

EXISTENCE RESULTS FOR FRACTIONAL IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is mainly concerned with the existence of solutions of impulsive fractional neutral functional differential equations in Banach Spaces. The results are obtained by using Krasnoselskii's fixed point theorem.

1. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling real world problems.

In the consequence, fractional differential equations have been of great interest. For details, see the monographs of Kilbas et al.[1], Lakshimkantham et al.[2], Miller and Ross [3], Podlubny [4] and the papers in [5, 6, 7, 8, 9, 10] and the references therein. On the otherhand, the theory of impulsive differential equations is also an important area of research which has been investigated in the last few years by great number of mathematicians. We recall that the impulsive differential equations may better model phenomena and dynamical processes subject to a great changes in short times issued, for instance, in biotechnology, automatics, population dynamics, economics and robotics. To learn more about this kind of problems, we refer to the books [11, 12]. So we propose to study fractional differential equation subject to a finite number of impulses.

Recently, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works [13, 12]. Balachandran and Trujillo [7] investigated the non-local Cauchy problem for non-linear fractional integro differential equations in Banach Spaces and in [14], Balachandran, Kiruthika and Trujillo obtained existence results for fractional impulsive integrodifferential equations in Banach spaces and in [6], Agarwal, YongZhou,

2000 *Mathematics Subject Classification.* 34K37, 34K30, 34A37.

Key words and phrases. Fractional neutral Impulsive differential equations, Existence, Bounded delay, Fixed point theorem.

Submitted Feb. 23, 2012. Published July 1, 2012.

YunYunHe proved the existence results of fractional neutral functional differential equations which motivates our present work to study the existence of solutions of fractional impulsive neutral functional differential equations in Banach Spaces by using fixed point theorems.

In this paper, we consider the fractional impulsive neutral functional differential equations with bounded delay of the form

$$\begin{cases} {}^c D^\alpha(x(t) - g(t, x_t)) = A(t, x)x(t) + f(t, x_t), & t \in (t_0, \infty), t_0 \geq 0, t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), & t = t_k, k = 1, 2, \dots, m \\ x_{t_0} = \phi \end{cases} \quad (1)$$

where ${}^c D^\alpha$ is the standard Caputo's fractional derivative of order $0 < \alpha < 1$, $f, g : [t_0, +\infty) \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ are given functions, $a > 0$ and $\phi \in C([-r, 0], \mathbb{R}^n)$. If $x \in C([t_0 - r, t_0 + a], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + a]$, define x_t by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$. Let $A(t, x)$ be a bounded linear operator on a Banach space \mathbb{R}^n and $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h \rightarrow 0^-} x(t_k - h)$, $k = 1, 2, \dots, m$ for $t_0 < t_1 < t_2 < \dots < t_m$.

2. PRELIMINARIES

Let $J \subset \mathbb{R}$. Denote $C(J, \mathbb{R}^n)$ be the Banach space of all continuous functions from J into \mathbb{R}^n with the norm $\|x\| = \sup_{t \in J} |x(t)|$, where $|\cdot|$ denotes a suitable complete norm on \mathbb{R}^n . Let $\mathcal{B}(\mathbb{R}^n)$ denote the Banach space of bounded linear operators from \mathbb{R}^n to \mathbb{R}^n with the norm $\|A\|_{\mathcal{B}(\mathbb{R}^n)} = \sup\{\|A(y)\| : \|y\| = 1\}$.

Also, consider the Banach space

$PC(J, \mathbb{R}^n) = \{x : J \rightarrow \mathbb{R}^n : x \in C((t_k, t_{k+1}], \mathbb{R}^n), k = 0, 1, \dots, m \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-), k = 1, 2, \dots, m \text{ with } x(t_k^-) = x(t_k)\}$
with the norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$.

Definition 1([1]). The Riemann-Liouville fractional integral operator of order $q > 0$ with the lower limit t_0 for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s) ds, \quad t > t_0$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the gamma function.

