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EXISTENCE OF EXTREME SOLUTIONS FOR FRACTIONAL ORDER BOUNDARY VALUE PROBLEM USING UPPER AND LOWER SOLUTIONS METHOD IN REVERSE ORDER

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Abstract. In this paper, we consider the existence of extreme solutions of the boundary value problem for a fractional order differential equation $D^\delta u = f(t,u), \quad t \in [0,T], 0 < \delta < 1, 0 < T < \infty$, with a nonlinear boundary conditions $g(u(0)) = u(T)$. Under a lower solution α and un upper solution β with $\beta \le \alpha$, we establish existence results of extreme solutions by means of the method of upper and lower solutions and a monotone iterative technique.

1. Introduction

Fractional differential equations have been of great interest recently. This concerns due to its important applications to real world problems. Many problems in applied sciences such as engineering and physics can be modelled by differential equations of fractional order. Existence theory for solutions to boundary value problems for fractional differential equations have attracted the attention of many researcher quite recently. In [1]-[7], using the upper and lower solutions method, with the usual order for the lower and upper solutions, authors considered the existence of solution of initial value problems and boundary value problems for fractional differential equation. In [7], using the method of upper and lower solutions (in the usual order) and its associated monotone iterative, we present an existence theorem for fractional differential equation with nonlinear boundary value condition

$$\begin{cases} D^\delta u = f(t,u), \quad t \in [0,T], 0 < T < \infty, \\ g(u(0), u(T)) = 0, \end{cases} \tag{1}$$

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where D^{δ} is a regularized fractional derivative (the Caputo derivative) of order $0 < \delta < 1$ (see [8]) defined by

$$D^{\delta}u(t) = \frac{1}{\Gamma(1-\delta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\delta}u(\tau)d\tau - \frac{u(0)}{\Gamma(1-\delta)}t^{-\delta}, 0 < \delta < 1, \tag{2}$$

where

$$\frac{1}{\Gamma(1-\delta)}\int_0^t (t-\tau)^{-\delta}u(\tau)d\tau = I^{1-\delta}u(t)$$

is the Riemann-Liouville fractional integral of order $1-\delta$; see [8].

In [9], by the method of upper and lower solutions in the reverse order and a monotone iterative technique, authors present an existence theorem for a nonlinear ordinary differential equation of first order with nonlinear boundary conditions.

Motivated by [1]-[7], [9], in [10], by means of upper and lower solutions method, in reverse order, we obtained the existence result of solution to boundary value problem

$$\begin{cases} D^{\delta}u - du = r(t), \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T), \end{cases} \tag{3}$$

where D^{δ} is a regularized fractional derivative (the Caputo derivative) of order $0 < \delta < 1$ (see [8] defined by

$$D^{\delta}u(t) = \frac{1}{\Gamma(1-\delta)}\int_0^t (t-\tau)^{-\delta}u^{'}(\tau)d\tau, 0 < \delta < 1, \tag{4}$$

$d \geq 0$ is a constant, $r \in C^1[0,T]$, $h \in C^1(R,R)$.

In this paper, by means of upper and lower solutions method, in reverse order, we will consider the existence of extreme solutions of the fractional differential equation with nonlinear boundary value condition

$$\begin{cases} D^{\delta}u = f(t,u), \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T), \end{cases} \tag{5}$$

where $f : [0,T] \times R \to R$ is a continuous function, $g : R \to R$ is continuous differential with respect to its all variables, and D^{δ} is the Caputo derivative of order $0 < \delta < 1$ defined by (2).

Definition 1.1 In this paper, we call a function $u(t)$ a solution of problem (5) if $u(t) \in C[0,T]$ satisfying problem (5).

2. Comparison principle

The following are some fundamental properties and an existence result of solution for linear initial value problem for fractional differential equation, which are important for us in the following analysis.

Lemma 2.1. ([8]) If $f(t) \in C[0,T]$ and $0 < \alpha < 1$, then

$$I^\alpha D^\alpha f(t) = f(t) - f(0).$$

Lemma 2.2. ([8]) The linear initial value problem

$$\begin{cases} D^\alpha u + Mu = q(t), \quad t \in (0,T], \\ \\ u(0) = u_0, \end{cases} \tag{6}$$

where M is a constant and $q \in C([0,T] \times R)$, has the following integral representation of solution

$$u(t) = u_0 E_{\alpha,1}(-Mt^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-M(t-s)^\alpha) q(s) ds, \tag{7}$$

where $E_{\alpha,1}(-Mt^\alpha), E_{\alpha,\alpha}(-Mt^\alpha)$ are Mittag-Leffler functions [8].

