

A PREDICTOR-CORRECTOR SCHEME FOR SOLVING NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. Adams-Bashforth-Moulton algorithm has been extended to solve delay differential equations of fractional order. Numerical illustrations are presented to demonstrate utility of the method. Chaotic behaviour is observed in one dimensional delayed systems of fractional order. We further find the smallest fractional order for the chaotic behaviour. It is also observed that the phase portraits get stretched in some cases as the order of the derivative is reduced.

1. INTRODUCTION

Due to its applicability in a variety of fields Fractional Calculus (FC) is receiving importance in various branches of Science and Engineering. Unlike ordinary derivative operator, fractional derivative operator (FDO) is non-local in nature. Due to non-local nature of FDO, it can formulate processes having memory and hereditary properties. Fractional calculus is finding applications especially in viscoelasticity, anomalous diffusion process, electrochemistry, fluid flow and so on [4, 17, 18].

Delay differential equation (DDE) is a differential equation in which the derivative of the function at any time depends on the solution at previous time. Introduction of delay in the model enriches its dynamics and allows a precise description of the real life phenomena. DDEs are proved useful in control systems [8], lasers, traffic models [3], metal cutting, epidemiology, neuroscience, population dynamics [11], chemical kinetics [7] etc. Even in one dimensional systems interesting phenomena like chaos are observed (cf. Example 1). In DDE one has to provide history of the system over the delay interval $[-\tau, 0]$ as the initial condition. Due to this reason delay systems are infinite dimensional in nature.

Because of infinite dimensionality the DDEs are difficult to analyse analytically [9] and hence the numerical solutions play an important role.

Existence and uniqueness theorems on fractional delay differential equations are discussed in [1, 10, 14, 15]. In this paper we extend the fractional predictor-corrector scheme to solve DDEs of fractional order. Some numerical examples are presented to explain the method.

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2. PRELIMINARIES

2.1. Fractional calculus. We present in this section some basic definitions and properties [13, 17, 18].

Definition 2.1. A real function $f(t)$, $t > 0$ is said to be in space C_α , $\alpha \in \mathfrak{R}$ if there exists a real number $p (> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty)$.

Definition 2.2. A real function $f(t)$, $t > 0$ is said to be in space C_α^m , $m \in \mathbb{N} \cup \{0\}$ if $f^{(m)} \in C_\alpha$.

Definition 2.3. Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann-Liouville integral of order μ , $\mu > 0$ is given by

$$I_t^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} f(\tau) d\tau, \quad t > 0. \quad (1)$$

Definition 2.4. The (left sided) Caputo fractional derivative of f , $f \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$, is defined as:

$$\begin{aligned} D_t^\mu f(t) &= \frac{d^m}{dt^m} f(t), & \mu = m \\ &= I_t^{m-\mu} \frac{d^m f(t)}{dt^m}, & m-1 < \mu < m, \quad m \in \mathbb{N}. \end{aligned} \quad (2)$$

Note that for $m-1 < \mu \leq m$, $m \in \mathbb{N}$,

$$I_t^\mu D_t^\mu f(t) = f(t) - \sum_{k=0}^{m-1} \frac{d^k f}{dt^k}(0) \frac{t^k}{k!}, \quad (3)$$

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \quad (4)$$

2.2. Predictor-corrector scheme for fractional differential equations. The numerical methods used for solving ODEs can not be used directly to solve fractional differential equations (FDEs) because of nonlocal nature of the FDO. A modification in Adams-Bashforth-Moulton algorithm is proposed by Diethelm et al. in [5, 6] to solve FDEs.

Consider the initial value problem (IVP)

$$D_t^\alpha y(t) = f(t, y(t)), \quad 0 \leq t \leq T, \quad (5)$$

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \dots, m-1, \quad \alpha \in (m-1, m], \quad (6)$$

where f is in general a nonlinear function of its arguments. The IVP (5)–(6) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (7)$$

Consider the uniform grid $\{t_n = nh/n = 0, 1, \dots, N\}$ for some integer N and $h := T/N$. Let $y_h(t_n)$ denote the approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j)$, $j = 1, 2, \dots, n$ and want to obtain $y_h(t_{n+1})$ by means of the equation [5, 6]

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^P(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, y_h(t_j)) \quad (8)$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \leq j \leq n, \\ 1, & \text{if } j = n+1. \end{cases} \quad (9)$$

The preliminary approximation $y_h^P(t_{n+1})$ is called predictor and is given by

$$y_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, y_n(t_j)), \quad (10)$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} ((n+1-j)^\alpha - (n-j)^\alpha). \quad (11)$$

Error in this method is

$$\max_{j=0,1,\dots,N} |y(t_j) - y_h(t_j)| = O(h^p), \quad (12)$$

where $p = \min(2, 1 + \alpha)$.

