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GENERALIZED SOLUTIONS FOR FRACTIONAL NONLINEAR DISPERSIVE EQUATIONS

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ABSTRACT. In this paper, homotopy perturbation method is implemented to obtain generalized solutions of fractional nonlinear dispersive equations. Some examples including the time fractional cubic Boussinesq equation, time fractional Boussinesq-type equation, time-space fractional $K(2,2)$type equation and time-space fractional Kdv-Burgers are investigated and the obtained results reveal that the method is very effective and convenient to solve the fractional nonlinear dispersive equations.

1. INTRODUCTION

Nonlinear wave phenomena appear in a wide variety of scientific applications such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics and optical fibers. These nonlinear phenomena are often related to wave and dispersive equations. And in recent years, it has turned out that many phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science can be successfully modeled by the use of fractional derivatives[1, 2]. But these nonlinear fractional differential equation are difficult to get their exact solutions[3, 4, 5]. Several analytical and numerical methods have been proposed to solve fractional ordinary differential equations, and fractional partial differential equations. For examples, Adomian decomposition method[6, 7], variational iteration method[8, 9], differential transform method[10] and homotopy perturbation method[11, 12]. The homotopy perturbation method first introduced by He[13, 14] for solving linear or nonlinear partial differential equations. The method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. The method, which does not require a small parameter in an equation, has many advantages over the classical technique.

The aim of the present paper is to extend the application of the homotopy perturbation method to derive the solutions of fractional nonlinear dispersive equations[15], including the time-fractional cubic Boussinesq equation, time fractional Boussinesq-type equation, time-space fractional K(2,2) type equation and time-space fractional

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Kdv-Burgers, and these equations are as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - u_{xx} + 2(u^3)_{xx} + u_{xxxx} = 0, \quad 1 < \alpha \le 2 \tag{1}$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - u_{xx} + (u^2)_{xx} + [u(u)_{xx}]_{xx} = 0, \quad 1 < \alpha \le 2 \tag{2}$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + \frac{\partial^{3\beta} u^2(x,t)}{\partial x^{3\beta}} = 0, \quad 1 < \alpha, \beta \le 1 \tag{3}$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \varepsilon u \frac{\partial^\beta u^2(x,t)}{\partial x^\beta} - \upsilon \frac{\partial^{2\beta} u^2(x,t)}{\partial x^{2\beta}} + \eta \frac{\partial^{3\beta} u^2(x,t)}{\partial x^{3\beta}} = 0, \quad 1 < \alpha, \beta \le 1 \tag{4}$$

where $t > 0$ and the fractional derivatives are defined in Caputo sense, which will be introduce in next sections. Our work here stems mainly from homotopy perturbation method, that has been widely used in applied sciences, which is capable of handing a wider class of diffusion problems. Numerical solutions of fractional nonlinear dispersive equations shall be presented to demonstrate the effectiveness of the algorithm.

2. Fractional Calculus

Definition 1. The Riemann-Liouville fractional integral operator $J^\alpha (\alpha \ge 0)$ of a function $f(t)$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha \ge 0) \tag{5}$$

where $\Gamma(\cdot)$ is the well-known gamma function, and some properties of the operator J^α are as follows

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \quad (\alpha \ge 0, \beta \ge 0) \tag{6}$$

$$J^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}, \quad (\gamma \ge -1) \tag{7}$$

Definition 2. The Caputo fractional derivative D^α of a function $f(t)$ is defined as

$$_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(t) d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < Re(\alpha) \le n, n \in N) \tag{8}$$

the following are two basic properties of the Caputo fractional derivative

$$_0D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \tag{9}$$

$$(J^\alpha D^\alpha) f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \tag{10}$$

we have chosen to the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem. And some other properties of fractional derivative can be found in [1].

