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EXISTENCE RESULTS FOR INITIAL VALUE PROBLEMS WITH INTEGRAL CONDITION FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some sufficient conditions for the existence of solutions for a class of initial value problem with integral condition for an impulsive fractional differential equation. Some existence results are proved when the nonlinearity has a sub-linear growth in its state variable (see Corollary 3.1) or the growth of nonlinearity only depends upon the local properties of nonlinear term on a bounded set (see condition (3.2)). An example is also given to illustrate our main results.

1. Introduction

This paper deals with the existence of solutions to a class of initial value problem for an impulsive fractional order differential equation in the following form

$$^{c}D^{\alpha}y(t) = f(t, y(t)), \quad t \in J = [0,1],\ t \neq t_k, \tag{1.1}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \tag{1.2}$$

$$y(0) = \int_0^1 g(s)y(s)ds, \tag{1.3}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $g \in L^1(J, J)$, $I_k : \mathbb{R} \to \mathbb{R}$, and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra *et al* [8], Lakshmikantham *et al* [13] and the references therein.

Recently, differential equation of fractional order has been paid great attention due to its much application in various fields of science and engineering, see the monographs of Miller and Ross [14], Podlubny [16], and the papers of Agarwal *et al* [1, 2, 3], Ahmad *et al* [4, 5, 6], Babakhani and Daftardar-Gejji [7], Benchohra *et*

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al [9, 10, 11], Chang and Nieto [12], Lv *et al* [15], Wang *et al* [17, 18, 22], Zhou *et al* [19, 20, 21, 23, 24, 25] and the references therein.

The problem (1.1), (1.3) was studied by Lv, Liang and Xiao [15] in an Banach space E without impulsive conditions (1.2). In present paper, we consider the impulsive problem (1.1)-(1.3) in a finite real space $\mathbb{R}$. Our results can be easily applied to the cases when the nonlinearity has a sub-linear growth in its state variable (see Corollary 3.1) or the growth of nonlinearity only depends upon the local properties of nonlinear term on a bounded set (see (3.2)).

This paper is organized as follows. In Section 2 we introduce some preliminary results needed in the following sections. In Section 3 we present some existence results for the problem (1.1)-(1.3) by using the fractional calculus and suitable fixed point theorems.

2. Preliminaries

First, we recall some basic definitions. Consider the set of functions

$$PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \ldots, m \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+),\ k = 1, \ldots, m \text{ with } x(t_k^-) = x(t_k)\}.$$

This set is a Banach space with the norm [8]

$$\|x\|_{PC} = \sup_{t\in J} |x(t)|.$$

Set $J' := [0,1]\backslash\{t_1, \ldots, t_m\}$ and $\mathbb{R}^+ = (0,\infty)$, $\quad \mathbb{R}_+ = [0,\infty)$.
For basic facts about fractional derivative and fractional calculus one can refer the books [14, 16].

Definition 2.1. A real function f is said to be in the space C_α, $(\alpha \in \mathbb{R})$ if there exists a real number $p > \alpha$ such that $f(t) = t^p g(t)$ for some $g \in C(\mathbb{R}_+)$, and f is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$ $(m \in N)$.

Definition 2.2. The fractional integral of the function $f \in L^1([a,b],\mathbb{R}_+)$ of order $q \in \mathbb{R}_+$ is defined by

$$I_a^q f(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\, ds,$$

where Γ is the Gamma function. When $a = 0$, we write $I^q f(t) = f(t) * \varphi_q(t)$, where $\varphi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$ for $t > 0$, and $\varphi_q(t) = 0$ for $t \le 0$. Note that $\varphi_q(t) \to \delta(t)$ as $q \to 0$, where δ is the delta function.

Definition 2.3. The Riemann–Liouville fractional integral of order $q > 0$, of a function $f \in C_\mu$,
$(\mu \ge -1)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds, \quad \text{for } q > 0 \text{ and } t > 0,$$

and in the case $q = 0$ we put $I^0 f(x) = f(x)$.

Definition 2.4. The Riemann–Liouville fractional derivative of order $q > 0$, of a function f, is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}}\, ds,$$

for $n-1 < q < n$ and $n \in N$, where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$.

Definition 2.5. The Caputo derivative of fractional order q for a function $f(t)$ is defined by

$$({}^cD^q f)(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q-n+1}}\, ds,$$

for $n-1 < q < n$ and $n = [q]+1$, where $[q]$ denotes the integer part of the real number q.

Lemma 2.1. Let $q > 0$. Then we have ${}^cD^q(I^q f(t)) = f(t)$.

