}{\psi(\|x\|)\left[I^q\phi(1)+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^{p+q}\phi(\eta)\right]}\le 1.$$

In view of (A_4), there exists M such that $\|x\|\neq M$. Let us set

$$U=\{x\in C([0,1],\mathbb{R}):\|x\|<M\}.$$

Note that the operator $F:\overline{U}\to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\lambda F(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that F has a fixed point $x\in\overline{U}$ which is a solution of the problem (1). This completes the proof. □

In the special case when $p(t)=1$ and $\psi(|x|)=k|x|+N$ we have the following corollary.

Corollary 3.6. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that:*

(A_5) *there exist constants $0\le\kappa<\dfrac{1}{A}$, where A is given by (10) and $M>0$ such that*

$$|f(t,x)|\le\kappa|x|+M,\quad \text{for all}\quad t\in[0,1], x\in C[0,1].$$

Then the boundary value problem (1) has at least one solution.

Example 3.7. *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^{1/2}x(t)=\dfrac{1}{2(t+2)^2}\dfrac{|x|}{1+|x|}+1+\sin^2 t,\ \ t\in[0,1],\\ x(0)=\sqrt{3}I^{1/2}x\left(\dfrac{1}{3}\right). \end{cases} \tag{12}$$

Here, $q=1/2,\ \alpha=\sqrt{3},\ p=1/2,\ \eta=1/3$ and $f(t,x)=\dfrac{1}{2(t+2)^2}\dfrac{|x|}{1+|x|}+1+\sin^2 t$. As $\alpha=\sqrt{3}\neq\Gamma(p+1)/\eta^p=\Gamma(3/2)/(1/3)^{1/2}$ and $|f(t,x)-f(t,y)|\le\dfrac{1}{8}|x-y|$, therefore, (A_1) is satisfied with $L=\dfrac{1}{8}$. Since

$$\begin{aligned} LA &= L\left\{\frac{1}{\Gamma(q+1)}+\frac{\alpha\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\alpha\eta^p|}\right\}\\ &= \frac{1}{8}\left\{\frac{2}{\sqrt{\pi}}+\frac{\sqrt{3\pi}}{3(2-\sqrt{\pi})}\right\}\approx 0.7019863<1, \end{aligned}$$

by the conclusion of Theorem 3.1, the boundary value problem (12) has a unique solution on $[0,1]$.

Example 3.8. *Consider the following boundary value problem*

$$\begin{cases} {}^cD^{1/2}x(t)=\dfrac{1}{16\pi}\sin(2\pi x)+\dfrac{|x|}{2(1+|x|)}+\dfrac{1}{2},\ \ t\in[0,1],\\ x(0)=\sqrt{3}I^{1/2}x\left(\dfrac{1}{3}\right). \end{cases} \tag{13}$$

Here,

$$\Big|f(t,x)\Big|=\left|\frac{1}{16\pi}\sin(2\pi x)+\frac{|x|}{2(1+|x|)}+\frac{1}{2}\right|\le\frac{1}{8}|x|+1.$$

Clearly $M=1$ and $\kappa=\dfrac{1}{8}<\dfrac{1}{A}\approx 0.1780661$. Thus, all the conditions of Corollary 3.6 are satisfied and consequently the problem (13) has at least one solution.

4. Existence Results for multi-valued case

Let us recall some basic definitions on multi-valued maps [22], [27].

For a normed space $(X,\|\cdot\|)$, let $P_{cl}(X)=\{Y\in\mathcal{P}(X): Y \text{ is closed}\}$, $P_b(X)=\{Y\in\mathcal{P}(X): Y \text{ is bounded}\}$, $P_{cp}(X)=\{Y\in\mathcal{P}(X): Y \text{ is compact}\}$, and $P_{cp,c}(X)=\{Y\in\mathcal{P}(X): Y \text{ is compact and convex}\}$. A multi-valued map $G:X\to\mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. The map G is bounded on bounded sets if $G(\mathbb{B})=\cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B}\in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y|: y\in G(x)\}\}<\infty$). G is called upper semi-continuous (u.s.c.)

on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.
Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

4.1. **The lower semi-continuous case.** As a first result in the subsection, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [19] for lower semi-continuous maps with decomposable values.

Definition 4.1. *Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E.*

Definition 4.2. *Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.*

Definition 4.3. *A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.*

Definition 4.4. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 4.5. *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 4.6. *([24]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Definition 4.7. *A function $x \in C^2([0,1],\mathbb{R})$ is a solution of the problem (4) if $x(0) = \alpha I^p x(\eta)$, and there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and*

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s)ds. \tag{14}$$

Theorem 4.8. *Assume that (A_4) holds. In addition we suppose that the following conditions hold:*

- (H_1) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}^+)$ such that*

  $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|)$ *for each* $(t,x) \in [0,1] \times \mathbb{R}$;
- (H_2) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
  - *(a) $(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
  - *(b) $x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$;*

Then the boundary value problem (4) has at least one solution on $[0,1]$.