Remark 2.3 In particular, when $M = 0$, then initial value problem (6) has solution

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) ds.$$

Throughout this paper, we always assume that the following conditions are satisfied:

(H) Let $h(0) \le 0$, d is a constant with $0 \le d < \frac{\Gamma(1+\delta)}{2T^\delta}$, assume that there exists a constant $\frac{d\delta T^\delta}{\Gamma(1+\delta) - dT^\delta} < r < 1$ such that

$$r < h'(s) < 1, \quad s \in R. \tag{8}$$

And let

$$c(t) = 1 - \sin \frac{\pi t}{2T}, t \in [0,T].$$

Remark 2.4 It follows from $0 \le d < \frac{\Gamma(1+\delta)}{2T^\delta}$ that $\frac{d\delta T^\delta}{\Gamma(1+\delta) - dT^\delta} < 1$.

Lemma 2.4 Let (H) hold. Suppose $u \in [0,T]$ and satisfies the relations

$$\begin{cases} D^\delta u(t) \ge du(t), t \in [0,T], 0 < \delta < 1, \\ \\ h(u(0)) \ge u(T). \end{cases} \tag{9}$$

Then $u < 0$ for $t \in [0,T]$.

Remark 2.5 In [10], for the Caputo derivative defined by (4), we obtained this results under assumption (H) with $\frac{dT^\delta}{\Gamma(\delta) - dT^\delta} < r < 1$. Here, we generalize this result for constant $\frac{d\delta T^\delta}{\Gamma(1+\delta) - dT^\delta} < r < 1$.

Proof. We suppose to the contrary that $u(t) > 0$ for some $t \in [0,T]$. We consider the following two possible cases.
Case 1. $u(t) > 0$ for all $t \in [0,T]$. Then, by (9), we have that

$$D^\delta u(t) \ge 0, t \in [0,T]. \tag{10}$$

From Lemma 2.1, $h(0) \le 0$, we can obtain that

$$u(t) - u(0) \ge 0, 0 \le t \le T,$$

$$h^{'}(\xi)u(0) \geq h(u(0)) - h(0) + h(0) \geq u(T) \geq u(0),$$
$$(h^{'}(\xi) - 1)u(0) \geq 0,$$

where ξ is between $u(0)$ and 0. This is a contraction, since $0 < h^{'}(\xi) < 1$.

Case 2. There exist $t_1, t_2 \in [0,T]$ such that $u(t_1) > 0$ and $u(t_2) < 0$. Put $u(t_0) = \min_{t\in[0,T]} u(t) = -\lambda$, then $\lambda > 0$ and

$$D^\delta u(t) \geq -d\lambda, t \in [0,T].$$

Hence there exists $\eta \geq 0$ such that

$$D^\delta u(t) = -d\lambda + \eta, t \in [0,T]. \tag{11}$$

By Lemma 2.1, we have that

$$u(t) = u(0) + \frac{\eta - d\lambda}{\Gamma(1+\delta)} t^\delta, t \in [0,T]. \tag{12}$$

Thus, it holds that

$$u^{'}(t) = \frac{\eta - d\lambda}{\Gamma(\delta)} t^{\delta-1}, t \in (0,T]. \tag{13}$$

We will claim that

$$u(0) \leq -\lambda + \frac{d\lambda}{\Gamma(1+\delta)} T^\delta.$$

In fact, if $t_0 = 0$, then $u(0) = -\lambda \leq -\lambda + \frac{d\lambda}{\Gamma(1+\delta)} T^\delta$. If $t_0 > 0$, then it follows from (12) that

$$u(0) = u(t_0) + \frac{d\lambda - \eta}{\Gamma(1+\delta)} t_0^\delta \leq -\lambda + \frac{d\lambda}{\Gamma(1+\delta)} T^\delta.$$

By the above inequality and (H), we known that

$$u(0) \leq -\lambda + \frac{d\lambda}{\Gamma(1+\delta)} T^\delta = \frac{\lambda(dT^\delta - \Gamma(1+\delta))}{\Gamma(1+\delta)} < 0. \tag{14}$$

Hence, it follows from $u(0) < 0$ and $u(t_1) > 0$ that there exists $\bar{t} \in (0, t_1)$ such that $u(\bar{t}) = 0$. On the other hand, by (12), (14) we can obtain that

$$\begin{aligned} u(0) = \quad & u(\bar{t}) - \frac{\eta - d\lambda}{\Gamma(1+\delta)} \bar{t}^\delta \\ = \quad & -\frac{\eta - d\lambda}{\Gamma(1+\delta)} \bar{t}^\delta < 0, \end{aligned}$$

which implies that

$$\eta > d\lambda. \tag{15}$$

By (13), (15), it holds that

$$u^{'}(t) = \frac{\eta - d\lambda}{\Gamma(\delta)} t^{\delta-1} \geq \frac{\eta - d\lambda}{\Gamma(\delta)} T^{\delta-1} \geq -\frac{d\lambda}{\Gamma(\delta)} T^{\delta-1}. \tag{16}$$