Lemma 2.2. Let $q > 0$ and $n = [q]+1$. Then

$$I^q({}^cD^q f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Lemma 2.3. [15] If $Q(\tau) = \int_\tau^1 g(s)(s-\tau)^{q-1}\, ds$ for $\tau \in [0,1]$, and if $g \in L^1([0,1],[0,1])$, then

$$\frac{Q(\tau)}{\Gamma(q)} < e \quad \text{and} \quad \frac{\int_0^t (t-s)^{q-1}\, ds}{\Gamma(q)} < e.$$

As a consequence of Lemmas 2.2 and 2.3, we have the following result which is useful in what follows.

Lemma 2.4. Let $0 < \alpha \leq 1$ and $\mu = \int_0^1 g(s)ds$. let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation

$$y(t) = \begin{cases} \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau)h(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds & \text{if } t \in [0,t_1], \\ \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau)h(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}h(s)ds & \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1}h(s)ds + \sum_{i=1}^k I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{1}$$

where $k = 1, \ldots, m$, if and only if y is a solution of the fractional IVP

$${}^cD^\alpha y(t) = h(t), \quad t \in J', \tag{2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{3}$$

$$y(0) = \int_0^1 g(s)h(s)ds. \tag{4}$$

Proof. Assume y satisfies (2)-(4). If $t \in [0,t_1]$ then

$${}^cD^\alpha y(t) = h(t).$$

Lemma 2.2 implies

$$y(t) = \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau)h(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds.$$

If $t \in (t_1, t_2]$ then Lemma 2.2 implies

$$\begin{aligned}
y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds \\
&= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds \\
&= I_1(y(t_1^-)) + \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau) h(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds.
\end{aligned}$$

If $t \in (t_2, t_3]$ then from Lemma 2.2 we get

$$\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds \\
&= I_2(y(t_2^-)) + I_1(y(t_1^-)) + \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau) h(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds.
\end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again from Lemma 2.2 we get (1).

Conversely, assume that y satisfies the impulsive fractional integral equation (1). If $t \in [0, t_1]$ then $y(0) = y(0) = \int_0^1 g(s) h(s) ds$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$ {}^cD^\alpha y(t) = h(t), \quad \text{for each } t \in [0, t_1]. $$

If $t \in [t_k, t_{k+1})$, $k = 1, \ldots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$ {}^cD^\alpha y(t) = h(t), \text{for each } t \in [t_k, t_{k+1}). $$

Also, we can easily show that

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m. $$

3. Main Results

In this section, we prove some existences results for the problem (1.1)-(1.3).

Theorem 3.1. Suppose that

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exists a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}^+$ such that $|f(t, u)| \leq \psi(|u|)$ for each $(t, u) \in J \times \mathbb{R}$ and

$$ \liminf_{r \to +\infty} \frac{\psi(r)}{r} = \beta. $$

(H3) The functions $I_k \in C(\mathbb{R}, \mathbb{R})$ and there exist continuous nondecreasing functions $\varphi_k : \mathbb{R}_+ \to \mathbb{R}^+$ such that $|I_k(u)| \leq \varphi_k(|u|)$ for each $u \in \mathbb{R}$, $k = 1, \ldots, m$ and

$$ \liminf_{r \to +\infty} \frac{\varphi_k(r)}{r} = \gamma_k, \; k = 1, \ldots, m. $$

Then (1.1)-(1.3) has at least one solution on J provided that

$$\beta\left[\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right]+\sum_{k=1}^{m}\gamma_k<1. \tag{3.1}$$

Proof. We transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$\begin{aligned}F(y)(t) &= \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,y(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,y(s))ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,y(s))ds+\sum_{0<t_k<t}I_k(y(t_k^-)).\end{aligned}$$

From Lemma 2.4, the fixed points of the operator F are solutions of the problem (1.1)-(1.3). We shall apply Schauder's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps. Let $B_r = \{y \in PC(J,\mathbb{R}) : \|y\|_{PC} \le r\}$.

Step1: $F(B_r) \subseteq B_r$ for some $r > 0$.

