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# A RELIABLE TREATMENT OF HOMOTOPY PERTURBATION METHOD FOR THE SINE-GORDON EQUATION OF ARBITRARY (FRACTIONAL) ORDER

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Abstract. In this paper, the reliable treatment of homotopy perturbation method (HPM) [19] is applied to obtain the solution of the sine-Gordon partial differential equation of arbitrary (fractional) order. The advantage of this algorithm is its ability to provide the analytical or approximate solutions to nonlinear equations with the capability to overcome the difficulty that arises in calculating complicated integrals. The numerical results are presented to show the efficiency of this method.

## 1. Introduction

The sine-Gordon equation which first appeared in the study of the differential geometry of surfaces with Gaussian curvature $K = -1$ found wide applications in the propagation of fluxons in Josephson junctions between two superconductors [1], the motion of a rigid pendulum attached to a stretched wire [2], solid state physics, nonlinear optics, stability of fluid motions, dislocations in crystals [2] and other scientific fields. Due to its wide applications and important mathematical properties, a great deal of effort has been devoted to studying the different solutions and physical phenomena related to this equation [3]-[11].

In 1998, J. H. He proposed the homotopy perturbation method (HPM) for addressing nonlinear problems in [13] and [14]. This method has been the subject of extensive studies, and applied to different linear and nonlinear problems in [14]-[20]. The advantage of this method is solving nonlinear equations without invoking unrealistic assumptions, discretization or linearization. The HPM has the advantage of dealing directly with the problem without transformations, linearization, discretizations or any unrealistic assumption. The method yields a rapidly convergent series solution, and usually a few iterations lead to accurate approximation of the exact solution [18] and [21].

Recently, Momani and Odibat suggested a reliable algorithm for the HPM for dealing with nonlinear terms [19]. The advantage of this algorithm is its ability to provide the analytical or approximate solutions to nonlinear equations and overcome

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the difficulty that arising in calculating complicated integrals. Our aim here, is to apply the reliable treatment of HPM to obtain the solution of the initial value problem of the sine-Gordon equation of fractional order

$$D_t^{\alpha}\ u(x,t) = au_{xx}(x,t) + b\sin\left(\lambda u\left(x,t\right)\right),\ x \in R,\ t > 0,\ \alpha \in (1,2]\,, \tag{1}$$

subjected to the initial conditions

$$u^{(k)}(x,0) = g_k(x), \quad x \in R,\ k = 0,1. \tag{2}$$

The article begins by presenting some basic definitions of fractional derivatives in section two. The HPM and the reliable treatment of HPM are introduced in section three. In section four, some case studies of the nonlinear sine-Gordon equations of arbitrary (fractional) orders are presented to illustrate the validity of this approach and to show the effects of fractional order parameters involved on solution accuracy and behavior.

## 2. Basic definitions

**Definition 1.** *A real function $f(t)$, $t > 0$, is said to be in the space $C_\mu$, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space $C_\mu^m$ if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.*

**Definition 2.** *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t) \in C_\mu$, $\mu \geq -1$ is defined as* [22]

$$\begin{cases} J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,\ \alpha > 0,\ t > 0, \\ J^0 f(t) = f(t). \end{cases} \tag{3}$$

*The operator $J^\alpha$ satisfy the following properties. For $f \in C_\mu$, $\mu \geq -1$, $\alpha$, $\beta \geq 0$ and $\gamma > -1$:*

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}$.

**Definition 3.** *The fractional derivative in Caputo sense of $f(t) \in C_{-1}^m$, $m \in \mathbb{N}$, $t > 0$ is defined as*

$$^{C}D_t^{\beta} f(t) = \begin{cases} J^{m-\beta} \frac{d^m}{dt^m} f(t), & m-1 < \beta < m, \\ \frac{d^m}{dt^m} f(t), & \beta = m. \end{cases} \tag{4}$$

*The operator $^{C}D^{\beta}$ satisfy the following properties. For $f \in C_\mu^m$, $\mu \geq -1$, $\gamma, \beta \geq 0$:*

1. $^{C}D_t^{\beta}[J^\beta f(t)] = f(t)$,
2. $J^\beta[\,^{C}D_t^{\beta} f(t)] = f(t) - \sum\limits_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}$, $t > 0$,
3. $^{C}D_t^{\beta} t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} t^{\gamma-\beta}$.