Proof. It follows from (H_1) and (H_2) that F is of l.s.c. type. Then from Lemma 4.6, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \ 0 < q \le 1, \ t \in [0,1], \\ x(0) = \alpha I^p x(\eta), \ 0 < \eta < 1. \end{cases} \tag{15}$$

Observe that if $x \in C^1([0,1],\mathbb{R})$ is a solution of (15), then x is a solution to the problem (4). In order to transform the problem (15) into a fixed point problem, we define the operator $\overline{\Omega_F}$ as

$$\overline{\Omega_F}x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(x(s))ds.$$

$\overline{\Omega_F}$ is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
& |\overline{\Omega_F}(x_n)(t) - \overline{\Omega_F}(x)(t)| \\
= & \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [f(x_n(s)) - f(x(s))]\, ds \right. \\
& \left. + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} [f(x_n(s)) - f(x(s))] \right| \\
\le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(y_n(s)) - f(y(s))\|\, ds \\
& + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} \|f(x_n(s)) - f(x(s))\|.
\end{aligned}$$

Hence

$$\|\overline{\Omega_F}(x_n) - \overline{\Omega_F}(x)\| \le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(x_n)(s) - f(x)(s)\| \, ds + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} \|f(x_n(s)) - f(x(s))\|,$$

which tends to 0, as $n \to \infty$. Thus $\overline{\Omega_F}$ is continuous.

The remaining part of the proof is similar to that of Theorem 3.5. So we omit it. This completes the proof. □

Example 4.9. *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^{1/2}x(t) \in F(t, x(t)), \ 0 < t < 1, \\ x(0) = \sqrt{3} I^{1/2} x\left(\frac{1}{3}\right). \end{cases} \tag{16}$$

Here, $q = 1/2$, $p = 1/2$, $\alpha = \sqrt{3}$, $\eta = 1/3$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{|x|^3}{|x|^3 + 3} + 3t^3 + 5, \frac{|x|}{|x|+1} + t + 1\right].$$

Clearly $\alpha = \sqrt{3} \ne \Gamma(p+1)/\eta^p = \Gamma(3/2)/(1/3)^{1/2}$ and for $f \in F$, we have

$$|f| \le \max\left(\frac{|x|^3}{|x|^3+3} + t^3 + 1, \frac{|x|}{|x|+1} + t + 1\right) \le 3, \ x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le 3 = p(t)\psi(|x|), \ x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(|x|) = 3$.

Further, using the condition (H_3) we find that $M > 16.847672$. Clearly, all the conditions of Theorem 4.8 are satisfied. So there exists at least one solution of the problem (16) on $[0,1]$.

4.2. **The Lipschitz case.** Now we prove the existence of solutions for the problem (4) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [21].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [29]).

Definition 4.10. *A multivalued operator $N : X \to P_{cl}(X)$ is called:*

(**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \le \gamma d(x,y) \textit{ for each } x, y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 4.11. *(Covitz-Nadler, [21]) Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \ne \emptyset$.*

Definition 4.12. *A measurable multi-valued function $F : [0,1] \to \mathcal{P}(X)$ is said to be integrably bounded if there exists a function $h \in L^1([0,1], X)$ such that for all $v \in F(t)$, $\|v\| \le h(t)$ for a.e. $t \in [0,1]$.*

Theorem 4.13. *Assume that the following conditions hold:*

- (H_3) *$F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,1] \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*
- (H_4) *$H_d(F(t,x), F(t,\bar{x})) \le m(t)|x - \bar{x}|$ for almost all $t \in [0,1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.*

Then the boundary value problem (4) has at least one solution on $[0,1]$ if

$$\gamma = I^q m(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} I^{p+q} m(\eta) < 1.$$

Proof. Define the operator $\Omega_F : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ by

$$\Omega_F(x) = \left\{ \begin{array}{l} h \in C([0,1], \mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} f(s)ds, \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1], \mathbb{R})$ by the assumption (H_3), so F has a measurable selection (see Theorem III.6 [20]). Now we show that the operator Ω_F, satisfies the assumptions of Lemma 4.11. To show that $\Omega_F(x) \in P_{cl}((C[0,1], \mathbb{R}))$ for each $x \in C([0,1], \mathbb{R})$, let $\{u_n\}_{n \ge 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1], \mathbb{R})$. Then $u \in C([0,1], \mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} v_n(s)ds.$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$u_n(t) \to u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} v(s)ds.$$

Hence, $u \in \Omega_F(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \le \gamma \|x - \bar{x}\| \text{ for each } x, \bar{x} \in C([0,1], \mathbb{R}).$$