If it is not true, then for each $r > 0$, there exists a function $y^r(\cdot) \in B_r$ but $|F(y^r)(t)| > r$ for some $t \in J$. However, on the other hand, we have from (H2), (H3) and Lemma 2.3,

$$\begin{aligned} r &< |F(y^r)(t)|\\ &\le \frac{e}{1-\mu}\psi(r)+\frac{m\psi(r)}{\Gamma(\alpha+1)}+\frac{\psi(r)}{\Gamma(\alpha+1)}+\sum_{k=1}^{m}\varphi_k(r)\\ &\le \psi(r)\left[\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right]+\sum_{k=1}^{m}\varphi_k(r).\end{aligned}$$

Dividing both sides by r and let $r \to \infty$, we obtain

$$1 \le \beta\left[\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right]+\sum_{k=1}^{m}\gamma_k.$$

This contradicts (3.1). Hence for some positive number r, $F(B_r) \subseteq B_r$.

Step 2: $F : B_r \to B_r$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in B_r. Then for each $t \in J$

$$\begin{aligned}|F(y_n)(t)-F(y)(t)| &\le \frac{e}{(1-\mu)}\int_0^1 f(\tau,y_n(\tau))-f(\tau,y(\tau))|d\tau\\ &+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds\\ &+\sum_{0<t_k<t}|I_k(y_n(t_k^-))-I_k(y(t_k^-))|.\end{aligned}$$

It is obvious that

$$|f(t,y_n(t))-f(t,y(t))| \le 2\psi(r).$$

Since f and I_k, $k = 1, \ldots, m$ are continuous functions, we have by the dominated convergence theorem

$$\|F(y_n) - F(y)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Step 3: F maps B_r into an equicontiuous family.
Let $\tau_1 < \tau_2 \in J$, $y \in B_r$. Then

$$\begin{aligned}
&|F(y)(\tau_2) - F(y)(\tau_1)| \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||f(s, y(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}||f(s, y(s))|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(y(t_k^-))| \\
&\leq \frac{\psi(r)}{\Gamma(\alpha+1)} [2(\tau_2 - \tau_1)^{\alpha} + \tau_2^{\alpha} - \tau_1^{\alpha}] + \sum_{0<t_k<\tau_2-\tau_1} \varphi(r).
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero independent of $y \in B_r$.

By Steps 1-3 together with the Arzelá-Ascoli theorem, we show that $F : B_r \to B_r$ is completely continuous. As a consequence of Schauder's fixed point theorem, we conclude that F has a fixed point $y(\cdot) \in B_r$ which is a solution to the problem (1.1)-(1.3). This ends of the proof.

As an immediate result of Theorem 3.1, we can obtain the following interesting result when the nonlinearity f has sub-linear growth in the state variable.

Corollary 3.1. Assume that (H1) and the following conditions are satisfied:

(H2′) There exist constants $c_1 > 0, c_2 \geq 0$ and $\mu \in [0, 1)$ such that $|f(t, u)| \leq c_1 + c_2|u|^{\mu}$ for all $t \in [0, 1]$ and $u \in \mathbb{R}$.

(H3′) There exist constants $a_k > 0, b_k \geq 0$ and $\rho_k \in [0, 1)$ such that $|I_k(u)| \leq a_k + b_k|u|^{\rho_k}$ for each $u \in \mathbb{R}$, $k = 1, \ldots, m$.

Then the problem (1.1)-(1.3) admits at least one solution on J.

The following result is concerned with the growth of nonlinear term at the *height* of nonlinear term on a bounded set (see (3.2)).

Theorem 3.2. Assume that (H1) and the following conditions hold:

(H4) There exists a constant $r > 0$ such that

$$\max\{|f(t, u)| : (t, u) \in [0, 1] \times [-r, r]\} \leq \frac{1}{\left[\frac{e}{1-\mu} + \frac{m}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right]} \frac{r}{2} \tag{3.2}$$

and

$$\max\{|I_k(u)|, I_k \in C(\mathbb{R}, \mathbb{R}), k = 1, \ldots, m\} \leq \frac{r}{2m}.$$

Then the problem (1.1)-(1.3) has at least one solution $y(\cdot)$ on J satisfying $||y|| \leq r$.

Proof. Let F be defined as in Theorem 3.1 and $y \in B_r$. Then $|y(t)| \leq r$ and $|f(t, y(t))| \leq \frac{1}{\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}}\frac{r}{2}$. Therefore

$$\begin{aligned}
||Fy|| &\leq \left[\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right]\max\{|f(t,u)| : (t,u) \in [0,1]\times[-r,r]\} \\
&\quad +m\max\{|I_k(u)|, k=1,\dots,m\} \\
&\leq \left[\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right]\frac{1}{\left[\frac{e}{1-\mu}+\frac{m}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right]}\frac{r}{2}+m\frac{r}{2m} \\
&\leq r.
\end{aligned}$$

This implies that $F : B_r \to B_r$. Just as the proof of Theorem 3.1, the Schauder's fixed point theorem can be applied to complete the remainder of the proof.