Definition 2([1]). The Riemann-Liouville (R-L) derivative of order $q > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \rightarrow \mathbb{R}$ can be written as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{(n-q-1)} f(s) ds, \quad t > t_0, \quad n-1 < q < n.$$

The most important property of R-L fractional derivative is that for $t > t_0$ and $q > 0$, we have $D^q(I^q f(t)) = f(t)$, which means that R-L fractional differentiation operator is a left inverse to the R-L fractional integration operator of the same order q .

Definition 3([1]). The Caputo fractional derivative of order $q > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{(n-q-1)} f^{(n)}(s) ds = I^{(n-q)} f^{(n)}(t), \quad t > t_0, \quad n-1 < q < n.$$

We shall state some properties of the operators I^α and ${}^c D^\alpha$.

Proposition 1([1, 2]). For $\alpha, \beta > 0$ and f as a suitable function, we have

- (i) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$
- (ii) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t)$
- (iii) $I^\alpha (f(t) + g(t)) = I^\alpha f(t) + I^\alpha g(t)$
- (iv) $I^\alpha {}^c D^\alpha f(t) = f(t) - f(0), 0 < \alpha < 1$
- (v) ${}^c D^\alpha I^\alpha f(t) = f(t)$
- (vi) ${}^c D^\alpha f(t) = I^{(1-\alpha)} Df(t) = I^{(1-\alpha)} f'(t), 0 < \alpha < 1, D = \frac{d}{dt}$
- (vii) ${}^c D^\alpha {}^c D^\beta f(t) \neq {}^c D^{(\alpha+\beta)} f(t)$
- (viii) ${}^c D^\alpha {}^c D^\beta f(t) \neq {}^c D^\beta {}^c D^\alpha f(t)$

In [7], Balachandran and Trujillo observed that both the R-L and the Caputo fractional differential operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. For basic facts about fractional integrals and fractional derivatives one can refer to the books [1, 3, 4, 10].

Lemma 1 (Krasnoselskii's Fixed point theorem)([17]). Let X be a Banach space, let E be a bounded closed convex subset of X and let S, U be maps of E into X such that $Sx + Uy \in E$ for every pair $x, y \in E$. If S is a contraction and U is completely continuous, then the equation $Sx + Ux = x$ has a solution on E .

Lemma 2([15]). Let $q > 0$; then the differential equation ${}^c D^q h(t) = 0$ has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 0, 1, 2, \dots, n-1, \quad n = [q] + 1.$$

Lemma 3([15]). Let $q > 0$, then $I^q {}^c D^q h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, for some $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1, n = [q] + 1$.

3. EXISTENCE RESULTS

Let $I_0 = [t_0, t_0 + \delta]$
 $B(\delta, \gamma) = \{x \in C([t_0 - r, t_0 + \delta], \mathbb{R}^n) \mid x_{t_0} = \phi, \sup_{t_0 \leq t \leq t_0 + \delta} |x(t) - \phi(0)| \leq \gamma\}$
 where δ, γ are positive constants.

For the forthcoming results, we need the following hypothesis:

- (H₁) $f(t, \varphi)$ is measurable with respect to t on I_0 .
- (H₂) $f(t, \varphi)$ is continuous with respect to φ on $C([-r, 0], \mathbb{R}^n)$.
- (H₃) there exist $\alpha_1 \in (0, \alpha)$ and a real-valued function $m(t) \in L^{\frac{1}{\alpha_1}}(I_0)$ such that for any $x \in B(\delta, \gamma)$, $|f(t, x_t)| \leq m(t)$, for $t \in I_0$.
- (H₄) for any $x \in B(\delta, \gamma)$, $g(t, x_t) = g_1(t, x_t) + g_2(t, x_t)$.

- (H₅) g_1 is continuous and for any $x', x'' \in B(\delta, \gamma), t \in I_0$,
 $|g_1(t, x'_t) - g_1(t, x''_t)| \leq l \|x' - x''\|$, where $l \in (0, 1)$.
- (H₆) g_2 is completely continuous and for any bounded set Λ in $B(\delta, \gamma)$, the set
 $\{t \rightarrow g_2(t, x_t) : x \in \Lambda\}$ is equicontinuous in $PC(I_0, \mathbb{R}^n)$.
- (H₇) $A : I_0 \times \mathbb{R}^n \rightarrow \mathcal{B}(\mathbb{R}^n)$ is a continuous bounded linear operator and
there exists a constant $L_1 > 0$ such that $\|A(t, x) - A(t, y)\| \leq L_1 \|x - y\|$,
for all $x, y \in \mathbb{R}^n$.
- (H₈) The functions $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous and there exists a constant
 $0 < L_2 < 1$ such that $\|I_k(u) - I_k(v)\| \leq \frac{L_2}{m} \|u - v\|, u, v \in \mathbb{R}^n, k = 1, 2, \dots, m,$
 $m > 0$.

