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CERTAIN CLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH CHEBYSHEV POLYNOMIAL AND Q-DIFFERENCE OPERATOR

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Abstract. Using Chebyshev polynomials and q-differential operator, we introduce a novel class of bi-univalent functions within the open unit disk. Initial coefficient bounds for this class are established, emphasizing their significance in complex analysis. Analyticity ensures the representation of these functions through convergent power series, offering a robust tool for comprehending their behavior. The condition of univalence guarantees the one-to-one nature of these functions, preventing multiple mappings of the same point. Researchers employ bi-univalent functions to delve into diverse aspects of complex analysis, unraveling their properties and applications across various mathematical contexts. The exploration encompasses the investigation of geometric properties, including Fekete-Szegö inequality and coefficient bounds, unraveling the intricate interplay between analytic and geometric characteristics. In summary, the study of bi-univalent functions contributes to the depth of complex analysis, providing a nuanced understanding of the relationships between analyticity and univalence. This exploration lays the foundation for advancements in mathematical theory and applications.

1. Introduction

Let $\mathcal{A}$ be the class of functions:

$$E(\varkappa) = \varkappa + \sum_{k=2}^{\infty} a_k \varkappa^k, \tag{1}$$

which are analytic and univalent in $\mathbb{U} = \{\varkappa \in \mathbb{C} : |\varkappa| < 1\}$.

Every univalent function E has an inverse E^{-1} satisfies (Duren [19])

$$E^{-1}(E(\varkappa)) = \varkappa \quad (\varkappa = \mathbb{U}), \quad and \quad E\left(E^{-1}(\omega)\right) = \omega \quad |\omega| < r_0(f), r(f) \geq \frac{1}{4}, \tag{2}$$

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It is easy to see that the inverse function has the form

$$E^{-1}(\omega) = G(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \ldots \quad . \tag{3}$$

A function $E \in \mathcal{A}$ is called bi-univalent in $\mathbb{U}$ if both E and $E^{-1}$are univalent in $\mathbb{U}$. For more study see([8], [9], [25] and [28]).

Denote the class of such functions by $\sum$.

For examples of classes of bi-univalent functions see [2] and [26].

Mason [24] and Doha [18] defined the first and second kinds of Chebyshev polynomials by (see also [3], [20])

$$T_k(t) = \cos k\theta \text{ and } U_k(t) = \frac{\sin(k+1)\theta}{\sin\theta},$$

where k denotes the polynomial degree and $t = cos\theta$. The Chebyshev polynomials of the first and second kinds are orthogonal for $t \in [-1,1]$.

The function

$$\begin{aligned} H(\varkappa,t) &= \frac{1}{1-2t\varkappa+\varkappa^2} \qquad z \in \mathbb{U}. \\ &= 1 + \sum_{k=2}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha}\varkappa^k = 1 + 2\cos\alpha\varkappa + (3\cos^2\alpha - \sin^2\alpha)\varkappa^2 + \ldots, \end{aligned} \tag{4}$$

if $t = cos\alpha$, $\alpha \in (\frac{-\pi}{3}, \frac{\pi}{3})$.

Whittaker and Watson [19] also wrote the function $H(\varkappa,t)$ in the form

$$H(\varkappa,t) = 1 + U_1(t)\varkappa + U_2(t)\varkappa^2 + \ldots(\varkappa \in \mathbb{U}, t \in (-1,1)), \tag{5}$$

where $U_{k-1}(t) = \frac{\sin(k \arccos t)}{\sqrt{1-t^2}}$, for $k \in \mathbb{N} = \{1,2,\ldots\}$, are the second kind the Chebyshev inequality. They obtained that relationships

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t), \tag{6}$$

and

$$U_1(t) = 2t,\ U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t,\ U_4(t) = 16t^4 - 12t^2 + 1, \ldots \quad . \tag{7}$$

For $0 < q < 1$ the Jackson's q - derivative of $E(\varkappa) \in \mathcal{A}$ is, given by [22] (see also [4], [10], [11], [12], [13], [14])

$$D_qE(\varkappa) = \begin{cases} \frac{E(\varkappa)-E(q\varkappa)}{(1-q)\varkappa} & , \quad \varkappa \neq 0 \\ E'(\varkappa) & , \quad \varkappa = 0 \end{cases}. \tag{8}$$