There is $t_3 \in (t_1, T)$ such that

$$u(T) = u(t_1) + u^{'}(t_3)(T - t_1) \geq u^{'}(t_3)(T - t_1) \geq -\frac{d\lambda}{\Gamma(\delta)} T^{\delta-1}(T - t_1) \geq -\frac{d\lambda}{\Gamma(\delta)} T^\delta. \tag{17}$$

Thus, from (9), (17) and (H), we have that

$$u(0) \geq H(u(T)) \geq H(-\frac{d\lambda}{\Gamma(\delta)}T^{\delta}) = H(0) - H^{'}(\rho)\frac{d\lambda}{\Gamma(\delta)}T^{\delta},$$

where H denotes the inverse of h, ρ is between $-\frac{d\lambda}{\Gamma(\delta)}T^{\delta}$ and 0. Noticing that $H(0) \geq 0$ and $0 < H^{'} \leq r^{-1}$, we have that

$$r(-\lambda + \frac{d\lambda}{\Gamma(1+\delta)}T^{\delta}) \geq -\frac{d\lambda}{\Gamma(\delta)}T^{\delta},$$

implies that

$$r \leq \frac{d\delta T^{\delta}}{\Gamma(1+\delta) - dT^{\delta}}.$$

This contacts with (H). Hence, we complete the proof.

Remark 2.6. If $d > 0$, then we may demand $\frac{d\delta T^{\delta}}{\Gamma(1+\delta)-dT^{\delta}} < r \leq 1$.

Corollary 2.5. Let (H) hold. Suppose $u \in [0,T]$ and satisfies the relations

$$\begin{cases} D^{\delta}u \geq du - (D^{\delta}c(t) - dc(t))(H(u(T)) - u(0)), t \in [0,T], 0 < \delta < 1, \\ \\ h(u(0)) \leq u(T), \end{cases}$$

where H denotes the inverse of h. Then $u < 0$ for $t \in [0,T]$.

Proof. Let

$$w(t) = u(t) + c(t)(H(u(T)) - u(0)), t \in [0,T].$$

It follows from $0 < r < h^{'}(s) < 1, s \in R$ that h (also H) is nondecreasing, so, according to $h(u(0)) \leq u(T)$, we get that $w(t) \geq u(t)$ for all $t \in [0,T]$. It holds

$$D^{\delta}w(t) - dw(t) = D^{\delta}u(t) - du(t) + (D^{\delta}c(t) - dc(t))(H(u(T)) - u(0)) \geq 0, t \in [0,T],$$

and $w(0) = H(u(T))$, which imply that $h(w(0)) = u(T) = w(T)$. By Lemma 2.4, we obtain that $w(t) < 0$ for all $t \in [0,T]$, which implies that $u(t) < 0$ for all $t \in [0,T]$. We complete this proof.

Definition 2.1. Functions $\beta, \alpha \in C[0,T]$ are called upper and lower solutions of problem (5) if they satisfy

$$D^{\delta}\beta(t) \geq f(t,\beta) - b_{\beta}(t), t \in [0,T], \tag{18}$$

$$D^{\delta}\alpha(t) \leq f(t,\alpha) + a_{\alpha}(t), t \in [0,T], \tag{19}$$

where

$$b_{\beta}(t) = \begin{cases} 0, & h(\beta(0)) \geq \beta(T), \\ \\ (D^{\delta}c(t) - dc(t))(H(\beta(T)) - \beta(0)), & h(\beta(0)) \leq \beta(T), \end{cases} \tag{20}$$

$$a_{\alpha}(t) = \begin{cases} 0, & h(\alpha(0)) \leq \alpha(T), \\ \\ (D^{\delta}c(t) - dc(t))(\alpha(0) - H(\alpha(T))), & h(\alpha(0)) \geq \alpha(T), \end{cases} \tag{21}$$

where H denotes the reverse of h, $c(t) = 1 - \sin\frac{\pi t}{2T}, t \in [0, T]$.

3. Linear problem

Consider the following linear problem with nonlinear boundary condition

$$\begin{cases} D^\delta u - du = r(t), \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T), \end{cases} \tag{22}$$

where d is the constant in (H), $r \in C[0, T]$.

Theorem 3.1. Let (H) hold. Assume that (22) exist lower and upper solutions $\alpha, \beta \in [0, T]$ with $\beta(t) \le \alpha(t)$ on $[0, T]$. Then problem (22) exists one solution $u \in C[0, T]$ with $\beta(t) < u(t) < \alpha(t), t \in [0, T]$.

Remark 3.1. In [10], for the Caputo derivative defined by (4), we obtain this result under condition (H) with $\frac{dT^\delta}{\Gamma(1+\delta)-dT^\delta} < r < 1$. Here, in view of convenience, we give the proof of this Lemma, which is similar to that ones in [10].

