

ANALYTICAL SOLUTION OF FRACTIONAL BLACK-SCHOLES EUROPEAN OPTION PRICING EQUATION BY USING LAPLACE TRANSFORM

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ABSTRACT. In this paper, Laplace homotopy perturbation method, which is combined form of the Laplace transform and the homotopy perturbation method, is employed to obtain a quick and accurate solution to the fractional Black Scholes equation with boundary condition for a European option pricing problem. The Black-Scholes formula is used as a model for valuing European or American call and put options on a non-dividend paying stock. The proposed scheme finds the solutions without any discretization or restrictive assumptions and is free from round-off errors and therefore, reduces the numerical computations to a great extent. The analytical solution of the fractional Black Scholes equation is calculated in the form of a convergent power series with easily computable components. Two examples are presented.

1. INTRODUCTION

In 1973, Fischer Black and Myron Scholes [1] derived the famous theoretical valuation formula for options. The main conceptual idea of Black and Scholes lie in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. Such an approach strengthens the use of the no-arbitrage principle as well. Thus, the Black-Scholes formula is used as a model for valuing European (the option can be exercised only on a specified future date) or american (the option can be exercised at any time up to the date, the option expires) call and put options on a non-dividend paying stock by Manale and Mahomed [2]. Derivation of a closed-form solution to the Black-Scholes equation depends on the fundamentals solution of the heat equation. Hence, it is important, at this point, to transform the Black-Scholes equation to the heat equation by change of variables. Having found the closed form solution to the heat equation, it is possible to transform it back to find the corresponding solution of the Black-Scholes PDE. Financial models were generally formulated in terms of stochastic differential equations. However, it was soon found that under certain restrictions these models could written as linear evolutionary PDEs with variable coefficients by Gazizov and Ibragimov [3]. Thus,

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the Black-Scholes model for the value of an option is described by the equation

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 v}{\partial x^2} + r(t) x \frac{\partial v}{\partial x} - r(t) v = 0, \quad (x, t) \in R^+ \times (0, T) \quad (1)$$

where $v(x, t)$ is the European call option price at asset price x and at time t , K is the exercise price, T is the maturity, $r(t)$ is the risk free interest rate, and $\sigma(x, t)$ represents the volatility function of underlying asset. Let us denote by $v_c(x, t)$ and $v_p(x, t)$ the value of the European call and put options, respectively. Then, the payoff functions are

$$v_c(x, t) = \max(x - E, 0), \quad v_p(x, t) = \max(E - x, 0), \quad (2)$$

where E denotes the expiration price for the option and the function $\max(x, 0)$ gives the larger value between x and 0. During the past few decades, many researchers studied the existence of solutions of the Black Scholes model using many methods in [4, 5, 6, 7, 8, 9, 10, 11, 12].

The seeds of fractional calculus (that is, the theory of integrals and derivatives of any arbitrary real or complex order) were planted over 300 years ago. Since then, many researchers have contributed to this field. Recently, it has turned out that differential equations involving derivatives of non-integer order can be adequate models for various physical phenomena Podlubny [13]. The book by Oldham and Spanier [14] has played a key role in the development of the subject. Some fundamental results related to solving fractional differential equations may be found in Miller and Ross [15], Kilbas and Srivastava [16].

The LHPM basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear differential equations by manipulating the homotopy perturbation method which was first introduced and applied by He [17, 18, 19, 20, 21]. The proposed method is coupling of the Laplace transformation, the homotopy perturbation method and He's polynomials and is mainly due to Ghorbani [22, 23]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations by using various methods with combined the Laplace transform. Among these are the Laplace decomposition methods [24, 25], Laplace homotopy perturbation method [26, 27]. The LHPM method is very well suited to physical problems since it does not require unnecessary linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously.

In this paper, the LHPM is applied to solve fractional Black-Scholes equation by using He's polynomials and well known Laplace transform. We discuss how to solve fractional Black-Scholes equation by using LHPM.