From (8) ,we have

$$D_qE(\varkappa) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \varkappa^{k-1}, \tag{9}$$

where

$$[k]_q = \frac{1-q^k}{1-q}, \tag{10}$$

as $q \to 1^-$ then $[k]_q \to k$, hence we have

$$\lim_{q\to 1^-} D_qE(\varkappa) = E'(\varkappa). \tag{11}$$

For $E\in\mathcal{A}$, Aouf and Madian [5], [[6], with $p=1$], [23] and [7] defined the q-differential Câtaş operator by:

$$I_q^n(\lambda,\ell)E(\varkappa)=\varkappa+\sum_{k=2}^{\infty}\Psi_q^n(k,\lambda,\ell)a_k\varkappa^k, \tag{12}$$

where

$$\Psi_q^n(k,\lambda,\ell)=\left[\frac{[1+\ell]_q+\lambda\left([k+\ell]_q-[1+\ell]_q\right)}{[1+\ell]_q}\right]^n \tag{13}$$

$$\left(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \lambda,\ell\geq 0,\ 0<q<1\right),$$

and satisfies

$$I_q^n(\lambda,\ell)E(\varkappa)=(1-\lambda)I_n^{n-1}(\lambda,\ell)E(\varkappa)+\frac{\lambda}{[\ell+1]_q\varkappa^{\ell-1}}D_qz^\ell I_q^{n-1}(\lambda,\ell)E(\varkappa).$$

Note that $I_q^n(\lambda,\ell)E(\varkappa)$ generalizes many operators such as the following:

(i) $I_q^n(\lambda,0)E(\varkappa)=D_{q,\lambda}^nE(\varkappa)$ $($*see* Aouf et al.[15];

(ii) $I_q^n(1,0)E(\varkappa)=I_q^nE(\varkappa)$, see ([21] and [14]);

(iii) $lim_{q\to 1^-}I_q^n(\lambda,\ell)E(\varkappa)=I^n(\lambda,\ell)E(\varkappa)$ $($*see Câtas et al.* [17]$)$;

(iv) $lim_{q\to 1^-}I_q^n(\lambda,0)E(\varkappa)=D_\lambda^nE(\varkappa)$ $($*see* AI-Aboudi [1]$)$;

(v) $lim_{q\to 1^-}I_q^n(1,0)E(\varkappa)=I^nE(\varkappa)$ (see Sălăgean [27]).

Using the operator $T_{\gamma,t}^{q,n,\lambda,\ell}$ and Chebyshev polynomials, we define the following class:

Definition 1.1 For $0\leq\gamma\leq 1$, a function $E\in\sum$, is said to be in the class $T_{\gamma,t}^{q,n,\lambda,\ell}$ if the following subordination hold:

$$(1-\gamma)\frac{\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}{I_q^n(\lambda,\ell)E(\varkappa)}+\gamma\frac{D_q\left(\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}\prec H(\varkappa,t), \tag{14}$$

and

$$(1-\gamma)\frac{\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}{I_q^n(\lambda,\ell)G(\omega)}+\gamma\frac{D_q\left(\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}\prec H(\omega,t), \tag{15}$$

where $\varkappa$, ω $\in\mathbb{U}$ and G is given by (3).

Note that

(i) If $\gamma=0$, then $T_{0,t}^{q,n,\lambda,\ell}=T_t^{q,n,\lambda,\ell}$: $E\in T_t^{q,n,\lambda,\ell}$ satisfying

$$\frac{\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}{I_q^n(\lambda,\ell)E(\varkappa)}\prec H(\varkappa,t)\quad\text{and}\quad\frac{\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}{I_q^n(\lambda,\ell)G(\omega)}\prec H(\omega,t),$$

(ii) If $\gamma=1$, then $T_{1,t}^{q,n,\lambda,\ell}=I_t^{q,n,\lambda,\ell}$: $E\in I_t^{q,n,\lambda,\ell}$satisfying

$$\frac{D_q\left(zD_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}\prec H(\varkappa,t)\quad\text{and}\quad\frac{D_q\left(\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}\prec H(\omega,t),$$

(iii) If $n=0$ then $T_{\gamma,t}^{q,0,\lambda,\ell}=T_{\gamma,t}^{q,}$:

$$(1-\gamma)\frac{\varkappa D_qE(\varkappa)}{E(\varkappa)}+\gamma\frac{D_q\left(\varkappa D_qE(\varkappa)\right)}{D_qE(\varkappa)}\prec H(\varkappa,t), \tag{16}$$

and

$$(1-\gamma)\frac{\omega D_q G(\omega)}{G(\omega)}+\gamma\frac{D_q(\omega D_q G(\omega))}{D_q G(\omega)}\prec H(\omega,t). \tag{17}$$

2. Main Results

Unless indicated, assume that $n\in\mathbb{N}_0,\ \lambda,\ell\geq 0,\ 0<q<1,\ t\in(-1,1)$ and $E(\varkappa)\in\sum$.

