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SOME APPLICATIONS OF FRACTIONAL q-CALCULUS AND FRACTIONAL q-LEIBNIZ RULE

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Abstract. In this article, the fractional q-calculus and fractional q-Leibniz rule are used to generate certain infinite series expansions and transformations relating some of q-special functions of mathematical physics. Some of these expansions and transformations thus generated are known, while others appear to be new.

1. Introduction

The subject of fractional calculus (that is, integrals and derivatives of any real or complex order) has gained noticeable importance and popularity during the past three decades or so, due mainly to its demonstrated applications in many seemingly diverse fields of science and engineering. Much of the theory of fractional calculus is based upon the familiar Riemann-Liouville fractional derivative (or integral). Many works involving fractional calculus, especially in the area of closed-form summation of infinite series [1]. Recently, there was a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics, statistics and physics.

In this paper, a brief review of fractional q-calculus and fractional q-Leibniz rules for q-integrals of the product of two functions is mentioned and used to generate certain infinite series expansions and transformations relating q-special functions of mathematical physics. Some of these expansions and transformations thus generated are known, while others appear to be new. We first show a list of various definitions and notations in q-calculus which are useful to understand the subject of this paper and will be taken from the well known books in this field [2, 3], unless otherwise stated.
For any complex number a, the basic number and q-factorial are defined as

$$[a]_q = \frac{1-q^a}{1-q}, \ q \neq 1; \ [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \ n \in \mathbb{N}; \ [0]_q! = 1 \tag{1.1}$$

and the scalar q-shifted factorials are defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n \in \mathbb{N}. \tag{1.2}$$

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The limit, $\lim_{n\to\infty}(a;q)_n$, is denoted by $(a;q)_\infty$ provided $|q|<1$. This implies that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad n\in\mathbb{N}_0, \quad |q|<1 \tag{1.3}$$

and, for any complex number α

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \qquad |q|<1 \tag{1.4}$$

where the principal value of q^α is taken.
The q-binomial coefficient is defined for positive integers n,k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{1.5}$$

This definition can be generalized in the following way. For arbitrary complex α we have

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}(-1)^k q^{\alpha k-\binom{k}{2}} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(k+1)\Gamma_q(\alpha-k)} \tag{1.6}$$

where $\Gamma_q(z)$ is the q-gamma function defined by the representation

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad z\neq 0,-1,-2,...;\ |q|<1. \tag{1.7}$$

For non-negative integer values of the variable $z=n$, we have

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n} = [n]_q!, \qquad |q|<1. \tag{1.8}$$

The basic hypergeometric series is defined as

$$\begin{aligned} {}_r\phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_s \end{matrix};q,z\right] &= {}_r\phi_s(a_1,a_2,\cdots,a_r;b_1,b_2,\cdots,b_s;q,z) \\ &= \sum_{n=0}^{\infty}\frac{(a_1,a_2,\cdots,a_r;q)_n}{(q,b_1,b_2,\cdots,b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n \end{aligned} \tag{1.9}$$

for all complex variable z if $r\le s, 0<|q|<1$ and for $|z|<1$ if $r=s+1$.
The exponential function e^z has many different q-extensions such as

$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}z^n}{[n]_q!} = (-(1-q)z;q)_\infty, \tag{1.10}$$

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty}. \tag{1.11}$$

For the convergence of the second series, we need $|z|<|1-q|^{-1}$.
The basic hypergeometric series have many properties and identities, here we need the following identities

$${}_1\phi_0(a;-;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1, \tag{1.12}$$

$${}_1\phi_1(a;c;q,c/a) = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{1.13}$$

The q-derivative $D_q f(z)$ of a function f is given as

$$(D_q f)(z) = \frac{f(z) - f(zq)}{(1-q)z}, \quad q \neq 1, \quad z \neq 0, \quad (D_q f)(0) = f'(0) \tag{1.14}$$

provided $f'(0)$ exists. If f is differentiable then $D_q f(z)$ tends to $f'(z)$ as $q \to 1$. The Jackson q-integral from a to z is defined as

$$I_{q,a} f(z) = \int_a^z f(t) d_q t = \int_0^z f(t) d_q t - \int_0^a f(t) d_q t \tag{1.15}$$

where

$$\int_0^x f(t) d_q t = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k) \tag{1.16}$$

provided the sum converges absolutely. We will denote $I_{q,0} f(z)$ by $I_q f(z)$.