2. BASIC DEFINITIONS OF FRACTIONAL CALCULUS AND LAPLACE TRANSFORM

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper:

Definition 1. *A real function $f(t)$, $t > 0$ is said to be in the space C_μ , $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C(0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.*

Definition 2. The left sided Riemann-Liouville fractional integral operator of order $\mu \geq 0$, of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as follows [28-29]:

$$I^\mu f(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, & \mu > 0, t > 0, \\ f(t), & \mu = 0 \end{cases} \quad (3)$$

Where $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 3. The left sided Caputo fractional derivative of f , $f \in C_{-1}^m$, $m \in N \cup \{0\}$ is defined as follows [13, 30]:

$$D_*^\mu f(t) = \frac{\partial^\mu f(t)}{\partial t^\mu} = \begin{cases} I^{m-\mu} \left[\frac{\partial^m f(t)}{\partial t^m} \right], & m-1 < \mu < m, m \in N, \\ \frac{\partial^m f(t)}{\partial t^m}, & \mu = m, \end{cases} \quad (4)$$

Note that [13, 30]

$$I_t^\mu f(x, t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x, t)}{(t-s)^{1-\mu}} dt, \quad \mu > 0, t > 0, \quad (5)$$

$$D_*^\mu f(x, t) = I_t^{m-\mu} \frac{\partial^m f(x, t)}{\partial t^m} \quad m-1 < \mu \leq m, \quad (6)$$

Definition 4. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [31]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (7)$$

Definition 5. The Laplace transform of $f(t)$

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt. \quad (8)$$

Definition 6. The Laplace transform $L[f(t)]$ of the Riemann-Liouville fractional integral is defined as follows [15]:

$$L[I^\alpha f(t)] = s^{-\alpha} F(s). \quad (9)$$

Definition 7. The Laplace transform $L[f(t)]$ of the Caputo fractional derivative is defined as follows [15]:

$$L[D_*^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \quad n-1 < \alpha \leq n \quad (10)$$

3. FRACTIONAL LAPLACE TRANSFORM HOMOTOPY PERTURBATION METHOD

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation method, we consider the following nonlinear fractional differential equation:

$$\begin{aligned} D_t^\alpha u(x, t) + R[x]u(x, t) + N[x]u(x, t) &= q(x, t), \quad t > 0, x \in R, \quad 0 < \alpha \leq 1, \\ u(x, 0) &= h(x), \end{aligned} \quad (11)$$

where $D^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, $R[x]$ is the linear operator in x , $N[x]$ is the general nonlinear operator in x , and $q(x, t)$ are continuous functions. Now, the methodology consists of applying Laplace transform first on both sides of Eq. (11), we get

$$L[D_t^\alpha u(x, t)] + L[R[x]u(x, t) + N[x]u(x, t)] = L[q(x, t)], \quad (12)$$

Now, using the differentiation property of the Laplace transform, we have

$$L[u(x, t)] = s^{-1}h(x) - s^{-\alpha}L[q(x, t)] + s^{-\alpha}L[R[x]u(x, t) + N[x]u(x, t)], \quad (13)$$

Operating the inverse Laplace transform on both sides in Eq. (13), we get

$$u(x, t) = G(x, t) - L^{-1} \left(s^{-\alpha} L[R[x]u(x, t) + N[x]u(x, t)] \right), \quad (14)$$

where $G(x, t)$, represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p as given below

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t), \quad (15)$$

where the homotopy parameter p is considered as a small parameter ($p \in [0, 1]$). The nonlinear term can be decomposed as

$$Nu(x, t) = \sum_{n=0}^{\infty} p^n H_n(u), \quad (16)$$

where H_n are He's polynomials of $u_0, u_1, u_2, \dots, u_n$ and it can be calculated by the following formula

$$H_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, 3, \dots$$

Substituting Eq. (15) and (16) in Eq. (14) and using HPM [17, 18, 19, 20, 21], we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p \left(L^{-1} \left[s^{-\alpha} L \left[R \sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right), \quad (17)$$

This is coupling of the Laplace transform and homotopy perturbation method using He's polynomials. Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as

$$\begin{aligned} p^0 & : u_0(x, t) = G(x, t), \\ p^1 & : u_1(x, t) = L^{-1} \left(s^{-\alpha} L[R[x]u_0(x, t) + H_0(u)] \right), \\ p^2 & : u_2(x, t) = L^{-1} \left(s^{-\alpha} L[R[x]u_1(x, t) + H_1(u)] \right), \\ p^3 & : u_3(x, t) = L^{-1} \left(s^{-\alpha} L[R[x]u_2(x, t) + H_2(u)] \right), \\ p^4 & : u_4(x, t) = L^{-1} \left(s^{-\alpha} L[R[x]u_3(x, t) + H_3(u)] \right), \\ & \vdots \end{aligned}$$

Proceeding in this same manner, the rest of the components $u_n(x, t)$ can be completely obtained and the series solution is thus entirely determined. Finally, we

approximate the analytical solution $u(x, t)$ by truncated series

$$u(x, t) = \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n(x, t), \quad (18)$$

The above series solutions generally converge very rapidly.