3. Homotopy perturbation method

For convenience of the reader, we will present a review of the homotopy perturbation method[13, 14], let us consider the following nonlinear differential equation:

$$L(u) + N(u) = f(r), r \in \Omega \tag{11}$$

where L is a linear operator, while N is nonlinear operator and $f(r)$ is a known analytic function. The he's homotopy perturbation technique[12-14] defines the homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{12}$$

where$r \in R$ and $p \in [0,1]$ is an embedding parameter. In case Eq.(12) is a linear differential equation, $L(v) - L(u_0) = 0$ which is easy to solve; and when Eq.(12) turns out to be the original one (11), u_0 is an initial approximation which satisfies the boundary conditions. The basic assumption is that the solutions can be written as a power series in P

$$v = v_p v_1 + p^2 v_2 + \cdots , \tag{13}$$

the approximate solution of Eq(11), therefore, can be readily obtained:

$$u = \lim_{p\to} v = v_0 + v_1 + v_2 + \cdots , \tag{14}$$

The homotopy perturbation method depends on the proper selection of the initial approximation $v_0(x,t)$, In the following sections, we implement the above theory to derive generalized solutions for fractional nonlinear dispersive equation.

4. Generalized solutions of fractional nonlinear dispersive equations

In order to access the advantages and the accuracy of the homotopy perturbation method presented in this paper for fractional nonlinear dispersive equation, we have applied it to the following several problems. All the results are calculated by using the symbolic calculus software Mathematica.

Case 1: In this case, we first consider the time fractional cubic Boussinesq equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - u_{xx} + 2(u^3)_{xx} + u_{xxxx} = 0, \quad 1 < \alpha \le 2 \tag{15}$$

subject to the initial condition

$$u(x,0) = \frac{1}{x}, u_t(x,0) = -\frac{1}{x^2} \tag{16}$$

in view of the homotopy (12), we construct the homotopy

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = p[u_{xx} - 2(u^3)_{xx} - u_{xxxx}], \tag{17}$$

substituting (13) and the initial condition (16) into the homotopy (17) and equating the terms with identical powers of p, the first few components of the homotopy solution for Eq.(15) are derived as follows:

$$u_0 = \frac{1}{x} - \frac{t}{x^2} \tag{18}$$

$$u_1 = J^\alpha[u_{0xx} - 2(u_0^3)_{xx} - u_{0xxxx}]$$

$$= \frac{2t^{\alpha}}{x^3\Gamma(1+\alpha)} - \frac{6t^{1+\alpha}}{x^4\Gamma(2+\alpha)} - \frac{360t^{2+\alpha}}{x^7\Gamma(3+\alpha)} + \frac{504t^{3+\alpha}}{x^8\Gamma(4+\alpha)}, \tag{19}$$

$$u_2 = J^{\alpha}[u_{1xx} - 6(u_0^2u_1)_{xx} - u_{1xxxx}]$$

$$= \frac{360t^{2\alpha\alpha}}{x^7\Gamma(1+2\alpha)} + \frac{24t^{2\alpha}}{x^5\Gamma(1+2\alpha)} - \frac{3528t^{1+2\alpha}}{x^8\Gamma(2+2\alpha)} - \frac{120t^{1+2\alpha}}{x^6\Gamma(2+2\alpha)} + \frac{1008t^{1+2\alpha}\Gamma(2+\alpha)}{x^8\Gamma(1+\alpha)\Gamma(2+2\alpha)}$$

$$-\frac{1620000t^{2+2\alpha}}{x^{11}\Gamma(3+2\alpha)} - \frac{20160t^{2+2\alpha}}{x^9\Gamma(3+2\alpha)} - \frac{672t^{2+2\alpha}\Gamma(3+\alpha)}{x^9\Gamma(1+\alpha)\Gamma(3+2\alpha)} - \frac{4032t^{2+2\alpha}\Gamma(3+\alpha)}{x^9\Gamma(2+\alpha)\Gamma(3+2\alpha)}$$