3. MAIN RESULTS

In this section, we modify the Adams-Bashforth-Moulton predictor-corrector scheme described in Section 2.2 to solve delay differential equations of fractional order (FDDE). Consider the following FDDE

$$D_t^\alpha y(t) = f(t, y(t), y(t-\tau)), \quad t \in [0, T], \quad 0 < \alpha \leq 1 \quad (13)$$

$$y(t) = g(t), \quad t \in [-\tau, 0]. \quad (14)$$

Consider a uniform grid $\{t_n = nh : n = -k, -k+1, \dots, -1, 0, 1, \dots, N\}$ where k and N are integers such that $h = T/N$ and $h = \tau/k$. Let

$$y_h(t_j) = g(t_j), \quad j = -k, -k+1, \dots, -1, 0 \quad (15)$$

and note that

$$y_h(t_j - \tau) = y_h(jh - kh) = y_h(t_{j-k}), \quad j = 0, 1, \dots, N. \quad (16)$$

Suppose we have already calculated approximations $y_h(t_j) \approx y(t_j)$, ($j = -k, -k+1, \dots, -1, 0, 1, \dots, n$) and we want to calculate $y_h(t_{n+1})$ using

$$y(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \xi)^{\alpha-1} f(\xi, y(\xi), y(\xi - \tau)) d\xi. \quad (17)$$

Note that equation (17) is obtained by applying $I_{t_{n+1}}^\alpha$ on both sides of (13) and using (14). We use approximations $y_h(t_n)$ for $y(t_n)$ in (17). Further the integral in equation (17) is evaluated using product trapezoidal quadrature formula. The corrector formula is thus

$$\begin{aligned} y_h(t_{n+1}) &= g(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h(t_{n+1}), y_h(t_{n+1} - \tau)) \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)) \\ &= g(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h(t_{n+1}), y_h(t_{n+1-k})) \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \end{aligned} \quad (18)$$

where $a_{j,n+1}$ are given by (9). The unknown term $y_h(t_{n+1})$ appears on both sides of (18) and due to nonlinearity of f equation (18) can not be solved explicitly for $y_h(t_{n+1})$. So we replace the term $y_h(t_{n+1})$ on the right hand side by an approximation $y_h^P(t_{n+1})$, called predictor. Product rectangle rule is used in (17) to evaluate predictor term

$$\begin{aligned} y_h^P(t_{n+1}) &= g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)) \\ &= g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \end{aligned} \quad (19)$$

where $b_{j,n+1}$ is given by (11).

4. ILLUSTRATIVE EXAMPLES

Example 4.1. Consider a fractional order version of the DDE given in [21]

$$D_t^\alpha y(t) = \frac{2y(t-2)}{1+y(t-2)^{9.65}} - y(t), \quad (20)$$

$$y(t) = 0.5, t \leq 0. \quad (21)$$

We have taken the step size $h = 0.01$ in this example. Fig. 1(a) shows the solution $y(t)$ of system (20)–(21) for $\alpha = 0.97$, whereas Fig. 1(b) shows phase portrait of the system i.e. plot of $y(t)$ versus $y(t-2)$ for the same value of α . It may be observed from these figures that the system shows aperiodic (chaotic) behaviour. In the following experiments we have decreased the value of α and observed that the system becomes periodic for $\alpha < 0.87$. The periodic behaviour of the system can be observed in Fig. 1(c) and 1(d) where we have considered $\alpha = 0.85$.