Proof. We define functions $p(t), q(t) \in C[0, T]$ as following

$$p(t) = \begin{cases} \alpha(t), & h(\alpha(0)) \le \alpha(T), \\ \\ \alpha(t) - c(t)(\alpha(0) - H(\alpha(T))), & h(\alpha(0)) \ge \alpha(T), \end{cases}$$

$$q(t) = \begin{cases} \beta(t), & h(\beta(0)) \ge \beta(T), \\ \\ \beta(t) + c(t)(H(\beta(T)) - \beta(0)), & h(\beta(0)) \le \beta(T), \end{cases}$$

where H denotes the reverse of h. We will claim that p, q satisfy the following results

$$p(T) = \alpha(T), \qquad h(p(0)) \le p(T). \tag{23}$$

$$q(T) = \beta(T), \qquad h(q(0)) \ge q(T). \tag{24}$$

$$D^\delta p(t) - dp(t) \le r(t), t \in [0, T], \tag{25}$$

$$D^\delta q(t) - dq(t) \ge r(t), t \in [0, T]. \tag{26}$$

And

$$q(t) \le p(t) \quad \text{on} \quad [0, T]. \tag{27}$$

If $h(\alpha(0)) \le \alpha(T)$, then $h(p(0)) = h(\alpha(0)) \le \alpha(T) = p(T)$; if $h(\alpha(0)) \ge \alpha(T)$, then $h(p(0)) = h(\alpha(0)-\alpha(0)+H(\alpha(T))) = \alpha(T) = p(T)$, that is, it holds $h(p(0)) \le p(T)$. Analogously, $h(q(0)) \ge q(T)$. Moreover, if $h(\alpha(0)) \le \alpha(T)$, then $p(t) = \alpha(t)$, which means that

$$D^\delta p(t) - dp(t) = D^\delta \alpha(t) - d\alpha(t) \le r(t), t \in [0, T];$$

if $h(\alpha(0)) \geq \alpha(T)$, then $p(t) = \alpha(t) - c(t)(\alpha(0) - H(\alpha(T)))$, which implies that

$$\begin{aligned} & D^{\delta}p(t) - dp(t) \\ = \quad & D^{\delta}\alpha(t) - (\alpha(0) - H(\alpha(T)))D^{\delta}c(t) - d\alpha(t) + (\alpha(0) - H(\alpha(T)))dc(t) \\ = \quad & D^{\delta}\alpha(t) - d\alpha(t) - a_{\alpha}(t) \leq r(t), t \in [0, T]. \end{aligned}$$

Analogously, if $h(\beta(0)) \geq \beta(T)$, then $q(t) = \beta(t)$, which means that

$$D^{\delta}q(t) - dq(t) = D^{\delta}\beta(t) - d\beta(t) \geq r(t), t \in [0, T];$$

if $h(\beta(0)) \leq \beta(T)$, then $q(t) = \beta(t) + c(t)(H(\beta(T)) - \beta(0))$, which implies that

$$\begin{aligned} & D^{\delta}q(t) - dq(t) \\ = \quad & D^{\delta}\beta(t) + (H(\beta(T)) - \beta(0))D^{\delta}c(t) - d\beta(t) - (H(\beta(T)) - \beta(0))dc(t) \\ = \quad & D^{\delta}\beta(t) - d\beta(t) + b_{\beta}(t) \geq r(t), t \in [0, T]. \end{aligned}$$

Hence, (23)-(26) hold.

Further, by Lemma 2.4, we can easy to see that $q(t) \leq p(t)$ on $[0, T]$. In fact, let $m(t) = q(t) - p(t), t \in [0, T]$; then by (23)-(26), we have that

$$\begin{cases} D^{\delta}m(t) - dm(t) \geq r(t) - r(t) = 0, t \in [0, T], \\ m(T) = q(T) - p(T) \leq h(q(0)) - h(p(0)) = h^{'}(\eta)m(0), \end{cases}$$

where η is between $q(0)$ and $p(0)$). By Lemma 2.4, we have that $q(t) < p(t)$ for all $t \in [0, T]$.

Now we consider the problem

$$\begin{cases} D^{\delta}u(t) - du(t) = r(t), t \in [0, T], \\ u(T) = \lambda, \end{cases} \tag{28}$$

where $\lambda \in R$. We will show that (28) has a unique solution $u(t, \lambda)$ and u is continuous in λ.

From Lemma 2.2, we know that (28) has a unique solution

$$u(t) = \mu E_{\delta,1}(dt^{\delta}) + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(d(t-s)^{\delta}) r(s) ds, t \in [0, T],$$

where

$$\mu = \frac{\lambda - \int_0^T (T-s)^{\delta-1} E_{\delta,\delta}(d(T-s)^{\delta}) r(s) ds}{E_{\delta,1}(dT^{\delta})}.$$

It follows from the properties of $E_{\delta,1}(dt^{\delta}), E_{\delta,\delta}(dt^{\delta})$ and $r \in C[0, T]$ that solution $u \in C[0, T]$.