The q-gamma function has a q-integral representation

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} t^{\alpha-1} E_q(-qt) d_q t, \qquad \Re(\alpha) > 0. \tag{1.17}$$

El-Shahed and Salem [4] defined q-analogues of the incomplete gamma function and its complementary, respectively, as

$$\gamma_q(\alpha, z) = \int_0^z t^{\alpha-1} E_q(-qt) d_q t, \qquad \Re(\alpha) > 0, \tag{1.18}$$

$$\Gamma_q(\alpha, z) = \int_z^{\frac{1}{1-q}} t^{\alpha-1} E_q(-qt) d_q t, \qquad \Re(\alpha) > 0. \tag{1.19}$$

There is an important relation among q-gamma function and incomplete q-gamma functions comes from their definitions

$$\Gamma_q(\alpha, z) + \gamma_q(\alpha, z) = \Gamma_q(\alpha), \qquad \alpha \neq 0, -1, -2, \cdots. \tag{1.20}$$

Some identities for incomplete q-gamma function and its complementary which have been studied and proved in [4], will be listed below

$$\Gamma_q(1, z) = E_q(-z), \qquad \gamma_q(1, z) = 1 - E_q(-z), \tag{1.21}$$

$$\gamma_q(\alpha+1, z) = [\alpha]_q \gamma_q(\alpha, z) - z^\alpha E_q(-z), \tag{1.22}$$

$$\Gamma_q(\alpha+1, z) = [\alpha]_q \Gamma_q(\alpha, z) + z^\alpha E_q(-z), \tag{1.23}$$

$$\gamma_q(\alpha, z) = [\alpha]_q^{-1} z^\alpha \; {}_1\phi_1(q^\alpha; q^{\alpha+1}; q, zq(1-q)). \tag{1.24}$$

Salem [5] proved that $\gamma_q(\alpha, z)$ is an entire function for fixed complex variable z and for all complex $\alpha \neq 0, -1, -2, \cdots$ on the open unit disc $|q| < 1$ and $\Gamma_q(\alpha, z)$ can also be continued analytically for all complex numbers $\alpha, z; |\arg(z)| < \pi$ by means of the expansions

$$\Gamma_q(0, z) = E_1(z, q) = \frac{1-q}{\ln q} \ln z - \gamma_q + \sum_{k=1}^{\infty} \frac{q^k \gamma_q(k, z)}{[k]_q!} \tag{1.25}$$

$$= \frac{1-q}{\ln q} \ln z - \gamma_q + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{\frac{k(k+1)}{2}} z^k}{k]_q \cdot [k]_q!} \tag{1.26}$$

where $E_1(z, q)$ is the q-analogue of the exponential integral and $\gamma_q = ((1-q)/\ln q)\Gamma'_q(1)$ denotes the q-analogue of the Euler-Mascheroni constant. Here, $\Gamma'_q(1) = \frac{d}{dz}[\Gamma_q(z)]_{z=1}$.

2. A brief review of fractional q-calculus

The usual starting point for a definition of fractional operators in q-calculus taken in [6, 7, 8], is the q-analogue of the Riemann-Liouville fractional integral

$$I_q^\alpha f(z) = \frac{z^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^z (tq/z;q)_{\alpha-1} f(t) d_q t. \tag{2.1}$$

This q-integral was motivated from the q-analogue of the Cauchy formula for a repeated q-integral

$$\begin{aligned} I_{q,a}^\alpha f(z) &= \int_a^z d_q t \int_a^t d_q t_{n-1} \int_a^{t_{n-1}} d_q t_{n-2} \cdots \int_a^{t_2} f(t_1) d_q t_1 \\ &= \frac{z^{n-1}}{[n-1]_q!} \int_0^z (tq/z;q)_{n-1} f(t) d_q t. \end{aligned} \tag{2.2}$$