Lemma 4 If there exist $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ such that (H₁) – (H₃) are satisfied, then for $t \in (t_0, t_0 + \delta]$, the equation (1) is equivalent to the following equation.

$$x(t) = \begin{cases} \phi(0) + g(t, x_t) - g(t_0, \phi) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s, x_s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} A(s, x) x(s) ds, & t \in [t_0, t_1] \\ \phi(0) - g(t_0, \phi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x_s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x) x(s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} A(s, x) x(s) ds + \sum_{t_0 < t_k < t} I_k(x(t_k^-)), & t \in (t_k, t_{k+1}] \end{cases} \quad (2)$$

$$x_{t_0} = \phi$$

Proof. Suppose that x satisfies the equation (1), then we have
if $t \in [t_0, t_1]$,

$$x(t) = g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} A(s, x) x(s) ds - c_0 \quad (3)$$

for some $c_0 \in \mathbb{R}$.

If $t \in (t_1, t_2]$,

$$x(t) = g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} A(s, x) x(s) ds - d_0$$

for some $d_0 \in \mathbb{R}$.

Consider,

$$\begin{aligned} x(t_1^+) &= g(t_1, x_{t_1}) - d_0 \\ x(t_1^-) &= g(t_1, x_{t_1}) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, x_s) ds + \\ &\quad \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} A(s, x) x(s) ds - c_0 \end{aligned}$$

We know that $\Delta x|_{t=t_1} = x(t_1^+) - x(t_1^-)$ and $\Delta x|_{t=t_1} = I_1(x(t_1^-))$.

$$\begin{aligned}
x(t) &= g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + I_1(x(t_1^-)) + x(t_1^-) - g(t_1, x_{t_1}) \\
&= g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + I_1(x(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} \\
&\quad A(s, x) x(s) ds - c_0
\end{aligned}$$

If $t \in (t_2, t_3]$,

$$x(t) = g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} A(s, x) x(s) ds - e_0$$

for some $e_0 \in \mathbb{R}$.

$$\begin{aligned}
x(t_2^+) &= g(t_2, x_{t_2}) - e_0 \\
x(t_2^-) &= g(t_2, x_{t_2}) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, x_s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} A(s, x) x(s) ds + I_1(x(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} f(s, x_s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} A(s, x) x(s) ds - c_0
\end{aligned}$$

In view of $\Delta x|_{t=t_2} = x(t_2^+) - x(t_2^-)$ and $\Delta x|_{t=t_2} = I_2(x(t_2^-))$, we have

$$\begin{aligned}
x(t) &= g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + x(t_2^+) - g(t_2, x_{t_2}) \\
&= g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + I_2(x(t_2^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + I_1(x(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} f(s, x_s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} A(s, x) x(s) ds - c_0
\end{aligned}$$

Applying initial conditions on equation (3), we have

$$\begin{aligned}
x(t) &= \phi(0) - g(t_0, \phi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, x_s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t - s)^{\alpha-1} f(s, x_s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} A(s, x) x(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t - s)^{\alpha-1} A(s, x) x(s) ds + I_1(x(t_1^-)) + I_2(x(t_2^-)).
\end{aligned}$$

Proceeding like this, we get

$$\begin{aligned}
x(t) &= \phi(0) - g(t_0, \phi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x_s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x) x(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s, x) x(s) ds + \sum_{t_0 < t_k < t} I_k(x(t_k^-)).
\end{aligned}$$

Conversely, assume that x is a solution of equation (2).