## 3. The homotopy perturbation method (HPM)

Consider the following equation

$$A\left(u\left(x,t\right)\right)-f(r)=0,\ \ r\in\Omega, \tag{5}$$

with boundary conditions

$$B(u,\partial u/\partial n)=0,\ \ r\in\Gamma, \tag{6}$$

where $A$ is a general differential operator, $u(x,t)$ is the unknown function and $x$ and $t$ denote spatial and temporal independent variables, respectively. $B$ is a boundary operator, $f(r)$ is a known analytic function, and $\Gamma$ is the boundary of the domain $\Omega$. The operator $A$ can be generally divided into linear and nonlinear parts, say $L$ and $N$. Therefore (5) can be written as

$$L\left(u\right)+N(u)-f(r)=0. \tag{7}$$

In [12], He constructed a homotopy $v(r,p):\Omega\times[0,1]\rightarrow R$ which satisfies

$$H(v,p)=(1-p)\left[L(v)-L(u_0)\right]+p\left[L(v)+N(v)-f(r)\right]=0,r\in\Omega, \tag{8}$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0,r\in\Omega, \tag{9}$$

where $p\in[0,1]$ is an embedding parameter, $u_0$ is an initial guess of $u(x,t)$ which satisfies the boundary conditions. Obviously, from (8) and (9) one has

$$H(v,0)=L(v)-L(u_0), \tag{10}$$

$$H(v,1)=L\left(u\right)+N(u)-f(r)=0. \tag{11}$$

Changing $p$ from zero to unity is just that change of $v(r,p)$ from $u_0(r)$ to $u\left(r\right)$. Expanding $v(r,p)$ in Taylor series with respect to $p$, one has

$$v=v_0+pv_1+p^2v_2+\cdots\ . \tag{12}$$

Setting $p=1$, results in the approximate solution of (5)

$$u=\lim_{p\rightarrow 1}v=v_0+v_1+v_2+\ldots\ . \tag{13}$$

The reliable treatment of the classical HPM suggested by Momani and Odibat [19] is presented for nonlinear function $N(u)$ which is assumed to be an analytic function and has the following Taylor series expansion

$$N(u)=\sum_{i=0}^{\infty}a_iu^i. \tag{14}$$

According to [19], the following homotopy is constructed for problem (1)

$$D_t^{\alpha}\ u=pau_{xx}+b\sum_{i=0}^{\infty}p^ia_iu^i,\qquad\quad p\in[0,1]\ . \tag{15}$$

The basic assumption is that the solution of (15) we can be written as a power series in $p$:

$$u = u_0 + pu_1 + p^2u_2 + .... \tag{16}$$

Substituting (16) into (15) and equating the terms with identical powers of $p$, we obtain a series of linear equations in $u_0$, $u_{1,}$ $u_2, ...$ which can be solved by symbolic computation software such as Mathematica. Finally, we approximate the solution $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ by the truncated series

$$\phi_n(x,t) = \sum_{i=1}^{n-1} u_i(x,t). \tag{17}$$

## 4. Numerical implementation

In this section, we present some numerical examples to validate the solution scheme. All the results are calculated using the symbolic software Mathematica.

**Example 1.** *Consider the fractional-order nonlinear sine–Gordon equation*

$$\begin{cases} D_t^{\alpha}\ u(x,t) = au_{xx}(x,t) + b\sin\left(\lambda u\left(x,t\right)\right), \ x \in R, t > 0, \ \alpha \in (1,2], \\ \quad u(x,0) = g_1\left(x\right), \ u_t(x,0) = g_2\left(x\right). \end{cases} \tag{18}$$

We approximate $\sin(\lambda u)$ by two terms of its Taylor series, $\sin(\lambda u) \simeq u - \frac{u^3}{3!}$.