The reduction of the multiple q-integral to a single one was considered by Al-Salam [9]. Formally replacing n by α to get (2.1) when $a = 0$. Now the q-analogue of the Riemann-Liouville integral (2.1) seems somewhat reasonable as a definition for fractional q-integral. The Jackson q-integral (1.16) can be used to get

$$I_q^\alpha f(z) = z^\alpha (1-q)^\alpha \sum_{k=0}^\infty \frac{(q^\alpha;q)_k}{(q;q)_k} q^k f(zq^k) \tag{2.3}$$

$$= z^\alpha (1-q)^\alpha \sum_{k=0}^\infty (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q q^{\frac{k(k+1)}{2}+\alpha k} f(zq^k). \tag{2.4}$$

There are two fractional q-Leibniz rules, the first rule has been derived by Al-Salam and Verma [7] as

$$I_q^\alpha \{f(z)g(z)\} = \sum_{k=0}^\infty \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q D_q^k f(zq^{-\alpha-k}) I_q^{\alpha+k} g(z). \tag{2.5}$$

Their proof for fractional q-Leibniz rule was derived based on the q-type interpolation series such formula in a slightly different form of q-Taylor series. Obviously, (2.5) is valid whenever the functions $f(z)$ and $g(z)$ are such that the series in (2.3) and (2.4) are absolutely convergent. In the case $g(z) = 1$, they obtained the fractional q-integral

$$I_q^\alpha f(z) = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^\infty \frac{(-1)^k q^{-\frac{k(k-1)}{2}-\alpha k} z^{\alpha+k}}{[k]_q! [\alpha+k]_q} D_q^k f(zq^{-\alpha-k}). \tag{2.6}$$

Agarwal [10] defined the second q-extension of the Leibniz rule for the fractional q-integrals for a product of two functions in terms of a series involving fractional q-integrals of the individual functions in the following manner

$$I_q^\alpha \{f(z)g(z)\} = \sum_{k=0}^\infty \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q D_q^k f(z) I_q^{\alpha+k} g(zq^k) \tag{2.7}$$

where $f(z)$ and $g(z)$ are two regular functions such that

$$f(z) = \sum_{r=0}^\infty a_r z^r, \quad |z| < R_1 \quad \text{and} \quad g(z) = \sum_{r=0}^\infty b_r z^r, \quad |z| < R_2$$

then for the result (2.7), $|z| < R = \min\{R_1, R_2\}$. Similarly, in the case $g(z) = 1$, yields

$$I_q^\alpha f(z) = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} z^{\alpha+k}}{[k]_q![\alpha+k]_q} D_q^k f(z). \tag{2.8}$$

3. Applications

Yadav and Purohit [11, 12] investigated the applications of q-Leibniz rule of fractional order q-derivatives and deduced several interesting transformations involving various basic hypergeometric functions of one variable including the basic analogue of Fox's H-function. In the present section, the fractional q-integral formula (2.3) is used to represent some q-special functions expressed as fractional q-integrals by assigning specific values to the function f and to the parameter α. After representing these expressions, it is useful to find some transformations by joining the results. Also the fractional q-Leibniz rule (2.7) is used to generate certain infinite series expansions relating q-special functions of mathematical physics by assigning specific values to the functions f and g, and to the parameter α.

3.1. q-Special functions expressed as fractional q-integrals.

For this purpose it is convenient to get a list of q-special functions in terms of fractional q-integrals using the fractional q-integral (2.3) after assigning the values of the function f and the parameter α, and simplifying the results using some well known transformations as follow

Basic Gauss hypergeometric function

$$ {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) = \frac{z^{1-\gamma}\Gamma_q(\gamma)}{\Gamma_q(\beta)} I_q^{\gamma-\beta}\left\{\frac{z^{\beta-1}}{(z;q)_\alpha}\right\}, \tag{3.1}$$
$$\Re(\gamma) > \Re(\beta) > 0; |z| < 1,$$

$$ {}_2\phi_1(q^{\gamma-\alpha}, z(1-q); 0; q, q^\alpha) = \frac{z^{1-\gamma}E_q(-z)}{(1-q)^{\gamma-\alpha}} I_q^{\gamma-\alpha}\left\{z^{\alpha-1}e_q(z)\right\}, \tag{3.2}$$
$$\Re(\gamma) > \Re(\alpha) > 0; |z| < |1-q|^{-1}.$$