$$-\frac{3659040t^{3+2\alpha}}{x^{12}\Gamma(4+2\alpha)} + \frac{36288t^{3+2\alpha}}{x^10\Gamma(4+2\alpha)} + \frac{2592t^{3+2\alpha}\Gamma(4+\alpha)}{x^{10}\Gamma(2+\alpha)\Gamma(4+2\alpha)} - \frac{475200t^{3+2\alpha}\Gamma(4+\alpha)}{x^{12}\Gamma(3+\alpha)\Gamma(4+2\alpha)}$$

$$+\frac{285120t^{4+2\alpha}\Gamma(5+\alpha)}{x^{13}\Gamma(3+\alpha)\Gamma(5+2\alpha)} + \frac{798336t^{4+2\alpha}\Gamma(5+\alpha)}{x^{13}\Gamma(4+\alpha)\Gamma(5+2\alpha)} - \frac{471744t^{5+2\alpha}\Gamma(6+\alpha)}{x^{14}\Gamma(4+\alpha)\Gamma(6+2\alpha)}, \tag{20}$$

$$\vdots$$

$$u(x,0) = \frac{1}{x} - \frac{t}{x^2} + \frac{2t^{\alpha}}{x^3\Gamma(1+\alpha)} - \frac{6t^{1+\alpha}}{x^4\Gamma(2+\alpha)} - \frac{360t^{2+\alpha}}{x^7\Gamma(3+\alpha)} + \frac{504t^{3+\alpha}}{x^8\Gamma(4+\alpha)} \tag{21}$$

if we take $\alpha = 2$, the first few components of the homotopy solution of Eq.(15) as follows:

$$u_0 = \frac{1}{x} - \frac{t}{x^2} \tag{22}$$

$$u_1 = \frac{t^2}{x^3} - \frac{t^3}{x^4} - \frac{15t^4}{x^7} + \frac{21t^5}{5x^8}, \tag{23}$$

$$u_2 = \frac{t^4}{x^5} - \frac{t^5}{x^6} + \frac{15t^4}{x^7} - \frac{21t^5}{5x^8} - \frac{308t^6}{5x^9} + \frac{612t^7}{35x^{10}} - \frac{2250t^6}{x^{11}} + \frac{1782t^7}{7x^{12}} + \frac{11583t^8}{35x^{13}} - \frac{273t^9}{5x^{14}}, \tag{24}$$

we have the solution of Eq.(15) in a series form for $\alpha = 2$

$$u(x,t) = \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{t^3}{x^4} + \frac{t^4}{x^5} - \frac{t^5}{x^6} + \cdots, \tag{25}$$

and the solution in closed form is

$$u(x,t) = \frac{1}{x+t}. \tag{26}$$

Case 2: In this case, we consider the time fractional Boussinesq-type equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - u_{xx} + (u^2)_{xx} + [u(u)_{xx}]_{xx} = 0, \quad 1 < \alpha \le 2 \tag{27}$$

subject to the initial condition

$$u(x,0) = -2(c^2-1)\sinh^2(\frac{1}{2}x), u_t(x,0) = (c^2-1)\sinh(x), \tag{28}$$

in view of the homotopy (12), we construct the homotopy

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = p[u_{xx} + (u^2)_{xx} + [u(u)_{xx}]_{xx}], \tag{29}$$

substituting (13) and the initial condition (27) into the homotopy (28) and equating the terms with identical powers of p, the first few components of the homotopy solution for Eq.(26) are derived as follows:

$$u_0 = -(c^2-1)[\cosh(x) - t\sinh(x) - 1], \tag{30}$$

$$\begin{aligned} u_1 &= J^{\alpha}[u_{0xx} + (u_0^2)_{xx} + [u_0(u_0)_{xx}]_{xx}] \\ &= c^2(c^2-1)(-\frac{t^{\alpha}\cosh(x)}{\Gamma(1+\alpha)} + \frac{t^{1+\alpha}\sinh(x)}{\Gamma(2+\alpha)}), \end{aligned} \tag{31}$$