According to the homotopy (15), we obtain the following set of linear partial differential equations of fractional-order

$$\begin{array}{lll} p^0 & : & D_t^{\alpha}\ u_0 = 0, & u(x,0) = g1\left(x\right), u_t(x,0) = g2\left(x\right), \\ p^1 & : & D_t^{\alpha}\ u_1 = aD_x^{\alpha}u_0 + b\lambda u_0, & u_1(x,0) = 0, \ u_{2t}(x,0) = 0, \\ p^2 & : & D_t^{\alpha}\ u_2 = aD_x^{\alpha}u_1 + b\lambda u_1, & u_2(x,0) = 0, \ u_{2t}(x,0) = 0, \\ p^3 & : & D_t^{\alpha}\ u_3 = aD_x^{\alpha}u_2 + b\lambda u_2 - \dfrac{b\lambda^3}{3!}u_0^3, & u_3(x,0) = 0, \ u_{3t}(x,0) = 0, \\ & \vdots & \end{array} \tag{19}$$

Solving (19) for $u_0$, $u_1$, $u_2$, $\cdots$, the first few components of the homotopy perturbation solution for (18) are derived as follows

$$u_0(x,t) = g_1\left(x\right) + tg_2\left(x\right),$$

$$u_1(x,t) = \frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)}\left(b\lambda g_1\left(x\right) + ag_1^{(2)}\left(x\right)\right) + \frac{t^{\alpha+1}}{\Gamma\left(\alpha+2\right)}\left(b\lambda g_2\left(x\right) + ag_1^{(2)}\left(x\right)\right),$$

$$\begin{aligned} u_2(x,t) &= \frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}\left(b\lambda\left(b\lambda g_1\left(x\right) + ag_1^{(2)}\left(x\right)\right) + a\left(b\lambda g_1^{(2)}\left(x\right) + ag_1^{(4)}\left(x\right)\right)\right) \\ &\quad + \frac{t^{2\alpha+1}}{\Gamma\left(2\alpha+2\right)}\left(\left(b\lambda g_2^{(2)}\left(x\right) + ag_2^{(4)}\left(x\right)\right) + b\lambda\left(b\lambda g_2\left(x\right) + ag_2^{(2)}\left(x\right)\right)\right), \end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{-1}{6}b\lambda^3 g_1^3(x)\right) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\left(\frac{1}{2}b\lambda^3 g_1^2(x)\, g_2(x)\right) - \\
&\quad \frac{t^{\alpha+2}}{\Gamma(\alpha+3)}\left(b\lambda^3 g_1(x)\, g_2^2(x)\right) - \frac{t^{\alpha+3}}{\Gamma(\alpha+4)}\left(b\lambda^3 g_2^3(x)\right), \\
&\quad \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\left(b^3\lambda^3 g_1(x) + 3ab^2\lambda^2 g_1^{(2)}(x) + 3a^2b\lambda g_1^{(4)}(x) + a^3 g_1^{(6)}(x)\right) + \\
&\quad \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\left(b^3\lambda^3 g_2(x) + 3ab^2\lambda^2 g_2^{(2)}(x) + 3a^2b\lambda g_2^{(4)}(x) + a^3 g_2^{(6)}(x)\right),
\end{aligned}
$$

$$\vdots$$

and the solution is thus obtained as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots.$$

For numerical comparison purpose, we consider (18) with tow different initial conditions indicated in the following table with fixed values of $a = 1$, $b = -1$ and $\lambda = 1$.

| | $\alpha \in (1,2]$ | |
|---|---|---|
| *Case* | $g_1(x)$ | $g_2(x)$ |
| 1 | 0 | $4\,\mathrm{sech}(x)$ |
| 2 | $\pi + \epsilon\cos(\mu x)$ | 0 |