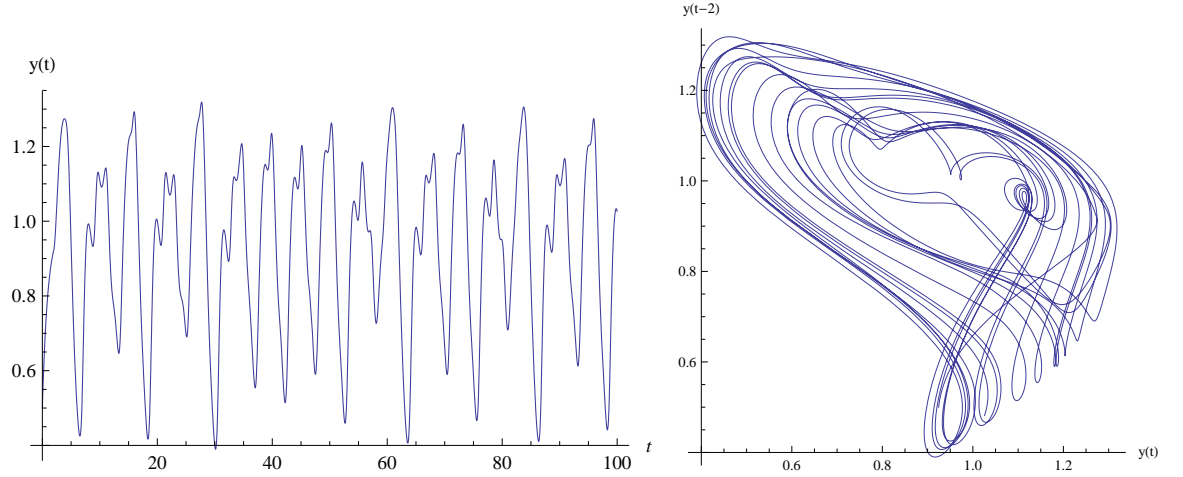


Fig.1(a): $\alpha = 0.97$

Fig.1(b): $\alpha = 0.97$

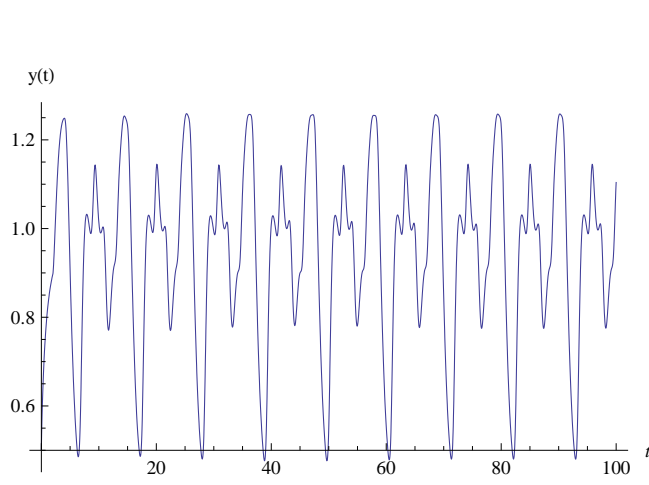


Fig.1(c): $\alpha = 0.85$

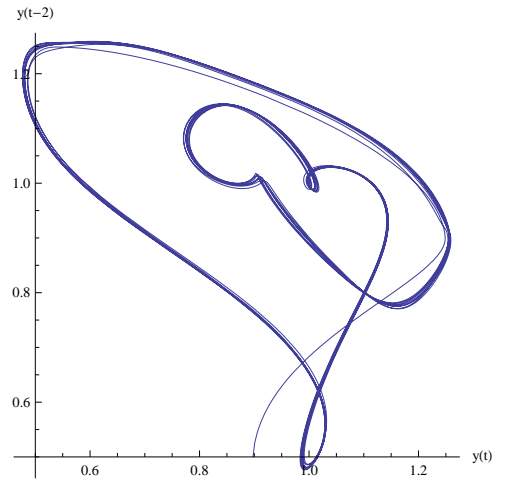


Fig.1(d): $\alpha = 0.85$

Example 4.2. *In this example we consider the fractional order version of the four year life cycle of a population of lemmings[20]*

$$D_t^\alpha y(t) = 3.5y(t) \left(1 - \frac{y(t-0.74)}{19} \right), \quad y(0) = 19.00001, \quad (22)$$

$$y(t) = 19, \quad t < 0. \quad (23)$$

Fig. 2(a) shows the evolution of the system (22)–(24) for $\alpha = 0.97$. Plot of $y(t)$ versus $y(t - 0.74)$ is drawn in Fig. 2(b) for the value $\alpha = 0.97$. It is observed that the phase portrait gets stretched as the value of α decreases. This stretching is towards positive side of the axes. Figs. 2(c), 2(d), 2(e) and 2(f) show the stretching phenomena for the values $\alpha = 0.90, 0.87, 0.83, 0.765$ respectively.