Let $u(t,\lambda_1), u(t,\lambda_2)$ be solutions of problems

$$\begin{cases} D^\delta u(t) - du(t) = r(t), t \in [0,T], \\ u(T) = \lambda_i, i = 1,2, \end{cases} \tag{29}$$

where $\lambda_i \in R, i = 1,2$; then, we have

$$\|u(t,\lambda_1) - u(t,\lambda_2)\| \leq \frac{|\lambda_1 - \lambda_2|}{E_{\delta,1}(dT^\delta)},$$

which implies that u is continuous in λ.

We show that

$$q(0) \leq u(0,\lambda) \leq p(0), \quad \text{for any } \lambda \in [h(q(0)), h(p(0))], \tag{30}$$

where $u(t,\lambda)$ is the unique solution of (29).

Let $Rm(t) = u(t,\lambda) - p(t)$(here $r < R < 1$, r is the constant in (H)). Suppose that $u(0,\lambda) > p(0)$; then $Rm(0) = u(0,\lambda) - p(0) > 0$,

$$m(T) = R^{-1}(u(T,\lambda) - p(T)) \leq R^{-1}(u(T,\lambda) - h(p(0))) = R^{-1}(\lambda - h(p(0)) \leq 0,$$

and

$$D^\delta m(t) - dm(t) \geq 0.$$

The Lemma 2.4 assures that $u(t,\lambda) < p(t)$ on $[0,T]$, which contradicts with $u(0,\lambda) > p(0)$. In a similar way, we obtain that $q(0) < u(0,\lambda)$. In fact, let $Rm(t) = q(t) - u(t,\lambda)$ (R is the same as the previous). Suppose that $q(0) > u(0,\lambda)$, then $Rm(0) = q(0) - u(0,\lambda) > 0$,

$$m(T) = R^{-1}(q(T) - u(T,\lambda)) \leq R^{-1}(h(q(0)) - u(T,\lambda)) = R^{-1}(h(q(0)) - \lambda) \leq 0,$$

and

$$D^\delta m(t) - dm(t) \geq 0.$$

Lemma 2.4 assures that $q(t) < u(t,\lambda)$ on $[0,T]$, which contradicts with $q(0) > u(0,\lambda)$.

Let $k(\lambda) = h(u(0,\lambda)) - \lambda$. From (30), we have

$$k(h(q(0)))k(h(p(0))) = (h(u(0,h(q(0))) - h(q(0)))(h(u(0,h(p(0))) - h(p(0))) \leq 0.$$

Since k is continuous in λ, then there exists a $\lambda_0 \in [h(q(0), h(p(0))]$ such that $h(u(0,\lambda_0)) = \lambda_0 = u(T)$. Hence, $u(t,\lambda_0)$ is the unique solution of (22).

Now, we will claim that the solution $\beta(t) < u(t,\lambda_0) < \alpha(t)$ for all $t \in [0,T]$. From the previous argument, we know that

$$h(q(0)) \leq u(T,\lambda_0) = \lambda_0 \leq h(p(0)).$$

Let $m(t) = u(t,\lambda_0) - \alpha(t), t \in [0,T]$. If $h(\alpha(0)) \leq \alpha(T)$, then $a_\alpha(t) = 0$. We have that

$$\begin{cases} D^\delta m(t) - dm(t) = r(t) - D^\delta \alpha(t) + d\alpha(t) \geq 0, t \in [0,T]; \\ m(T) = u(T,\lambda_0) - \alpha(T) \leq h(u(0,\lambda_0)) - h(\alpha(0)) = h^{'}(\xi_1)m(0); \end{cases}$$

where ξ_1 is between $u(0,\lambda_0)$ and $\alpha(0)$. The Lemma 2.4 assures that $m(t) = u(t,\lambda_0) - \alpha(t) < 0$ for all $t \in [0,T]$. If $h(\alpha(0)) \geq \alpha(T)$, then $a_\alpha(t) = (D^\delta c(t) - dc(t))(\alpha(0) - H(\alpha(T)))$, then, we have that

$$\begin{aligned}
D^\delta m(t) - dm(t) = \quad & r(t) - D^\delta \alpha(t) + d\alpha(t) \geq -a_\alpha(t) \\
= \quad & -(D^\delta c(t) - dc(t))(\alpha(0) - H(\alpha(T))) \\
= \quad & -(D^\delta c(t) - dc(t))(\alpha(0) - H(\alpha(T)) - u(0,\lambda_0) + H(u(T,\lambda_0))) \\
= \quad & -(D^\delta c(t) - dc(t))(\frac{1}{H'(\xi_2)} m(T) - m(0)), t \in [0,T],
\end{aligned}$$