If $t \in [t_0, t_1]$, then $x_{t_0} = \phi$ and using the fact that ${}^c D^\alpha$ is the left inverse of I^α , we get

$${}^c D^\alpha(x(t) - g(t, x_t)) = A(t, x)x(t) + f(t, x_t) \text{ for each } t \in [t_0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$ and using the fact that ${}^c D^\alpha C = 0$, where C is a constant, we get ${}^c D^\alpha(x(t) - g(t, x_t)) = A(t, x)x(t) + f(t, x_t)$ for each $t \in (t_k, t_{k+1}]$.

Also, we can easily show that $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, m$. Therefore, x is a solution of equation (1).

Also by (H_1) and (H_2) , we get that $f(t, x_t)$ is Lebesgue measurable on I_0 . A direct calculation gives that $(t - s)^{\alpha-1} \in L^{\frac{1}{1-\alpha_1}}([t_0, t])$, for $t \in I_0$. By Holder's inequality and by (H_3) , we get that

$$\int_{t_0}^t |(t - s)^{\alpha-1} f(s, x_s)| ds \leq \| (t - s)^{\alpha-1} \|_{L^{\frac{1}{1-\alpha_1}}([t_0, t])} \| m \|_{L^{\frac{1}{\alpha_1}}(I_0)}$$

which means that $(t - s)^{\alpha-1} f(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in I_0$ and $x \in B(\delta, \gamma)$, where

$$\| F \|_{L^p(J)} = \left(\int_J |F(t)|^p dt \right)^{\frac{1}{p}}, \text{ for any } L^p\text{-integrable function } F : J \rightarrow \mathbb{R}.$$

Theorem 1 Assume that there exist $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ such that $(H_1) - (H_8)$ are satisfied, then the IVP(1) has at least one solution

$$\begin{cases} x(t) = \phi(0) - g(t_0, \phi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x_s) ds \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x) x(s) ds \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} A(s, x) x(s) ds + \sum_{t_0 < t_k < t} I_k(x(t_k^-)). \\ x_{t_0} = \phi \end{cases} \quad (4)$$

on $[t_0, t_0 + \eta]$ for some positive number η .

Proof. From (H_4) , equation (4) is of the form,

$$\begin{cases} x(t) = \phi(0) - g_1(t_0, \phi) - g_2(t_0, \phi) + g_1(t, x_t) + g_2(t, x_t) \\ \quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x_s) ds \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x) x(s) ds \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} A(s, x) x(s) ds + \sum_{t_0 < t_k < t} I_k(x(t_k^-)), \quad t \in I_0 \\ x_{t_0} = \phi \end{cases} \quad (5)$$

Let $\tilde{\phi} \in B(\delta, \gamma)$ be defined as $\tilde{\phi}_{t_0} = \phi$, $\tilde{\phi}(t_0 + t) = \phi(0)$, $\forall t \in [0, \delta]$.

If x is a solution of the equation (1), let us assume that $x(t_0 + t) = \tilde{\phi}(t_0 + t) + y(t)$, $t \in [-r, \delta]$, then we have $x_{t_0+t} = \tilde{\phi}_{t_0+t} + y_t$, $t \in [0, \delta]$.

Therefore, the above equation becomes,

$$\begin{aligned} y(t) &= -g_1(t_0, \phi) - g_2(t_0, \phi) + g_1(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) + g_2(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)}) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)}) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(t_0 + s, x) x(t_0 + s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} A(t_0 + s, x) x(t_0 + s) ds \\ &\quad + \sum_{t_0 < t_k < t} I_k(x(t_k^-)), \quad t \in [0, \delta]. \end{aligned} \quad (6)$$

According to (H_5) and (H_6) , g_1 and g_2 are continuous, x_t is continuous in t , there exists $\delta' > 0$, when $0 < t < \delta'$,

$$|g_1(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) - g_1(t_0, \phi)| < \frac{\gamma}{5}$$

and

$$|g_2(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) - g_2(t_0, \phi)| < \frac{\gamma}{5}$$

Choose

$$\eta = \left\{ \delta, \delta', \left(\frac{\gamma \Gamma(\alpha) (1 + \beta)^{1-\alpha_1}}{5M(m+1)} \right)^{\frac{1}{(1+\beta)(1-\alpha_1)}}, \left(\frac{\Gamma(\alpha+1)}{5(L_1 \gamma + k)(m+1)} \right)^{\frac{1}{\alpha}} \right\}$$

where $\beta = \frac{\alpha-1}{1-\alpha_1} \in (-1, 0)$ and $M = \|m\|_{L^{\frac{1}{\alpha_1}}(I_0)}$.