**Case 1**

Substituting the initial conditions, we obtain

$$
\begin{aligned}
u_0(x,t) &= 4t\,\mathrm{sech}(x), \\
u_1(x,t) &= t^{\alpha+1}\frac{4\,\mathrm{sech}(x)}{\Gamma(\alpha+2)}\left(\tanh^2(x) - \overset{2}{\mathrm{sech}}(x) - 1\right), \\
u_2(x,t) &= t^{2\alpha+1}\frac{4\,\mathrm{sech}(x)}{\Gamma(2\alpha+2)}(5\,\overset{4}{\mathrm{sech}}(x) + \left(-1+\tanh^2(x)\right)^2 \\
&\quad -2\,\overset{2}{\mathrm{sech}}(x)\left(-1+9\tanh^2(x)\right)),
\end{aligned}
$$

Where the exact solution is given by $u(x,t) = 4\arctan(\mathrm{sech}(x)\,t)$ [24]

Figure (1) gives the comparison between the HPM $3^{rd}$-order approximate solution of problem (18) in case1 with $\alpha = 1.99$, 1.95, 1.90 and 1.85 and the solution of corresponding problem of integer order, denoted by $u_2$, given in [24] at $t = 0.5$.

**Case 2**

Substituting the initial conditions, we obtain

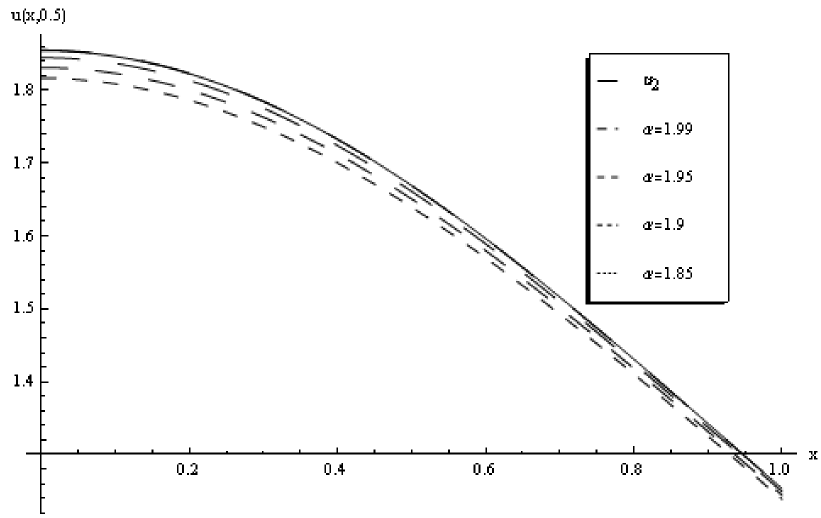


FIGURE 1. $u(x, 0.5)$ of case1 for $3^{rd}$-order HPM approximation as parameterized by $\alpha$.

$$
\begin{aligned}
u_0(x,t) &= \pi + \epsilon\cos(\mu x),\\
u_1(x,t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(-\pi + \epsilon\left(-1-\mu^2\right)\cos(\mu x)\right)\\
u_2(x,t) &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(\pi + \epsilon\left(-1-\mu^2\right)^2\cos(\mu x)\right),\\
u_3(x,t) &= \frac{t^{3\alpha}\left(-\pi + \epsilon\left(-1-\mu^2\right)^3\cos(\mu x)\right)}{\Gamma(3\alpha+1)} + \frac{t^{\alpha}\left(-\pi+\epsilon\cos(\mu x)\right)^3}{\Gamma(\alpha+1)},\\
&\vdots
\end{aligned}
$$

For numerical comparison purpose, we substitute $\epsilon = 0.05$ and $\mu = \frac{1}{\sqrt{2}}$ as chosen in [21]. Figure (2) gives the HPM $3^{rd}$-order approximate solution of problem (18) in case2 with $\alpha = 1.99$, 1.95, 1.90 and 1.85 at $t = 0.5$ whereas Figure (3) gives the HPM $3^{rd}$-order approximate solution with $\alpha = 1.99$ which coincides with the figure given in [21] for the corresponding integer order problem.