$$\begin{aligned} u_2 &= J^{\alpha}[u_{1xx} + 2(u_0u_1)_{xx} + [u_0u_{1xx} + u_1u_{0xx}]_{xx}] \\ &= c^4(c^2-1)(-\frac{t^{\alpha}\cosh(x)}{\Gamma(1+2\alpha)} + \frac{t^{1+2\alpha}\sinh(x)}{\Gamma(2+2\alpha)}), \end{aligned} \tag{32}$$

$$\vdots$$

and so on, in the same manner the rest of components can be obtained using Mathematica package, Consequently, we have the solution of Eq.(27) in a series form

$$\begin{aligned} u(x,t) &= -(c^2-1)[\cosh(x) - t\sinh(x) - 1] + c^2(c^2-1)(-\frac{t^{\alpha}\cosh(x)}{\Gamma(1+\alpha)} + \frac{t^{1+\alpha}\sinh(x)}{\Gamma(2+\alpha)}) + \cdots \\ &= -(c^2-1)[\cosh(x)(1 + \frac{c^2}{\Gamma(1+\alpha)}t^{\alpha} + \frac{c^4}{\Gamma(1+2\alpha)}t^{2\alpha} + \cdots) - 1] \\ &\quad + (c^2-1)[\sinh(x)(t + \frac{c^2}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{c^4}{\Gamma(2+2\alpha)}t^{1+2\alpha} + \cdots)], \end{aligned} \tag{33}$$

if we take $\alpha = 2$, the first few components of the homotopy solution of Eq.(27) as follows:

$$u_0 = u_0 = -(c^2-1)[\cosh(x) - t\sinh(x) - 1], \tag{34}$$

$$u_1 = c^2(c^2-1)[-\frac{t^2\cosh(x)}{2!} + \frac{t^3\sinh(x)}{3!}], \tag{35}$$

$$u_2 = c^4(c^2-1)[-\frac{t^4\cosh(x)}{4!} + \frac{t^5\sinh(x)}{5!}], \tag{36}$$

$$\cdots$$

$$u_n = c^{2n}(c^2-1)(-\frac{t^{2n}\cosh(x)}{(2n)!} + \frac{t^{2n+1}\sinh(x)}{(2n+1)!}), \tag{37}$$

we have the solution of Eq.(27) in a series form for $\alpha = 2$,

$$\begin{aligned} u(x,t) &= -(c^2-1)[\cosh(x) - t\sinh(x) - 1] + c^2(c^2-1)(-\frac{t^{\alpha}\cosh(x)}{\Gamma(1+\alpha)} + \frac{t^{1+\alpha}\sinh(x)}{\Gamma(2+\alpha)}) + \cdots \\ &= -(c^2-1)[\cosh(x)(1 + \frac{c^2}{\Gamma(1+\alpha)}t^{\alpha} + \frac{c^4}{\Gamma(1+2\alpha)}t^{2\alpha} + \cdots) - 1] \end{aligned}$$

$$u(x,t) = -(c^2-1)[\cosh(x)(1 + \frac{c^2t^2}{2!} + \frac{c^4t^4}{4!} + \cdots) - 1] + (c^2-1)[t\sinh(x)(1 + \frac{c^2t^2}{3!} + \frac{c^4t^4}{5!} + \cdots)], \tag{38}$$