Basic confluent hypergeometric function

$$ {}_1\phi_1(q^\alpha; q^\gamma; q, z(1-q)) = \frac{z^{1-\gamma}\Gamma_q(\gamma)}{\Gamma_q(\alpha)} I_q^{\gamma-\alpha}\left\{z^{\alpha-1}E_q(-z)\right\}, \quad \Re(\gamma) > \Re(\alpha) > 0. \tag{3.3}$$

Incomplete q-gamma function

$$\gamma_q(\alpha, z) = \Gamma_q(\alpha)E_q(-z)I_q^\alpha\{e_q(z)\}, \qquad \Re(\alpha) > 0;\ |z| < |1-q|^{-1}. \tag{3.4}$$

The q-Laguerre function

$$L_\nu^{(\alpha)}(z;q) = \frac{z^{-\alpha}(-z;q)_\infty}{\Gamma_q(\nu+1)} I_q^{-\nu}\left\{\frac{z^{\nu+\alpha}}{(-z;q)_\infty}\right\}, \qquad \Re(\nu+\alpha) > -1, \Re(\nu) < 0 \tag{3.5}$$

where $L_\nu^{(\alpha)}(z;q)$ denotes the q-Laguerre function which we can define it as

$$L_\nu^{(\alpha)}(z;q) = \begin{bmatrix}\alpha+\nu \\ \nu\end{bmatrix}_q {}_1\phi_1(q^{-\nu}; q^{\alpha+1}; q, -zq^{\alpha+\nu+1}). \tag{3.6}$$

To prove (3.1), substituting $f(z)=z^{\beta-1}/(z;q)_\alpha$ into (2.3) yields

$$\begin{aligned}I_q^{\gamma-\beta}\left\{\frac{z^{\beta-1}}{(z;q)_\alpha}\right\} &= z^{\gamma-1}(1-q)^{\gamma-\beta}\sum_{k=0}^{\infty}\frac{(q^{\gamma-\beta};q)_k(zq^{\alpha+k};q)_\infty}{(q;q)_k(zq^k;q)_\infty}q^{k\beta}\\ &= \frac{z^{\gamma-1}(1-q)^{\gamma-\beta}(zq^\alpha;q)_\infty}{(z;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{\gamma-\beta};q)_k(z;q)_k}{(q;q)_k(zq^\alpha;q)_k}q^{k\beta}\\ &= \frac{z^{\gamma-1}(1-q)^{\gamma-\beta}(zq^\alpha;q)_\infty}{(z;q)_\infty}\ {}_2\phi_1(q^{\gamma-\beta},z;zq^\alpha;q,q^\beta).\end{aligned}$$

Using the well known Heines transformation formulas for the ${}_2\phi_1$ series [10]

$${}_2\phi_1(a,b;c;q,z)=\frac{(az,b;q)_\infty}{(c,z;q)_\infty}\ {}_2\phi_1(c/b,z;az;q,b), \qquad |z|<1;\ |b|<1$$

would yield the desired result. The proof of other formulas is similar.

Also, we can obtain another list by using fractional q-Leibniz rule (2.7) after assigning the values of the functions f and g and the parameter α, and simplifying the results as follow

$${}_2\phi_2(q^{\gamma-\beta},q^\alpha;q^\gamma,zq^\alpha;q,zq^\beta)=\frac{z^{1-\gamma}\Gamma_q(\gamma)(z;q)_\alpha}{\Gamma_q(\beta)}I_q^{\gamma-\beta}\left\{\frac{z^{\beta-1}}{(z;q)_\alpha}\right\}, \tag{3.7}$$
$$\Re(\gamma)>\Re(\beta)>0;\ z\in\mathbb{C},$$