$$m(0) = u(0,\lambda_0) - \alpha(0) = H(u(T,\lambda_0)) - \alpha(0) \leq H(u(T,\lambda_0)) - H(\alpha(T)) = \frac{1}{h'(\xi_2)} m(T);$$

where ξ_2 is between $u(T,\lambda_0)$ and $\alpha(T)$. The corollary 2.5 assures that $m(t) = u(t,\lambda_0) - \alpha(t) < 0$ for all $t \in [0,T]$. As a result, it holds $u(t,\lambda_0) < \alpha(t)$ for all $t \in [0,T]$. Using a similar way, we can obtain that $\beta(t) < u(t,\lambda_0)$ for all $t \in [0,T]$. In fact, let $m(t) = \beta(t) - u(t,\lambda_0), t \in [0,T]$. If $h(\beta(0)) \geq \beta(T)$, then $b_\beta(t) = 0$. We have that

$$\begin{cases} D^\delta m(t) - dm(t) = D^\delta \beta(t) - d\beta(t) - r(t) \geq 0, t \in [0,T]; \\ m(T) = \beta(T) - u(T,\lambda_0) \leq h(\beta(0)) - h(u(0,\lambda_0)) = h'(\xi_3) m(0); \end{cases}$$

where ξ_3 is between $u(0,\lambda_0)$ and $\beta(0)$. The Lemma 2.4 assures that $m(t) = \beta(t) - u(t,\lambda_0) < 0$ for all $t \in [0,T]$. If $h(\beta(0)) \leq \beta(T)$, then $b_\beta(t) = (D^\delta c(t) - dc(t))(H(\beta(T)) - \beta(0))$, then, we have that

$$\begin{aligned}
D^\delta m(t) - dm(t) = \quad & D^\delta \beta(t) - d\beta(t) - r(t) \geq -b_\beta(t) \\
= \quad & (D^\delta c(t) - dc(t))(\beta(0) - H(\beta(T))) \\
= \quad & -(D^\delta c(t) - dc(t))(H(\beta(T)) - \beta(0) + u(0,\lambda_0) - H(u(T,\lambda_0))) \\
= \quad & -(D^\delta c(t) - dc(t))(\frac{1}{h'(\xi_4)} m(T) - m(0)), t \in [0,T],
\end{aligned}$$

$$m(0) = \beta(0) - u(0,\lambda_0) = \beta(0) - H(u(T,\lambda_0)) \leq H(\beta(T)) - H(u(T,\lambda_0)) = \frac{1}{h'(\xi_4)} m(T);$$

where ξ_4 is between $u(T,\lambda_0)$ and $\beta(T)$. Corollary 2.5 assures that $m(t) = \beta(t) - u(t,\lambda_0) < 0$ for all $t \in [0,T]$. As a result, it holds $\beta(t) < u(t,\lambda_0)$ for all $t \in [0,T]$. Thus, we complete this proof.

4. Main result

In this section, according to Theorem 3.1 and monotone iterative technique, we we shall consider the existence of extreme solutions of (5). Our main result is the following theorem.

Theorem 4.1. Let (H) hold. Assume that (5) exist lower and upper solutions $\alpha, \beta \in C[0,T]$ with $\beta(t) \leq \alpha(t)$ on $[0,T]$. And assume that $f : [0,T] \times R \to R$ is continuous differential with respect to its two variables, satisfying

$$f(t,x) - f(t,y) \leq d(x-y), \beta(t) \leq y \leq x \leq \alpha(t), t \in [0,T], \tag{31}$$

where d is the constant in assumption (H). Then problem (5) exists extreme solutions on $[\beta, \alpha] = \{u \in C[0,T] : \beta(t) \leq u(t) \leq \alpha(t), 0 \leq t \leq T\}$.

Proof. We consider the problem

$$\begin{cases} D^{\delta}u - du = f(t,\alpha) - d\alpha, \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T). \end{cases} \tag{32}$$

Since α, β are lower and upper solutions of (5), by (31), we have that

$$D^{\delta}\alpha(t) - d\alpha(t) \leq f(t,\alpha) - d\alpha + a_{\alpha}(t),$$

$$\begin{aligned} D^{\delta}\beta(t) - d\beta(t) \geq \quad & f(t,\beta) - d\beta - b_{\beta}(t) \\ \\ \geq \quad & f(t,\alpha) - d(\alpha-\beta) - d\beta - b_{\beta}(t) \\ \\ = \quad & f(t,\alpha) - d\alpha - b_{\beta}(t), \end{aligned}$$

which imply that α, β are lower and upper solutions of problem (32). Therefore, by Theorem 3.1, problem (32) has a solution $u_1(t)$ with $\beta(t) < u_1(t) < \alpha(t), t \in [0,T]$. Now, for the problem

$$\begin{cases} D^{\delta}u - du = f(t,u_1) - du_1, \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T). \end{cases} \tag{33}$$