*Case 3:*In this case, we first consider the time-space fractional K(2,2) type equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial^\beta u^2(x,t)}{\partial x^\beta}+\frac{\partial^{3\beta} u^2(x,t)}{\partial x^{3\beta}}=0,\quad 1<\alpha,\beta\leq 1 \tag{39}$$

subject to the initial condition

$$u(x,0)=x, \tag{40}$$

in view of the homotopy (12), we construct the homotopy

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=p[-\frac{\partial^\beta u^2(x,t)}{\partial x^\beta}-\frac{\partial^{3\beta} u^2(x,t)}{\partial x^{3\beta}}], \tag{41}$$

substituting (13) and the initial condition (37) into the homotopy (41) and equating the terms with identical powers of p, the first few components of the homotopy solution for Eq.(39) are derived as follows:

$$u_0=x, \tag{42}$$

$$\begin{aligned}u_1&=J^\alpha[-\frac{\partial^\beta u_0^2(x,t)}{\partial x^\beta}-\frac{\partial^{3\beta} u_0^2(x,t)}{\partial x^{3\beta}}]\\&=[-\frac{-2x^{2-\beta}}{\Gamma(3-\beta)}-\frac{-2x^{2-3\beta}}{\Gamma(3-3\beta)}]\frac{t^\alpha}{\Gamma(1+\alpha)},\end{aligned} \tag{43}$$

$$\begin{aligned}u_2&=J^\alpha[-2\frac{\partial^\beta u_0u_1}{\partial x^\beta}-2\frac{\partial^{3\beta} u_0u_1}{\partial x^{3\beta}}]\\&=\frac{4\Gamma(4-3\beta)t^{2\alpha}x^{3-4\beta}}{\Gamma(1+2\alpha)\Gamma(3-3\beta)\Gamma(4-4\beta)}+\frac{4\Gamma(4-\beta)t^{2\alpha}x^{3-2\beta}}{\Gamma(1+2\alpha)\Gamma(3-\beta)\Gamma(4-2\beta)}\\&+\frac{4\Gamma(4-3\beta)t^{2\alpha}x^{3-6\beta}}{\Gamma(1+2\alpha)\Gamma(3-3\beta)\Gamma(4-6\beta)}+\frac{4\Gamma(4-\beta)t^{2\alpha}x^{3-4\beta}}{\Gamma(1+2\alpha)\Gamma(3-\beta)\Gamma(4-4\beta)},\end{aligned} \tag{44}$$

$$u_3=J^\alpha[-\frac{\partial^\beta (u_1^2+2u_0u_2)}{\partial x^\beta}-\frac{\partial^{3\beta} (u_1^2+2u_0u_2)}{\partial x^{3\beta}}]$$