$${}_1\phi_1(q^{\gamma-\alpha};q^\gamma;q,zq^\alpha(1-q))=\frac{z^{1-\gamma}\Gamma_q(\gamma)E_q(-z)}{\Gamma_q(\alpha)}I_q^{\gamma-\alpha}\left\{z^{\alpha-1}e_q(z)\right\}, \tag{3.8}$$
$$\Re(\gamma)>\Re(\alpha)>0;|z|<|1-q|^{-1},$$

$${}_3\phi_2(q^{\gamma-\alpha},0,0;q^\gamma,z(1-q);q,zq^\alpha(1-q))=\frac{z^{1-\gamma}\Gamma_q(\gamma)e_q(z)}{\Gamma_q(\alpha)}I_q^{\gamma-\alpha}\left\{z^{\alpha-1}E_q(-z)\right\},$$
$$\Re(\gamma)>\Re(\alpha)>0;|zq^\alpha|<|1-q|^{-1}. \tag{3.9}$$

3.2. **Transformation formulas.** Three well known transformations can be obtained from comparing the relations (3.1), (3.2) and (3.3) with the relations (3.7), (3.8) and (3.9), respectively, as follow

$${}_2\phi_1(a,b;c;q,z)=\frac{(az;q)_\infty}{(z;q)_\infty}\ {}_2\phi_2(a,c/b;c,az;q,bz), \qquad |z|<1, \tag{3.10}$$

$${}_2\phi_1(a,b;0;q,z)=\frac{(az;q)_\infty}{(z;q)_\infty}\ {}_1\phi_1(a;az;q,bz), \qquad |z|<1, \tag{3.11}$$

$${}_1\phi_1(a;c;q,z)=(z;q)_\infty\ {}_3\phi_2(c/a,0,0;c,z;q,az), \qquad |az|<1. \tag{3.12}$$

3.3. **Series expansions.** The above results can be used to establish some new expansions associated with some of q-special functions. These expansions will be obtained by means of changing the positions of the functions f and g in the fractional q-Leibniz rule (2.7). We can derive these expansions as follow
i) *Expansion of basic Gauss hypergeometric function.*

$$\begin{aligned}{}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,z) &= \frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}\sum_{k=0}^{\infty}\frac{(q^{1-\beta};q)_kq^{\beta k}}{(q;q)_k[\gamma-\beta+k]_q}\\ &\times\ {}_2\phi_1(q,q^\alpha;q^{\gamma-\beta+k+1};q,zq^k), \qquad |z|<1\end{aligned} \tag{3.13}$$

or, equivalently

$$ {}_2\phi_1(a,b;c;q,z)=\frac{(b,qc/b;q)_\infty}{(q,c;q)_\infty}\sum_{k=0}^{\infty}\frac{(q/b,c/b;q)_k b^k}{(q,qc/b;q)_k}\,{}_2\phi_1(q,a;cq^{k+1}/b;q,zq^k),\quad |z|<1. \tag{3.14}$$

To prove the formula (3.13), we need

$$D_q^k z^{\beta-1}=(-1)^k q^{\beta k-\frac{k(k+1)}{2}}\frac{(q^{1-\beta};q)_k}{(1-q)^k}z^{\beta-k-1},\qquad \Re(\beta)>0, \tag{3.15}$$

$$I_q^{\lambda}\left\{\frac{1}{(z;q)_\alpha}\right\}=\frac{z^\lambda}{\Gamma_q(\lambda+1)}\,{}_2\phi_1(q,q^\alpha;q^{\lambda+1};q,z),\qquad |z|<1 \tag{3.16}$$

where the formula (3.16) comes from (3.1) by putting $\beta=1, 1-\gamma=-\lambda$. Substituting into the fractional q-Leibniz rule (2.7) would yield