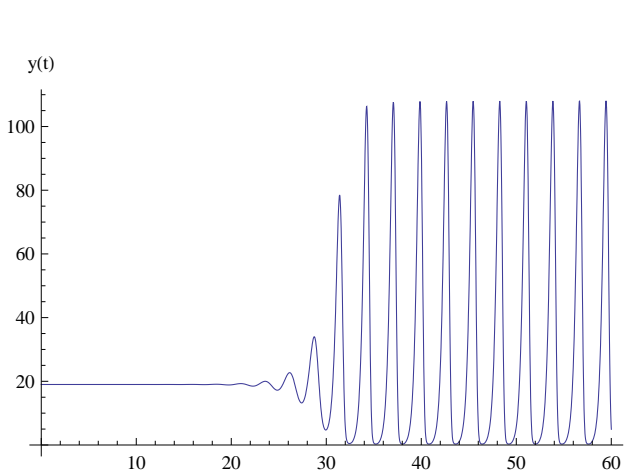


Fig.2(a): $\alpha = 0.97$

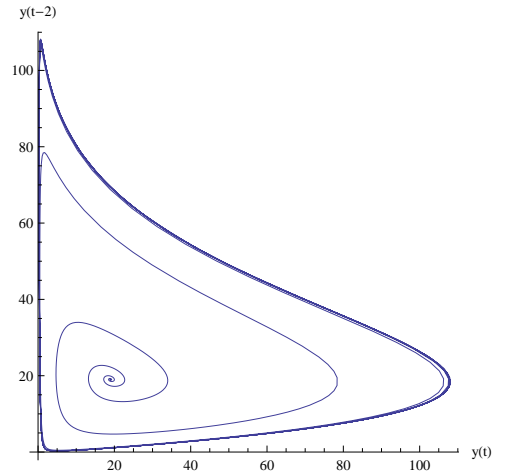
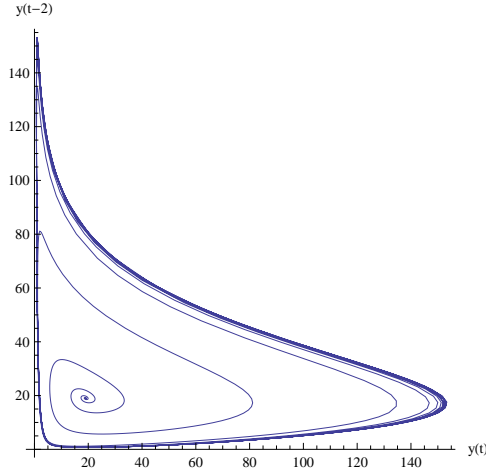
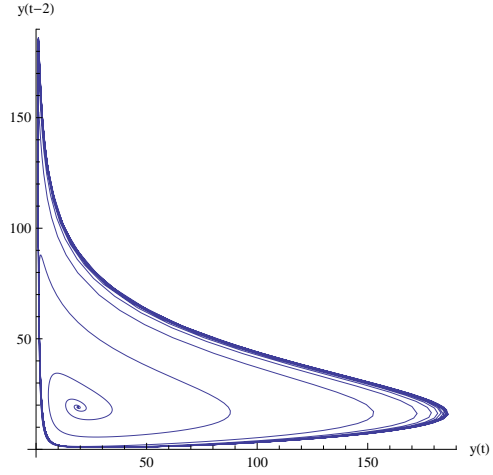
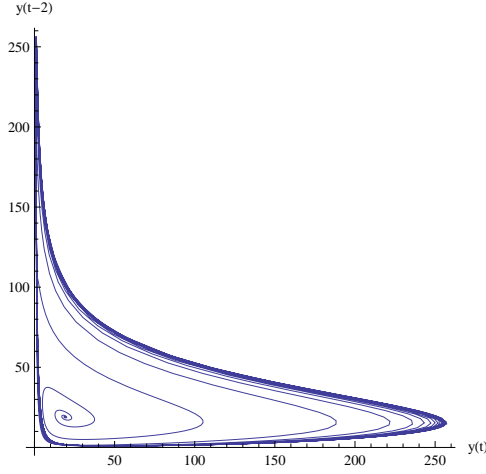
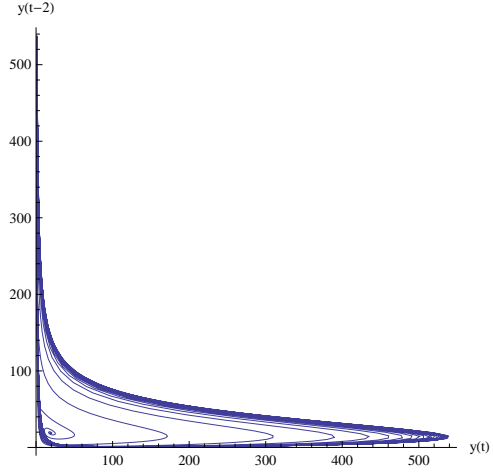