$$\begin{aligned}&=-\frac{4\Gamma(1+2\alpha)\Gamma(5-6\beta)t^{3\alpha}x^{4-9\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(5-9\beta)\Gamma^2(3-3\beta)}-\frac{4\Gamma(1+2\alpha)\Gamma(5-6\beta)t^{3\alpha}x^{4-7\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(5-7\beta)\Gamma^2(3-3\beta)}\\&-\frac{8\Gamma(5-6\beta)\Gamma(4-3\beta)t^{3\alpha}x^{4-9\beta}}{\Gamma(1+3\alpha)\Gamma(5-9\alpha)\Gamma(4-6\beta)\Gamma(3-3\beta)}-\frac{8\Gamma(5-6\beta)\Gamma(4-3\beta)t^{3\alpha}x^{4-7\beta}}{\Gamma(1+3\alpha)\Gamma(5-7\alpha)\Gamma(4-6\beta)\Gamma(3-3\beta)}\\&-\frac{8\Gamma(5-4\beta)\Gamma(4-3\beta)t^{3\alpha}x^{4-7\beta}}{\Gamma(1+3\alpha)\Gamma(5-7\beta)\Gamma(4-4\beta)\Gamma(3-3\beta)}-\frac{8\Gamma(5-4\beta)\Gamma(4-3\beta)t^{3\alpha}x^{4-5\beta}}{\Gamma(1+3\alpha)\Gamma(5-5\beta)\Gamma(4-4\beta)\Gamma(3-3\beta)}\\&-\frac{4\Gamma(5-2\beta)\Gamma(1+2\alpha)t^{3\alpha}x^{4-5\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(5-5\beta)\Gamma^2(3-\beta)}-\frac{4\Gamma(5-2\beta)\Gamma(1+2\alpha)t^{3\alpha}x^{4-3\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(5-3\beta)\Gamma^2(3-\beta)}\\&-\frac{8\Gamma(5-4\beta)\Gamma(1+2\alpha)t^{3\alpha}x^{4-7\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(5-7\beta)\Gamma(3-\beta)\Gamma(3-3\beta)}-\frac{8\Gamma(5-4\beta)\Gamma(1+2\alpha)t^{3\alpha}x^{4-5\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(5-5\beta)\Gamma(3-\beta)\Gamma(3-3\beta)}\\&-\frac{8\Gamma(5-4\beta)\Gamma(4-\beta)t^{3\alpha}x^{4-7\beta}}{\Gamma(1+3\alpha)\Gamma(5-7\beta)\Gamma(4-4\beta)\Gamma(3-\beta)}-\frac{8\Gamma(5-4\beta)\Gamma(4-\beta)t^{3\alpha}x^{4-5\beta}}{\Gamma(1+3\alpha)\Gamma(5-5\beta)\Gamma(4-4\beta)\Gamma(3-\beta)}\\&-\frac{8\Gamma(5-2\beta)\Gamma(4-\beta)t^{3\alpha}x^{4-5\beta}}{\Gamma(1+3\alpha)\Gamma(5-5\beta)\Gamma(4-2\beta)\Gamma(3-\beta)}-\frac{8\Gamma(5-2\beta)\Gamma(4-\beta)t^{3\alpha}x^{4-3\beta}}{\Gamma(1+3\alpha)\Gamma(5-3\beta)\Gamma(4-2\beta)\Gamma(3-\beta)},\end{aligned} \tag{45}$$

$$\vdots$$

and so on, in the same manner the rest of components can be obtained using Mathematica package, Consequently, we have the solution of Eq.(34) in a series form

$$u(x,t) = x + [-\frac{2x^{2-\beta}}{\Gamma(3-\beta)} - \frac{2x^{2-3\beta}}{\Gamma(3-3\beta)}]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{4\Gamma(4-3\beta)t^{2\alpha}x^{3-4\beta}}{\Gamma(1+2\alpha)\Gamma(3-3\beta)\Gamma(4-4\beta)} + \cdots, \tag{46}$$

if we take $\alpha = \beta = 1$, the homotopy solution is given by

$$u(x,t) = x(1 - 2t + 4t^2 - 8t^3 + \cdots), \tag{47}$$

Hence the exact solution of Eq.(39) for $\alpha = \beta = 1$ is given by

$$u(x,t) = \frac{x}{1+2t}, \tag{48}$$

Case 4: the time In this case, we consider the time-space fractional Kdv-Burgers equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + \varepsilon u\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} - \upsilon\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} + \eta\frac{\partial^{3\beta}u(x,t)}{\partial x^{3\beta}} = 0, \quad 1 < \alpha, \beta \leq 1 \tag{49}$$

subject to the initial condition

$$u(x,0) = x^3, \tag{50}$$

in view of the homotopy (12), we construct the homotopy

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = p[-\varepsilon u\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} + \upsilon\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - \eta\frac{\partial^{3\beta}u(x,t)}{\partial x^{3\beta}}], \tag{51}$$

substituting (13) and the initial condition (50) into the homotopy (51) and equating the terms with identical powers of p, the first few components of the homotopy solution for Eq.(49) are derived as follows:

$$u_0 = x^3, \tag{52}$$

$$u_1 = J^{\alpha}[-\varepsilon u_0\frac{\partial^{\beta}u_0}{\partial x^{\beta}} + \upsilon\frac{\partial^{2\beta}u_0}{\partial x^{2\beta}} - \eta\frac{\partial^{3\beta}u_0}{\partial x^{3\beta}}]$$
$$= -\frac{6\varepsilon t^{\alpha}x^{6-\beta}}{\Gamma(1+\alpha)\Gamma(4-\beta)} + \frac{6\upsilon t^{\alpha}x^{3-2\beta}}{\Gamma(1+\alpha)\Gamma(4-2\beta)} - \frac{6\eta t^{\alpha}x^{3-3\beta}}{\Gamma(1+\alpha)\Gamma(4-3\beta)}, \tag{53}$$