$$\begin{aligned}
I_q^{\gamma-\beta}\left\{\frac{z^{\beta-1}}{(z;q)_\alpha}\right\} &= \sum_{k=0}^{\infty}\begin{bmatrix}\beta-\gamma\\ k\end{bmatrix}_q D_q^k\{z^{\beta-1}\} I_q^{\gamma-\beta+k}\left\{\frac{1}{(zq^k;q)_\alpha}\right\}\\
&= \frac{z^{\gamma-1}}{\Gamma_q(\gamma-\beta+1)}\sum_{k=0}^{\infty}\frac{(q^{\gamma-\beta};q)_k(q^{1-\beta};q)_k}{(q;q)_k(q^{\gamma-\beta+1};q)_k}q^{\beta k}\,{}_2\phi_1(q,q^\alpha;q^{\gamma-\beta+k+1};q,zq^k)\\
&= \frac{z^{\gamma-1}}{\Gamma_q(\gamma-\beta)}\sum_{k=0}^{\infty}\frac{(q^{1-\beta};q)_k}{(q;q)_k[\gamma-\beta+k]_q}q^{\beta k}\,{}_2\phi_1(q,q^\alpha;q^{\gamma-\beta+k+1};q,zq^k).
\end{aligned}$$

Using the previous relation and (3.1) would yield the proof.
ii) *Expansions of basic confluent hypergeometric function.*
Similarly, by using (3.3), we can get

$$ {}_1\phi_1(q^\alpha;q^\gamma;q,z)=\frac{\Gamma_q(\gamma)}{\Gamma_q(\alpha)\Gamma_q(\gamma-\alpha)}\sum_{k=0}^{\infty}\frac{(q^{1-\alpha};q)_k q^{\alpha k}}{(q;q)_k[\gamma-\alpha+k]_q}\,{}_1\phi_1(q;q^{\gamma-\alpha+k+1};q,zq^k) \tag{3.17}$$

or, equivalently

$$ {}_1\phi_1(a;c;q,z)=\frac{(a,qc/a;q)_\infty}{(q,c;q)_\infty}\sum_{k=0}^{\infty}\frac{(q/a,c/a;q)_k a^k}{(q,qc/a;q)_k}\,{}_1\phi_1(q;cq^{k+1}/a;q,zq^k). \tag{3.18}$$

When $z=c/a$, we get the q-binomial theorem (1.12) via the identity (1.13).
iii) *Expansions in the incomplete q-gamma function.*
As above, substituting the relations (3.4) and (3.15) into fractional q-Leibniz rule (2.7) and comparing with (3.2) following by using the transformation (3.11) would yield for $\Re(\beta)>0$

$$ {}_1\phi_1(q^\beta;q^{\alpha+\beta};q,zq^\alpha(1-q))=\frac{z^{-\beta}\Gamma_q(\alpha+\beta)}{\Gamma_q(\alpha)\Gamma_q(\beta)}\sum_{k=0}^{\infty}\frac{(q^{1-\alpha};q)_k(z(1-q);q)_k}{(q;q)_k q^{k(k+\beta-\alpha)}z^k}\gamma_q(\beta+k,zq^k) \tag{3.19}$$

It is obvious that if $z=1/(1-q)$, the expansion (3.19) returns to the identity (1.13) and if $\alpha=1$, we arrive at the well known relation between incomplete q-gamma function and basic confluent hypergeometric function (1.24).

Another series expansion in terms of incomplete q-gamma function can be obtained from the fact $e_q(z)E_q(-z) = 1$ and the relation

$$I_q^{\alpha+k}\{1\} = \frac{z^{\alpha+k}}{\Gamma_q(\alpha+k+1)} \tag{3.20}$$

with taking $f(z) = E_q(-z)$ and $g(z) = e_q(z)$ in fractional q-Leibniz rule (2.7)

$$\sum_{k=0}^{\infty} \frac{\gamma_q(\alpha+k, zq^k)}{[k]_q!} q^{-\alpha k} = \frac{z^\alpha}{[\alpha]_q}. \tag{3.21}$$

The above relation can be rewritten as

$$\sum_{k=1}^{\infty} \frac{\gamma_q(\alpha+k, zq^k)}{[k]_q!} q^{-\alpha k} = \frac{z^\alpha - \Gamma_q(\alpha+1)}{[\alpha]_q} + \Gamma_q(\alpha, z). \tag{3.22}$$