Define $E(\eta, \gamma)$ as $E(\eta, \gamma) = \{y \in PC([-r, \eta], \mathbb{R}^n) | y(s) = 0 \text{ for } s \in [-r, 0] \text{ and } \|y\| \leq \gamma\}$.

Clearly, $E(\eta, \gamma)$ is a closed bounded and convex subset of $PC([-r, \delta], \mathbb{R}^n)$.

Now, we introduce the operators S and U on $E(\eta, \gamma)$ as follows:

$$\begin{aligned}
 Sy(t) &= \begin{cases} 0, & t \in [-r, 0] \\ -g_1(t_0, \phi) + g_1(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}), & t \in [0, \eta] \end{cases} \\
 Uy(t) &= \begin{cases} 0, & t \in [-r, 0] \\ -g_2(t_0, \phi) + g_2(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) \\ + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)}) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)}) ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(t_0 + s, x)x(t_0 + s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(t_0 + s, x)x(t_0 + s) ds + \sum_{t_0 < t_k < t} I_k(x(t_k^-)), & t \in [0, \eta]. \end{cases}
 \end{aligned}$$

Then, $y = Sy + Uy$ has a solution on $E(\eta, \gamma)$ iff y is a solution of equation (6).

Thus, $x(t_0 + t) = y(t) + \tilde{\phi}_{(t_0+t)}$

is a solution of equation (1), on $[0, \eta]$.

Therefore, if y has a fixed point in $E(\eta, \gamma)$, then there exists a solution of the equation (1).

So, now, we have to prove that $S + U$ has a fixed point in $E(\eta, \gamma)$.

For brevity, let us take $k = \sup_{t \in J} \|A(t, 0)\|$.

From (H_7) ,

$$\begin{aligned}
 \|A(t, x)\| &\leq \|A(t, x) - A(t, 0)\| + \|A(t, 0)\| \\
 &\leq L_1 \gamma + k.
 \end{aligned}$$

The proof is divided into three steps.

Step I. To prove $Sz + Uy \in E(\eta, \gamma)$ for every pair $z, y \in E(\eta, \gamma)$.

Since, $z, y \in E(\eta, \gamma)$, $(Sz + Uy)(t) = 0, t \in [-r, 0]$. Obviously, $Sz + Uy \in PC([-r, \eta], \mathbb{R}^n)$.

Consider,

$$\begin{aligned}
 |Sz(t) + Uy(t)| &\leq | -g_1(t_0, \phi) + g_1(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) | \\
 &\quad + | -g_2(t_0, \phi) + g_2(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}) | \\
 &\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(t_0 + s, x)\| \|x(t_0 + s)\| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(t_0 + s, x)\| \|x(t_0 + s)\| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} |(t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)})| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t |(t - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)})| ds + \sum_{t_0 < t_k < t} \|I_k(x(t_k^-))\|
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} (L_1\gamma + k)\gamma \int_{t_k}^t (t-s)^{\alpha-1} ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \left(\int_{t_{k-1}}^{t_k} (t_k - s)^{\frac{\alpha-1}{1-\alpha_1}} \right)^{1-\alpha_1} \left(\int_{t_{k-1}}^{t_k} m(s)^{\frac{1}{\alpha_1}} \right)^{\alpha_1} \\
& + \frac{1}{\Gamma(\alpha)} \left(\int_{t_k}^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} \right)^{1-\alpha_1} \left(\int_{t_k}^t m(s)^{\frac{1}{\alpha_1}} \right)^{\alpha_1} + L_2\gamma \\
\leq & \frac{2\gamma}{5} + \frac{(m+1)\eta^\alpha}{\Gamma(\alpha+1)} (L_1\gamma + k)\gamma \\
& + \frac{M(m+1)\eta^{(1+\beta)(1-\alpha_1)}}{\Gamma(\alpha)} \frac{1}{(1+\beta)^{1-\alpha_1}} + L_2\gamma \\
\leq & \gamma
\end{aligned}$$

Therefore, $\|Sz + Uy\| = \sup_{t \in [0, \eta]} |Sz(t) + Uy(t)| \leq \gamma$ which means that $Sz + Uy \in E(\eta, \gamma)$ for any $z, y \in E(\eta, \gamma)$.