By definition 2.1 and (31), we have that

$$\begin{aligned} D^{\delta}\beta(t) - d\beta(t) \geq \quad & f(t,\beta) - d\beta - b_{\beta}(t) \\ \\ \geq \quad & f(t,u_1) - d(u_1-\beta) - d\beta - b_{\beta}(t) \\ \\ = \quad & f(t,u_1) - du_1 - b_{\beta}(t), \end{aligned}$$

which implies that β is upper solution of problem (33). u_1 is a solution of problem (32), hence, by $h(u_1(0)) = u_1(T)$, it holds $a_{u_1}(t) = 0$, and that

$$\begin{aligned} D^{\delta}u_1(t) - du_1(t) = \quad & f(t,\alpha) - d\alpha \\ \\ \leq \quad & f(t,u_1) + d(\alpha - u_1) - d\alpha, \\ \\ = \quad & f(t,u_1) - du_1, \end{aligned}$$

which implies that u_1 is lower solution of problem (33). Therefore, by Theorem 3.1, problem (33) has a solution $u_2(t)$ with $\beta(t) < u_2(t) < u_1(t) < \alpha(t), t \in [0,T]$. In a similar way, we consider the problem

$$\begin{cases} D^\delta u - du = f(t,\beta) - d\beta, \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T). \end{cases} \tag{34}$$

Since α, β are lower and upper solutions of (5), by (31), we have that

$$\begin{aligned} D^\delta\alpha(t) - d\alpha(t) \leq & \quad f(t,\alpha) - d\alpha + a_\alpha(t) \\ \\ \leq & \quad f(t,\beta) + d(\alpha - \beta) - d\alpha + a_\alpha(t) \\ \\ = & \quad f(t,\beta) - d\beta + a_\alpha(t) \end{aligned}$$

$$D^\delta\beta(t) - d\beta(t) \geq f(t, \beta - d\beta - b_\beta(t),$$

which imply that α, β are lower and upper solutions of problem (34). Therefore, by Theorem 3.1, problem (34) has a solution $v_1(t)$ with $\beta(t) < v_1(t) < \alpha(t), t \in [0,T]$. Now, for the problem problem

$$\begin{cases} D^\delta u - du = f(t,v_1) - dv_1, \quad t \in [0,T], 0 < T < \infty, \\ \\ h(u(0)) = u(T). \end{cases} \tag{35}$$

By definition 2.1 and (31), we have that

$$\begin{aligned} D^\delta\alpha(t) - d\alpha(t) \leq & \quad f(t,\alpha) - d\alpha + a_\alpha(t), \\ \\ \leq & \quad f(t,v_1) + d(\alpha - v_1) - d\alpha + a_\alpha(t), \\ \\ = & \quad f(t,v_1) - dv_1 + a_\alpha(t), \end{aligned}$$

which implies that α is lower solutions of problem (35). v_1 is a solution of problem (34), hence, by $h(v_1(0)) = v_1(T)$, it holds $b_{v_1}(t) = 0$, and that

$$\begin{aligned} D^\delta v_1(t) - dv_1 = & \quad f(t,\beta) - d\beta \\ \\ \geq & \quad f(t,v_1) - d(v_1 - \beta) - d\beta \\ \\ = & \quad f(t,v_1) - dv_1, \end{aligned}$$

which implies that v_1 is upper solution of problem (35). Therefore, by Theorem 3.1, problem (35) has a solution $v_2(t)$ with $\beta(t) < v_1(t) < v_2(t) < \alpha(t), t \in [0,T]$.

Also, we can verify that $\beta(t) < v_1(t) < v_2(t) < u_2(t) < u_1(t) < \alpha(t), t \in 0,T]$. In fact, let $m(t) = v_1(t) - u_1(t), t \in [0,T]$. Since v_1, u_1 are solutions of (34) and (32) respectively, thus, we have that

$$\begin{aligned} D^\delta m(t) - dm(t) = & \quad f(t,\beta) - f(t,\alpha) - d\beta + d\alpha \\ \\ \geq & \quad -d(\alpha - \beta) + d(\alpha - \beta) = 0, \end{aligned}$$

$$m(T) = h(v_1(0)) - h(u_1(0)) = h^{'}(\xi_1)m(0),$$

where ξ_1 is between $v_1(0)$ and $u_1(0)$. Lemma 2.4 assures that $m(t) < 0$ on $[0,T]$, which means that $v_1(t) < u_1(t)$ for all $t \in [0,T]$. In a similar way, let $m(t) = v_2(t) - u_2(t), t \in [0,T]$. Since v_2, u_2 are solutions of (35) and (32) respectively, thus, we have that

$$\begin{aligned} D^\delta m(t) - dm(t) = & \quad f(t,v_1) - f(t,u_1) - dv_1 + du_1 \\ \\ \geq & \quad -d(u_1 - v_1) + d(u_1 - v_1) = 0, \end{aligned}$$

$$m(T) = h(v_2(0)) - h(u_2(0)) = h^{'}(\xi_2)m(0),$$

where ξ_2 is between $v_2(0)$ and $u_2(0)$. Lemma 2.4 assures that $m(t) < 0$ on $[0,T]$, which means that $v_2(t) < u_2(t)$ for all $t \in [0,T]$.