Taking the limit as $\alpha \to 0$ with applying l'Hôspital rule to the first term on the right hand side would yield

$$\Gamma_q(0, z) = \frac{1-q}{\ln q} \ln z - \gamma_q + \sum_{k=1}^{\infty} \frac{\gamma_q(k, zq^k)}{[k]_q!}. \tag{3.23}$$

From the equations (1.25) and (1.26), we can arrive at

$$\sum_{k=1}^{\infty} \frac{\gamma_q(k, zq^k)}{[k]_q!} = \sum_{k=1}^{\infty} \frac{q^k \gamma_q(k, z)}{[k]_q!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{\frac{k(k+1)}{2}} z^k}{[k]_q \cdot [k]_q!}. \tag{3.24}$$

iv) *Expansion of q-exponential function.*
In the case of choosing $f(z) = e_q(z)$ and $g(z) = E_q(-z)$ in (2.7), the q-exponential function can be expanded in basic confluent hypergeometric function as

$$E_q(-z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} (q^\alpha; q)_k z^k}{[k]_q! (q^{\alpha+1}; q)_k} \, {}_1\phi_1(q; q^{\alpha+k+1}; q, zq^k(1-q)) \tag{3.25}$$

or equivalently

$$(z; q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a; q)_k z^k}{(q; q)_k (aq; q)_k} \, {}_1\phi_1(q; aq^{k+1}; q, zq^k). \tag{3.26}$$

Notice that the previous expansion will return immediately to (1.10) when $a = 0$. Using the fact $e_q(z)E_q(-z) = 1$ and the relations (3.21) and (3.25) would yield the following equation

$$\sum_{k=0}^{\infty} \frac{\gamma_q(\alpha+k, zq^k)}{[k]_q!} q^{-\alpha k} = z^\alpha e_q(z) \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} z^k}{[k]_q! [\alpha+k]_q} \, {}_1\phi_1(q; q^{\alpha+k+1}; q, zq^k(1-q)) \tag{3.27}$$

which can also be rewritten as

$$\frac{\Gamma_q(\alpha+1) - z^\alpha e_q(z) \, {}_1\phi_1(q; q^{\alpha+1}; q, z(1-q))}{[\alpha]_q} - \Gamma_q(\alpha, z) + \sum_{k=1}^{\infty} \frac{\gamma_q(\alpha+k, zq^k)}{[k]_q!} q^{-\alpha k}$$
$$= z^\alpha e_q(z) \sum_{k=1}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} z^k}{[k]_q! [\alpha+k]_q} \, {}_1\phi_1(q; q^{\alpha+k+1}; q, zq^k(1-q)). \tag{3.28}$$

Taking the limit as $\alpha \to 0$ with applying l'Hôspital rule to the first term on the right hand side and taking into account the relation (3.23) would yield

$$\lim_{\alpha\to 0}\frac{d}{d\alpha}[\,{}_1\phi_1(q;q^{\alpha+1};q,z(1-q))] = \frac{\ln q}{1-q}\sum_{k=1}^{\infty}\frac{(-1)^k q^{\frac{k(k+1)}{2}}z^k}{[k]_q![k]_q}\,{}_1\phi_1(q;q^{k+1};q,zq^k(1-q)). \tag{3.29}$$

This implies that

$$\sum_{k=1}^{\infty}\frac{(-1)^k q^{\frac{k(k-1)}{2}}z^k}{[k]_q!}H_{k,q} = \sum_{k=1}^{\infty}\frac{(-1)^k q^{\frac{k(k+1)}{2}}z^k}{[k]_q!(1-q^k)}\,{}_1\phi_1(q;q^{k+1};q,zq^k(1-q)) \tag{3.30}$$

where $H_{k,q}$ is a q-harmonic number defined by [13] as

$$H_{k,q} = \sum_{i=1}^{k}\frac{q^i}{1-q^i}. \tag{3.31}$$

A useful generating function for q-harmonic number can be obtained by putting $z = 1/(1-q)$ and inserting (1.13) as

$$(q;q)_\infty = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(q;q)_k}H_{k,q}. \tag{3.32}$$