Step II. To prove S is a contraction on $E(\eta, \gamma)$.

Let $y', y'' \in E(\eta, \gamma)$, $y'_t + \tilde{\phi}_{(t_0+t)}$, $y''_t + \tilde{\phi}_{(t_0+t)} \in B(\delta, \gamma)$. From (H_5) , we have

$$\begin{aligned}
|Sy'(t) - Sy''(t)| & = |g_1(t_0 + t, y'_t + \tilde{\phi}_{(t_0+t)}) - g_1(t_0 + t, y''_t + \tilde{\phi}_{(t_0+t)})| \\
& \leq l \|y' - y''\|
\end{aligned}$$

which implies that $\|Sy' - Sy''\| \leq l \|y' - y''\|$, where $0 < l < 1$. So, S is a contraction on $E(\eta, \gamma)$.

Step III. To prove that U is a completely continuous operator.

Let

$$U_1 y(t) = \begin{cases} 0 & t \in [-r, 0] \\ -g_2(t_0, \phi) + g_2(t_0 + t, y_t + \tilde{\phi}_{(t_0+t)}), & t \in [0, \eta] \end{cases}$$

and

$$U_2 y(t) = \begin{cases} 0 & t \in [-r, 0] \\ \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)}) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}_{(t_0+s)}) ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(t_0 + s, x) x(t_0 + s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} A(t_0 + s, x) x(t_0 + s) ds + \sum_{t_0 < t_k < t} I_k(x(t_k^-)), & t \in [0, \eta]. \end{cases}$$

Clearly, $U = U_1 + U_2$.

Since, g_2 is completely continuous, U_1 is continuous and $\{U_1 y : y \in E(\eta, \gamma)\}$ is uniformly bounded. From (H_6) , we can conclude that U_1 is a completely continuous operator.

On the otherhand, for any $t \in [0, \eta]$, we have

$$\begin{aligned}
\|U_2y(t)\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} |(t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}(t_0 + s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t |(t - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}(t_0 + s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(t_0 + s, x)\| \|x(t_0 + s)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(t_0 + s, x)\| \|x(t_0 + s)\| ds \\
&\quad + \sum_{t_0 < t_k < t} \|I_k(x(t_k^-))\| \\
&\leq \frac{M(m+1)\eta^{(1+\beta)(1-\alpha_1)}}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} \\
&\quad + \frac{(m+1)\eta^\alpha}{\Gamma(\alpha+1)}(L_1\gamma + k)\gamma + L_2\gamma \\
&\leq \gamma
\end{aligned}$$

Hence, $\{U_2y : y \in E(\eta, \gamma)\}$ is uniformly bounded.

Next, we have to prove that $\{U_2y : y \in E(\eta, \gamma)\}$ is equicontinuous.