Next, we give a uniqueness result for the problem (1.1)-(1.3).

Theorem 3.3 Assume that

(H5) There exists a constant $l > 0$ such that $|f(t,u) - f(t,\overline{u})| \leq l|u-\overline{u}|$, for each $t \in J$, and each $u, \overline{u} \in \mathbb{R}$.

(H6) There exists a constant $l^* > 0$ such that $|I_k(u)-I_k(\overline{u})| \leq l^*|u-\overline{u}|$, for each $u, \overline{u} \in \mathbb{R}$ and $k = 1, \dots, m$. are satisfied.If $\mu = \int_0^1 g(s)ds$ and

$$\left[\frac{l(m+1)}{\Gamma(\alpha+1)}+\frac{el}{1-\mu}+ml^*\right] < 1. \tag{3.3}$$

Then (1.1)-(1.3) has a unique solution on J.

Proof. Let the operator F be defined as in Theorem 3.1. We shall use the Banach contraction principle to prove that F has a fixed point. We shall show that F *is a contraction.* Let $x, y \in PC(J,\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned}
&|F(x)(t) - F(y)(t)| \\
&\leq \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)|f(\tau,x(\tau)) - f(\tau,y(\tau))|d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds + \sum_{0<t_k<t}|I_k(x(t_k^-)) - I_k(y(t_k^-))| \\
&\leq \frac{el}{(1-\mu)}\int_0^1|x(\tau)) - y(\tau))|d\tau + \frac{l}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|x(s)-y(s)|ds \\
&\quad + \frac{l}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|x(s)-y(s)|ds + \sum_{k=1}^{m}l^*|x(t_k^-) - y(t_k^-)| \\
&\leq \frac{el}{(1-\mu)}\|x-y\| + \frac{ml}{\Gamma(\alpha+1)}\|x-y\| + \frac{l}{\Gamma(\alpha+1)}\|x-y\| + ml^*\|x-y\|.
\end{aligned}$$

Therefore,

$$\|F(x) - F(y)\| \leq \left[\frac{l(m+1)}{\Gamma(\alpha+1)}+\frac{el}{(1-\mu)}+ml^*\right]\|x-y\|.$$

Consequently by (3.3), F is a contraction. As a consequence of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1)-(1.3). This achieves the proof.

4. An Example

In this section we give an example to illustrate our main results. Let us consider the following impulsive fractional initial-value problem,

$$^{c}D^{\alpha}y(t) = \frac{e^{t}|y(t)|}{(9+e^{t})(1+|y(t)|)}, \quad t \in J := [0,1],\ t \neq \frac{1}{2},\ 0 < \alpha \leq 1, \tag{5}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^{-})|}{3+|y(\frac{1}{2}^{-})|}, \tag{6}$$

$$y(0) = \int_0^1 \frac{|y(s)|}{5+|y(s)|}ds. \tag{7}$$

Set

$$f(t,u) = \frac{e^{t}u}{(9+e^{t})(1+u)}, \quad (t,u) \in J \times [0,\infty), \quad g(s) = \frac{u(s)}{5+u(s)}$$

and

$$I_k(u) = \frac{u}{3+u}, \quad u \in R_+.$$

Let $x, y \in R_+$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{e^{-t}}{(9+e^{t})}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| \\ &= \frac{e^{-t}|x-y|}{(9+e^{t})(1+x)(1+y)} \\ &\leq \frac{e^{-t}}{(9+e^{t})}|x-y| \\ &\leq \frac{1}{10}|x-y|. \end{aligned}$$

Hence the condition (H5) holds with $l = 1/10$. Let $x, y \in R_+$. Then we have

$$|I_k(x) - I_k(y)| = \Big|\frac{x}{3+x} - \frac{y}{3+y}\Big| = \frac{3|x-y|}{(3+x)(3+y)} \leq \frac{1}{3}|x-y|.$$

Hence the condition (H6) holds with $l^{*} = 1/3$. We shall check that condition with $T = 1$ and $m = 1$. Indeed

$$\left[\frac{l(m+1)}{\Gamma(\alpha+1)} + ml^{*}\right] < 1 \Longleftrightarrow \Gamma(\alpha+1) > \frac{3}{10}, \tag{8}$$

which is satisfied for some $\alpha \in (0,1]$. Then by Theorem 3.2 and Theorem 3.3, the problem (5)-(7) has a unique solution on $[0,1]$ for values of α satisfying (8).

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