The relation (3.30) after replacing z by $-z$ when $q \to 1^-$ will return to the well known exponential generating function for a harmonic number [14]

$$\sum_{k=1}^{\infty}\frac{z^k}{k!}H_k = e^z\sum_{k=1}^{\infty}\frac{\gamma(k,z)}{k!} = e^z(E_1(z)+\gamma+\ln z) \tag{3.33}$$

where $E_1(z)$ is the exponential integral and γ is the Euler-Mascheroni constant.
v) *Expansion of q-Laguerre function.*
In the case of $f(z) = z^{\alpha+\nu}$ and $g(z) = 1/(-z;q)_\infty$ in fractional q-Leibniz rule (2.7) after simplifying the results and using the relation (3.5), we can derive an expansion of q-Laguerre function (3.6) as

$$L_\nu^{(\alpha)}(z;q) = \sum_{k=0}^{\infty}\frac{(q^{-\alpha-\nu};q)_k(-z;q)_k}{(q;q)_k}(-1)^k q^{\alpha k+\frac{k(k+1)}{2}}L_{\nu-k}^{(k-\nu)}(zq^k;q) \tag{3.34}$$

which can also be rewritten as

$$\begin{aligned} L_\nu^{(\alpha)}(-z(1-q);q) = {}& \frac{z^\nu}{\Gamma_q(-\nu)\Gamma_q(\nu+1)}\sum_{k=0}^{\infty}\frac{(q^{-\alpha-\nu};q)_k(z(1-q);q)_k}{(q;q)_k z^k} \\ & \times q^{k(\alpha+2\nu-k+1)}\gamma_q(k-\nu,zq^k), \qquad \Re(\nu)<0. \end{aligned} \tag{3.35}$$

vi) *q-Analogue of the Hadamard expansion.*
Finally, the expansions (3.13), (3.19) and (3.35) after assigning a suitable values for its parameters can be converted to the expansion

$$\begin{aligned} {}_1\phi_1(q^{\nu+1/2};q^{2\nu+1};q,zq^{\nu+1/2}(1-q)) = {}& \frac{\Gamma_q(2\nu+1)E_q(-z)}{z^{\nu+1/2}\Gamma_q^2(\nu+1/2)}\sum_{k=0}^{\infty}\frac{(q^{1/2-\nu};q)_k q^{-k^2}}{(q;q)_k z^k} \\ & \times e_q(zq^k)\gamma_q(\nu+k+1/2,zq^k), \qquad \Re(\nu) > -\frac{1}{2}, \quad |z|<|1-q|^{-1} \end{aligned} \tag{3.36}$$

which can be considered a q-analogue of the Hadamard expansion [15]

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n}{n!(2z)^n} \frac{\gamma(n+\nu+1/2, 2z)}{\Gamma(\nu+1/2)}, \qquad \Re(\nu) > -\frac{1}{2} \tag{3.37}$$

where $I_\nu(z)$ is the modified Bessel function defined as

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}. \tag{3.38}$$

Indeed, in view of the familiar relation [16]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} e^{\pm z} \ {}_1F_1(\nu+\frac{1}{2}; 2\nu+1; \mp 2z)$$

and the relation [17]

$$\gamma(\alpha, z) = \alpha^{-1} z^\alpha e^{-z} \ {}_1F_1(1; \alpha+1; z).$$

It is easy to rewrite the Hadamard expansion (3.37) in its equivalent form

$$\begin{aligned} {}_1F_1(\nu+\frac{1}{2}; 2\nu+1; -z) &= \frac{\Gamma(2\nu+1)}{z^{\nu+1/2}\Gamma^2(\nu+1/2)} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n}{n!z^n} \\ &\times \gamma(n+\nu+\frac{1}{2}, z), \qquad \Re(\nu) > -\frac{1}{2}. \end{aligned} \tag{3.39}$$

It is obvious that the expansion (3.36) tends to the equivalent Hadamard expansion (3.39) as $q \to 1$.

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