For any $0 \leq t_1 < t_2 \leq \eta$ and $y \in E(\eta, \gamma)$, we get that

$$\begin{aligned}
|U_2y(t_2) - U_2y(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t_2 - t_1} \int_{t_{k-1}}^{t_k} |(t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}(t_0 + s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] |f(t_0 + s, y_s + \tilde{\phi}(t_0 + s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(t_0 + s, y_s + \tilde{\phi}(t_0 + s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t_2 - t_1} \int_{t_{k-1}}^{t_k} |(t_k - s)^{\alpha-1} A(t_0 + s, x)x(t_0 + s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \|A(t_0 + s, x)\| \|x(t_0 + s)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|A(t_0 + s, x)\| \|x(t_0 + s)\| ds \\
&\quad + \sum_{t_0 < t_k < t_2 - t_1} \|I_k(x(t_k^-))\|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t_2 - t_1} \int_{t_{k-1}}^{t_k} |(t_k - s)^{\alpha-1} f(t_0 + s, y_s + \tilde{\phi}(t_0 + s))| ds \\ &\quad + \frac{M}{\Gamma(\alpha)} \left(\int_{t_k}^{t_1} (t_1 - s)^\beta - (t_2 - s)^\beta ds \right)^{1-\alpha_1} \\ &\quad + \frac{M}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2 - s)^\beta ds \right)^{1-\alpha_1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k < t_2 - t_1} \int_{t_{k-1}}^{t_k} |(t_k - s)^{\alpha-1} A(t_0 + s, x)x(t_0 + s)| ds \\ &\quad + \frac{(L_1\gamma + k)\gamma}{\Gamma(\alpha + 1)} [(t_2 - t_k)^\alpha - (t_1 - t_k)^\alpha + 2(t_2 - t_1)^\alpha] \\ &\quad + \sum_{t_0 < t_k < t_2 - t_1} \|I_k(x(t_k^-))\| \end{aligned}$$

The right-hand side tends to zero as $t_2 - t_1 \rightarrow 0$, which means that $\{U_2 y : y \in E(\eta, \gamma)\}$ is equicontinuous. Also, it is clear that U_2 is continuous. So, U_2 is a completely continuous operator. Thus, $U = U_1 + U_2$ is a completely continuous operator.

Therefore, by Krasnoselskii’s fixed point theorem $S + U$ has a fixed point on $E(\eta, \gamma)$ and hence the equation (1) has a solution $x(t) = \phi(0) + y(t - t_0)$, for all $t \in [t_0, t_0 + \eta]$. Since, η is arbitrary, the equation (1) has a solution $x(t) = \phi(0) + y(t - t_0)$ for all $t \in [t_0, \infty)$. This completes the proof.

In the case where $g_1 = 0$, we get the following result.

Theorem 2 Assume that there exist $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ such that $(H_1) - (H_3)$ hold and $(H_5)'$ g is continuous and for any $x', x'' \in B(\delta, \gamma)$, $t \in I_0$ $|g_1(t, x'_t) - g_1(t, x''_t)| \leq l \|x' - x''\|$, where $l \in (0, 1)$. Then the equation (1) has atleast one solution on $[t_0, t_0 + \eta]$ for some positive number η .

In the case where $g_2 = 0$, we have the following result.

Theorem 3 Assume that there exist $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ such that $(H_1) - (H_3)$ hold and $(H_6)'$ g is completely continuous and for any bounded set Λ in $B(\delta, \gamma)$, the set $\{t \rightarrow g(t, x_t) : x \in \Lambda\}$ is equicontinuous on $PC(I_0, \mathbb{R}^n)$. Then the equation (1) has atleast one solution on $[t_0, t_0 + \eta]$ for some positive number η .

4. EXAMPLE

Consider the following fractional differential equation with impulsive conditions of the form

$$\begin{cases} {}^c D^\alpha(x(t) - \frac{|x|+t}{(t+9)(|x|+1)}) = \frac{1}{9} \sin x(t)x(t) + \frac{|x|}{(t+1)(|x|+9)}, & t \in (0, \infty), \text{ where } t_0 = 0 \\ \Delta x|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2}^-)|}{9+|x(\frac{1}{2}^-)|} \\ x_0 = \phi[-1, 0] \end{cases} \tag{7}$$

where $0 < \alpha < 1$. Take $\delta = 1$ so that $I_0 = [0, 1]$.

Also, we choose $\alpha = \frac{1}{2}, \alpha_1 = \frac{1}{4}$ so that $\beta = \frac{-2}{3}, \eta = 1$ and $\gamma = 1$.

Clearly, we have $L_1 = \frac{1}{9}, l = \frac{1}{9}, \frac{L_2}{m} = \frac{1}{9}$.

Here $k = \frac{1}{9}$ and $M = 0.0816543 < \frac{1}{12}$.

If we take $m = 1$, all the conditions of theorem [1] are satisfied. Hence, by the conclusion of theorem [1], the problem (7) has a solution on $[0, 1]$.

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