4. NUMERICAL EXAMPLES

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the homotopy perturbation method with coupling of the Laplace transform. The simplicity and accuracy of the proposed method is illustrated through the following numerical examples.

Example 1. *We consider the following fractional Black-Scholes option pricing equation [12] as follows:*

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + (k-1) \frac{\partial v}{\partial x} - kv, \quad , \quad 0 < \alpha \leq 1, \quad (19)$$

with initial condition $v(x, 0) = \max(e^x - 1, 0)$. Notice that this system of equations contains just two dimensionless parameters $k = 2r/\sigma^2$, where k represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry $\frac{1}{2}\sigma^2 T$, even though there are four dimensional parameters, E , T , σ^2 and r , in the original statements of the problem.

Now, applying the aforesaid method subject to the initial condition, we have

$$L[v(x, t)] = \frac{1}{s} \max(e^x - 1, 0) + \frac{1}{s^\alpha} L[v_{xx} + (k-1)v_x - kv], \quad (20)$$

Operating the Inverse Laplace transform on both sides in (20), we have

$$v(x, t) = \max(e^x - 1, 0) + L^{-1} \left(\frac{1}{s^\alpha} L[v_{xx} + (k-1)v_x - kv] \right), \quad (21)$$

Now, we apply the homotopy perturbation method [17, 18, 19, 20, 21], we get

$$\sum_{n=0}^{\infty} p^n v_n(x, t) = \max(e^x - 1, 0) + p \left(L^{-1} \left(\frac{1}{s^\alpha} L \left[\sum_{n=0}^{\infty} p^n H_n(v) \right] \right) \right), \quad (22)$$

Where $H_n(v)$ are He's polynomials Ghorbani [22, 23]. The components of He's polynomials are given by the recursive relation

$$H_n(v) = v_{nxx} + (k-1)v_{nx} + kv_n, \quad n \geq 0, \quad n \in N. \quad (23)$$

Equating the corresponding power of p on both sides in equation (22), we get

$$\begin{aligned}
p^0 & : v_0(x, t) = \max(e^x - 1, 0), \\
p^1 & : v_1(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_0(v)] \right) = -\max(e^x, 0) \frac{(-kt^\alpha)}{\Gamma(\alpha + 1)} + \max(e^x - 1, 0) \frac{(-kt^\alpha)}{\Gamma(\alpha + 1)}, \\
p^2 & : v_2(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_1(v)] \right) = -\max(e^x, 0) \frac{(-kt^\alpha)^2}{\Gamma(2\alpha + 1)} + \max(e^x - 1, 0) \frac{(-kt^\alpha)^2}{\Gamma(2\alpha + 1)}, \\
p^3 & : v_3(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_2(v)] \right) = -\max(e^x, 0) \frac{(-kt^\alpha)^3}{\Gamma(3\alpha + 1)} + \max(e^x - 1, 0) \frac{(-kt^\alpha)^3}{\Gamma(3\alpha + 1)}, \\
& \vdots \\
p^n & : v_n(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_{n-1}(v)] \right) = -\max(e^x, 0) \frac{(-kt^\alpha)^n}{\Gamma(n\alpha + 1)} + \max(e^x - 1, 0) \frac{(-kt^\alpha)^n}{\Gamma(n\alpha + 1)}.
\end{aligned} \tag{24}$$

So that the solution $v(x, t)$ of the problem given by

$$v(x, t) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n v_n(x, t) = \max(e^x - 1, 0) E_\alpha(-kt^\alpha) + \max(e^x, 0) (1 - E_\alpha(-kt^\alpha)), \tag{25}$$

where $E_\alpha(z)$ is Mittag-Leffler function in one parameter. Eq. (25) represents the closed form solution of the fractional Black Scholes equation Eq. (19). Now for the standard case $\alpha = 1$, this series has the closed form of the solution $v(x, t) = \max(e^x - 1, 0) e^{-kt} + \max(e^x, 0) (1 - e^{-kt})$, which is an exact solution of the given Black Scholes equation (19) for $\alpha = 1$.