Hence, from the previous arguments, we can obtain the sequences $\{u_k\}_{k\in N}$, $\{v_k\}_{k\in N}$ such that

$$\beta = v_0 < v_1 < v_2 < \cdots < v_n < u_n < \cdots < u_2 < u_1 < u_0 = \alpha, \tag{36}$$

from the following problems (37) and (38)

$$\begin{cases} D^\delta v_k - dv_k = f(t, v_{k-1}) - dv_{k-1}, & t \in [0,T], 0 < T < \infty, \\ \\ h(v_k(0)) = v_k(T), \end{cases} \tag{37}$$

$$\begin{cases} D^\delta u_k - du_k = f(t, u_{k-1}) - du_{k-1}, & t \in [0,T], 0 < T < \infty, \\ \\ h(u_k(0)) = u_k(T), \end{cases} \tag{38}$$

$k = 1, 2, \cdots$.
Hence there exist u, v such that

$$lim_{n\to\infty} u_n(t) = u(t), \qquad lim_{n\to\infty} v_n(t) = v(t) \tag{39}$$

uniformly on $t \in [0,T]$. From the previous arguments, we know that $u_n, v_n \in C^1[0,T], n = 1, 2, \cdots$. Clearly, u, v satisfy

$$\begin{cases} D^\delta u = f(t,u), & t \in [0,T], \\ \\ h(u(0)) = u(T), \end{cases} \tag{40}$$

$$\begin{cases} D^\delta v = f(t,v), & t \in [0,T], \\ \\ h(v(0)) = v(T), \end{cases} \tag{41}$$

which implies that u, v are two solutions of (5).

Finally, we show that if $w \in [\beta, \alpha]$ is any solution of (5), then $v(t) \leq w(t) \leq u(t)$ on $[0,T]$. Since $w(t)$ is a solution (5), $w(T) = h(w(0))$, hence $b_w(t) = 0$, thus we can know that

$$\begin{aligned} D^\delta w(t) - dw(t) = & \quad f(t,w) - dw \\ \geq & \quad f(t,\alpha) - d(\alpha - w) - dw, \\ = & \quad f(t,\alpha) - d\alpha, \end{aligned}$$

which implies that w is upper solution of (32). From the previous arguments, we know that α is lower solution of (32). Therefore, by Theorem 3.1, problem (32) has a solution u_1 with $w(t) < u_1(t) < \alpha(t), t \in [0,T]$. By a similar way, we can obtain that

$$\begin{aligned} D^\delta w(t) - dw(t) = &\quad f(t,w) - dw \\ \geq &\quad f(t,u_1) - d(u_1 - w) - dw, \\ = &\quad f(t,u_1) - du_1, \end{aligned}$$

which implies that w is upper solution of (33). From the previous arguments, we know that u_1 is lower solution of (33). Therefore, by Theorem 3.1, problem (33) has a solution u_1 with $w(t) < u_2(t) < u_1(t) < \alpha(t), t \in [0,T]$. By the similar arguments, we know that problem (34) has a solution $\beta(t) < v_1(t) < w(t), t \in [0,T]$, problem (35) has a solution $\beta(t) < v_2(t) < v_1(t) < w(t), t \in [0,T]$. Hence, we can claim that

$$v_n(t) < w(t) < u_n(t), t \in [0,T], n = 1, \cdots, \tag{42}$$

with $v_0 = \beta, u_0 = \alpha$. From (42), we have that $v(t) \leq w(t) \leq u(t)$ on $[0,T]$, which indicates that u, v are maximal and minimal solutions of (5), respectively. This completes the proof.

Example 1. Consider the following problem

$$\begin{cases} D^\delta u = f(t,u), \quad t \in [0,T], 0 < \delta < 1, 0 < T < +\infty, \\ \\ h(u(0)) = u(T), \end{cases} \tag{43}$$

where $f(t,u) = d \sin u$, $h(u(0)) = ru(0)$, d, r are constants satisfying (H).

Clearly, we can know that $\beta(t) = 0, \alpha(t) = 1$ are upper and lower solutions of (43), respectively. Also, f satisfied condition (31). Hence, Theorem 4.1 implies that (43) has maximal and minimal solutions on $[0,1]$.

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