Fig.2(b): $\alpha = 0.97$

Fig.2(c): $\alpha = 0.90$ Fig.2(d): $\alpha = 0.87$ Fig.2(e): $\alpha = 0.83$ Fig.2(f): $\alpha = 0.765$

Example 4.3. Consider fractional version of four dimensional enzyme kinetics with an inhibitor molecule [16]

$$D_t^\alpha y_1(t) = 10.5 - \frac{y_1(t)}{1 + 0.0005 y_4^3(t-4)}, \quad (24)$$

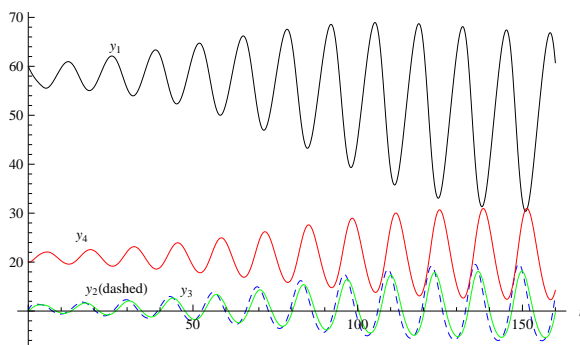
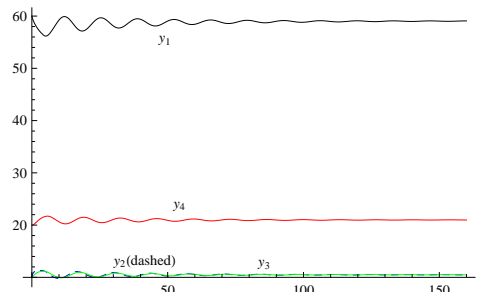
$$D_t^\alpha y_2(t) = \frac{y_1(t)}{1 + 0.0005 y_4^3(t-4)} - y_2(t), \quad (25)$$

$$D_t^\alpha y_3(t) = y_2(t) - y_3(t), \quad (26)$$

$$D_t^\alpha y_4(t) = y_3(t) - 0.5 y_4(t), \quad (27)$$

$$y(t) = [60, 10, 10, 20]^T, \quad t \leq 0. \quad (28)$$

For $0.88 < \alpha \leq 1$, the height of oscillations of $y_i(t)$ ($1 \leq i \leq 4$) increases as t increases. For $\alpha < 0.88$ the system settles down for sufficiently large t . Figs. 3(a), 3(b) show the solutions $y_i(t)$ for $\alpha = 0.95$ and $\alpha = 0.83$ respectively.

Fig.3(a): $\alpha = 0.95$ Fig.3(b): $\alpha = 0.83$

5. CONCLUSIONS

Adams-Bashforth-Moulton method is extended to solve fractional differential equations involving delay. Some interesting fractional delay differential equations arising in Biology have been solved. It is observed that even one dimensional delayed systems of fractional order show chaotic behaviour, and below some critical order, the system changes its nature and becomes periodic. In some cases it is observed that the phase portrait gets stretched as the order of the derivative is reduced.

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