## 5. Conclusion

The reliable treatment HPM is applied to obtain the solution of the sine-Gordon partial differential equation of arbitrary (fractional) order. The main advantage of this algorithm is the capability to overcome the difficulty that arising in calculating complicated integrals. The graphs illustrate the continuation of the solution of fractional-order sine-Gordon equation to the solution of the corresponding second-order problem when the fractional order parameters approach their integer limits.

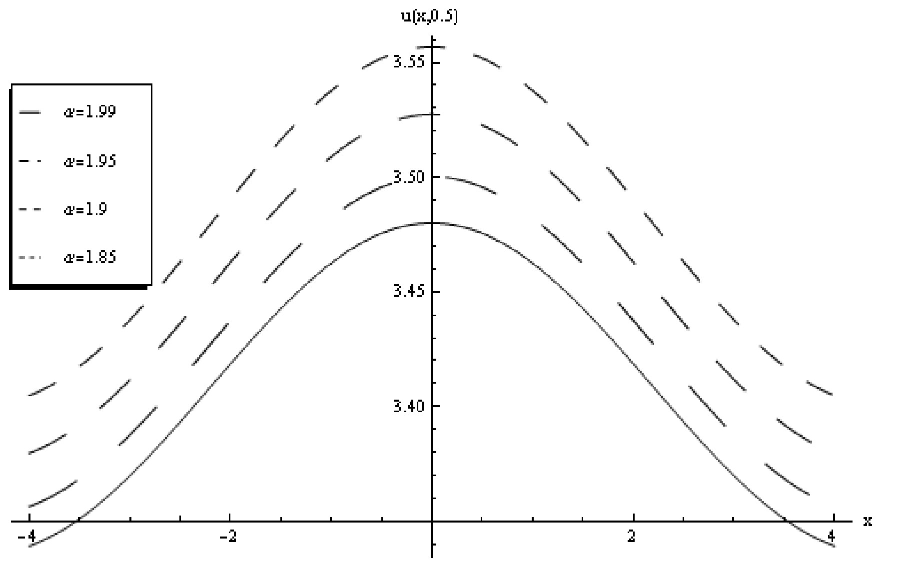


FIGURE 2. $u(x, 0.5)$ of case2 for $3^{rd}$-order HPM approximation as parameterized by $\alpha$.

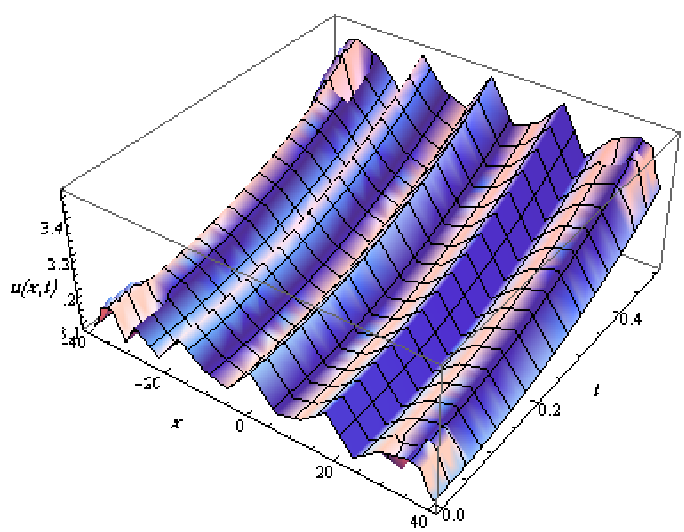


FIGURE 3. Solution of case2 for $3^{rd}$-order HPM approximation at $\alpha = 1.99$.

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