Example 2. In this example, we consider the following generalized fractional Black - Scholes equation [6] as follows:

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + 0.08(2 + \sin x)^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06 x \frac{\partial v}{\partial x} - 0.06 v = 0, \quad 0 < \alpha \leq 1, \tag{26}$$

with initial condition $v(x, 0) = \max(x - 25e^{-0.06}, 0)$. The methodology consists of the applying Laplace transform on both sides to Eq. (26), we get

$$L[v(x, t)] = \frac{1}{s} \max(x - 25e^{-0.06}, 0) - \frac{1}{s^\alpha} L [0.08(2 + \sin x)^2 x^2 v_{xx} + 0.06 x v_x - 0.06 v], \tag{27}$$

Now, applying the inverse Laplace transform on both sides to Eq. (27), we get

$$v(x, t) = \max(x - 25e^{-0.06}, 0) - L^{-1} \left(\frac{1}{s^\alpha} L [0.08(2 + \sin x)^2 x^2 v_{xx} + 0.06 x v_x - 0.06 v] \right), \tag{28}$$

Now, we apply the homotopy perturbation method [17, 18, 19, 20, 21], we have

$$\sum_{n=0}^{\infty} p^n v_n(x, t) = \max(x - 25e^{-0.06}, 0) - p \left(L^{-1} \left(\frac{1}{s^\alpha} L \left[\sum_{n=0}^{\infty} p^n H_n(v) \right] \right) \right), \tag{29}$$

The components of He's polynomials are given by relation

$$H_n(v) = 0.08(2 + \sin x)^2 x^2 v_{nxx} + 0.06 x v_{nx} - 0.06 v_n, \quad n \geq 0, \tag{30}$$

Equating the corresponding power of p on both sides in equation (29), we get

$$\begin{aligned}
 p^0 & : v_0(x, t) = \max(x - 25e^{-0.06}, 0), \\
 p^1 & : v_1(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_0(v)] \right) = -x \frac{(-0.06t^\alpha)}{\Gamma(\alpha + 1)} + \max(x - 25e^{-0.06}, 0) \frac{(-0.06t^\alpha)}{\Gamma(\alpha + 1)}, \\
 p^2 & : v_2(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_1(v)] \right) = -x \frac{(-0.06t^\alpha)^2}{\Gamma(2\alpha + 1)} + \max(x - 25e^{-0.06}, 0) \frac{(-0.06t^\alpha)^2}{\Gamma(2\alpha + 1)}, \\
 p^3 & : v_3(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_2(v)] \right) = -x \frac{(-0.06t^\alpha)^3}{\Gamma(3\alpha + 1)} + \max(x - 25e^{-0.06}, 0) \frac{(-0.06t^\alpha)^3}{\Gamma(3\alpha + 1)}, \\
 p^n & : v_n(x, t) = L^{-1} \left(\frac{1}{s^\alpha} L[H_{n-1}(v)] \right) = -x \frac{(-0.06t^\alpha)^n}{\Gamma(n\alpha + 1)} + \max(x - 25e^{-0.06}, 0) \frac{(-0.06t^\alpha)^n}{\Gamma(n\alpha + 1)},
 \end{aligned}$$

So that the solution $v(x, t)$ of the problem given as

$$v(x, t) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n v_n(x, t) = x(1 - E_\alpha(-0.06t^\alpha)) + \max(x - 25e^{-0.06}, 0) E_\alpha(-0.06t^\alpha), \tag{31}$$

This is the exact solution of the given option pricing equation (26). Now the solution of the generalized Black Scholes equation (26) at $\alpha = 1$ is $v(x, t) = x(1 - e^{-0.06t}) + \max(x - 25e^{-0.06}, 0)e^{-0.06t}$. which is an exact solution of the given Black Scholes equation (19) for $\alpha = 1$.

5. CONCLUSION

The main study of this work is to provide analytical solution of the fractional Black-Scholes option pricing equation by homotopy perturbation method with coupling of the Laplace transform, and the two examples from literature [6, 12] are presented to determine the efficiency and simplicity of the proposed method. The main advantage of this method is to overcome the deficiency that is mainly caused by unsatisfied conditions. Thus, it can be concluded that the LHPM methodology is very powerful and efficient in finding approximate solutions as well as numerical solutions.

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