Let $x, \bar{x} \in C([0,1], \mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$h_1(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} v_1(s)ds.$$

By (H_4), we have

$$H_d(F(t,x), F(t,\bar{x})) \le m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|, \; t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [20]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} v_2(s)ds.$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| \;&\leq\; \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\ &\quad + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} |v_1(s) - v_2(s)|ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|h_1 - h_2\| \;&\leq\; \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} m(s)ds + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} I^{p+q} m(\eta) \\ &= I^q m(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} I^{p+q} m(\eta). \end{aligned}$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega_F(x), \Omega_F(\bar{x})) \;&\leq\; \gamma\|x - \bar{x}\| \\ &\leq \left\{ I^q m(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} I^{p+q} m(\eta) \right\} \|x - \bar{x}\|. \end{aligned}$$

Since Ω_F is a contraction, it follows by Lemma 4.11 that Ω_F has a fixed point x which is a solution of (4). This completes the proof. $\square$

Acknowledgments. The author would like to express his thanks to the referee for his/her helpful comments and suggestions.

References

[1] R. P. Agarwal, B. Andrade, C. Cuevas, Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations, *Nonlinear Anal. Real World Appl.* **11** (2010), 3532-3554.

[2] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.* **23** (2010), 390-394.

[3] B. Ahmad, A. Alsaedi, B. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal. Real World Appl.* **9** (2008), 1727-1740.

[4] B. Ahmad, T. Hayat, S. Asghar, Diffraction of a plane wave by an elastic knife-edge adjacent to a strip, *Canad. Appl. Math. Quart.* 9(2001) 303-316.

[5] B. Ahmad, J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher order nonlinear fractional differential equations, *Abstr. Appl. Anal., 2009 (2009), Article ID 494720, 9 pages.*

[6] B. Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, *Bound. Value Probl. 2009, Art. ID 708576, 11 pp.*

[7] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009) 1838-1843.

[8] B. Ahmad, S. K. Ntouyas, Existence results for nonlinear fractional differential equations with four-point nonlocal type integral boundary conditions, *Afr. Diaspora J. Math.*,**11** (2011), 29-39.
[9] B. Ahmad, S.K. Ntouyas, Some existence results for boundary value problems for fractional differential inclusions with non-separated boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* 2010, No. 71, 1-17.
[10] B. Ahmad, S. K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Differ. Equ.*, Volume 2011, Article ID 107384, 11 pages.
[11] B. Ahmad, S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, *Appl. Math. Comput.* **217** (2010), 480-487.
[12] S. Asghar, B. Ahmad, M. Ayub, Diffraction from an absorbing half plane due to a finite cylindrical source, *Acustica-Acta Acustica* 82(1996), 365-367.
[13] Z.B. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal.* **72** (2010), 916-924.
[14] K. Balachandran, J. J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces, *Nonlinear Anal.* **72** (2010) 4587-4593.
[15] D. Baleanu, K. Diethelm, E. Scalas, J. J.Trujillo, *Fractional calculus models and numerical methods. Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, 2012.
[16] D. Baleanu, O. G. Mustafa, On the global existence of solutions to a class of fractional differential equations, *Comp. Math. Appl.* **59** (2010), 1835-1841.
[17] D. Baleanu, O. G. Mustafa, D. O'Regan, A Nagumo-like uniqueness theorem for fractional differential equations, *J. Phys. A, Math. Theor.* **44**, No. 39, Article ID 392003, 6 p. (2011).
[18] A. Boucherif, Second-order boundary value problems with integral boundary conditions, *Nonlinear Anal.* **70** (2009), 364-371.
[19] A. Bressan, G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.
[20] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
[21] H. Covitz, S. B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.
[22] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.
[23] A.M.A. El-Sayed, E.M. Hamdallah, Kh.W. El-Kadeky, Monotonic positive solutions of nonlocal boundary value problems for a second-order functional differential equation, *Abstr. Appl. Anal. 2012, Article ID 489353, 12 p. (2012).*
[24] M. Frigon, Théorèmes d'existence de solutions d'inclusions différentielles, *Topological Methods in Differential Equations and Inclusions* (edited by A. Granas and M. Frigon), NATO ASI Series C, Vol. 472, Kluwer Acad. Publ., Dordrecht, (1995), 51-87.
[25] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2005.
[26] A. Guezane-Lakoud, R. Khaldi, Solvability of a fractional boundary value problem with fractional integral condition, *Nonlinear Anal.* **75** (2012), 2692-2700.
[27] Sh. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis, Theory I*, Kluwer, Dordrecht, 1997.
[28] A. A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
[29] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.
[30] M. A. Krasnoselskii, Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk* **10** (1955), 123-127.
[31] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[32] S. G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail address: sntouyas@uoi.gr