$$u_2 = J^{\alpha}[-\varepsilon u_0\frac{\partial^{\beta}u_1}{\partial x^{\beta}} - \varepsilon u_1\frac{\partial^{\beta}u_0}{\partial x^{\beta}} + \upsilon\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} - \eta\frac{\partial^{3\beta}u_1}{\partial x^{3\beta}}]$$
$$\frac{6t^{2\alpha}x^{3-6\beta}\eta^2}{\Gamma(1+2\alpha)\Gamma(4-6\beta)} - \frac{12t^{2\alpha}x^{3-5\beta}\eta\upsilon}{\Gamma(1+2\alpha)\Gamma(4-5\beta)} + \frac{6t^{2\alpha}x^{6-4\beta}\varepsilon\eta}{\Gamma(1+2\alpha)\Gamma(4-4\beta)}$$
$$\frac{6t^{2\alpha}x^{3-4\beta}\upsilon^2}{\Gamma(1+2\alpha)\Gamma(4-4\beta)} - \frac{6t^{2\alpha}x^{6-3\beta}\varepsilon\upsilon}{\Gamma(1+2\alpha)\Gamma(4-3\beta)} + \frac{36t^{2\alpha}x^{9-2\beta}\varepsilon^2}{\Gamma(1+2\alpha)\Gamma^2(4-\beta)}$$
$$+\frac{36t^{2\alpha}x^{6-4\beta}\eta\varepsilon}{\Gamma(1+2\alpha)\Gamma(4-3\beta)\Gamma(4-\beta)} - \frac{36t^{2\alpha}x^{6-3\beta}\varepsilon\upsilon}{\Gamma(1+2\alpha)\Gamma(4-2\beta)\Gamma(4-\beta)}$$
$$+\frac{6t^{2\alpha}x^{6-4\beta}\eta\varepsilon\Gamma(7-\beta)}{\Gamma(1+2\alpha)\Gamma(7-4\beta)\Gamma(4-\beta)} - \frac{6t^{2\alpha}x^{6-3\beta}\upsilon\varepsilon\Gamma(7-\beta)}{\Gamma(1+2\alpha)\Gamma(7-3\beta)\Gamma(4-\beta)}$$
$$+ \frac{6t^{2\alpha}x^{9-2\beta}\varepsilon^2\Gamma(7-\beta)}{\Gamma(1+2\alpha)\Gamma(7-2\beta)\Gamma(4-\beta)}, \tag{54}$$

and so on, in the same manner the rest of components can be obtained using Mathematica package, Consequently, we have the solution of Eq.(49) in a series form

$$u(x,t) = x^3 - \frac{6\varepsilon t^\alpha x^{6-\beta}}{\Gamma(1+\alpha)\Gamma(4-\beta)} + \frac{6\upsilon t^\alpha x^{3-2\beta}}{\Gamma(1+\alpha)\Gamma(4-2\beta)} - \frac{6\eta t^\alpha x^{3-3\beta}}{\Gamma(1+\alpha)\Gamma(4-3\beta)} + \cdots, \tag{55}$$

5. Conclusion

In this paper, approximate solutions for the time fractional cubic Boussinesq equation, time fractional Boussinesq-type equation, time-space fractional K(2.2) type equation and time-space fractional Kdv-Burgers have been obtained, and the homotopy perturbation method was successfully used to these solutions. The reliability of this method and reduction in computations give this method a wider applicability. The corresponding solutions are obtained according to the recurrence relation using Mahtematica.

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