Theorem 2.1 Let $E(\varkappa)\in T_{\gamma,t}^{q,n,\lambda,\ell}$. Then

$$|a_2|\leq\frac{4t\sqrt{t}}{\sqrt{2\left[\begin{array}{c}-q\left(1-\gamma+\gamma[2]_q^2\right)\left(\Psi_q^n(2,\lambda,\ell)\right)^2\\+\left([3]_q-1\right)\left(1-\gamma+\gamma[3]_q\right)\Psi_q^n(3,\lambda,\ell)\end{array}\right]4t^2\\-\left(4t^2-1\right)q^2(1+\gamma q)^2\left(\Psi_q^n(2,\lambda,\ell)\right)^2}}, \tag{18}$$

and

$$|a_3|\leq\left(\frac{2t}{q(1+\gamma q)\left(\Psi_q^n(2,\lambda,\ell)\right)}\right)^2+\frac{2t}{\left([3]_q-1\right)\left(1-\gamma+[3]_q\gamma\right)\Psi_q^n(3,\lambda,\ell)}.$$

Proof. Let $E(\varkappa)\in T_{\gamma,t}^{q,n,\lambda,\ell}$. From (14) and (15), we have

$$(1-\gamma)\frac{\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}{I_q^n(\lambda,\ell)E(\varkappa)}+\gamma\frac{D_q\left(\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}=H(p(\varkappa),t), \tag{19}$$

and

$$(1-\gamma)\frac{\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}{I_q^n(\lambda,\ell)G(\omega)}+\gamma\frac{D_q\left(\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}=H(q(\omega),t), \tag{20}$$

for

$$p(\varkappa)=c_1\varkappa+c_2\varkappa^2+...(\varkappa\in\mathbb{U}), \tag{21}$$

$$q(\omega)=d_1\omega+d_2\omega^2+...(\omega\in\mathbb{U}), \tag{22}$$

such that $|p(\varkappa)|<1\ (z\in\mathbb{U})$ and $|q(\omega)|<1\ (\omega\in\mathbb{U})$, hence

$$|c_j|\leq 1 \text{ and } |d_j|\leq 1 \text{ for all } \quad j\in\mathbb{N}. \tag{23}$$

From (19), (20), (21), and (22), we have

$$\begin{aligned}&(1-\gamma)\frac{\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}{I_q^n(\lambda,\ell)E(\varkappa)}+\gamma\frac{D_q\left(\varkappa D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)E(\varkappa)\right)}\\&=1+U_1(t)c_1\varkappa+\left(U_1(t)c_2+U_2(t)c_1^2\right)\varkappa^2+...\end{aligned}$$

$$\begin{aligned}&\Rightarrow(1-\gamma)\frac{1+\sum_{k=2}^{\infty}[k]_q\Psi_q^n(k,\lambda,\ell)a_k\varkappa^{k-1}}{1+\sum_{k=2}^{\infty}\Psi_q^n(k,\lambda,\ell)a_k\varkappa^{k-1}}+\gamma\frac{1+\sum_{k=2}^{\infty}[k]_q^2\Psi_q^n(k,\lambda,\ell)a_k\varkappa^{k-1}}{1+\sum_{k=2}^{\infty}[k]_q\Psi_q^n(k,\lambda,\ell)a_k\varkappa^{k-1}}\\&=1+U_1(t)c_1\varkappa+\left(U_1(t)c_2+U_2(t)c_1^2\right)\varkappa^2+...,\end{aligned} \tag{24}$$

and

$$\begin{aligned}&(1-\gamma)\frac{\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}{I_q^n(\lambda,\ell)G(\omega)}+\gamma\frac{D_q\left(\omega D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)\right)}{D_q\left(I_q^n(\lambda,\ell)G(\omega)\right)}\\&=1+U_1(t)d_1\omega+\left(U_1(t)d_2+U_2(t)d_1{}^2\right)\omega^2+...\ .\end{aligned}$$

$$\begin{aligned}&\Rightarrow(1-\gamma)\frac{1-[2]_q\Psi_q^n(2,\lambda,\ell)a_2\omega+[3]_q\left(2a_2^2-a_3\right)\Psi_q^n(3,\lambda,\ell)\omega^2+...}{1-\Psi_q^n(2,\lambda,\ell)a_2\omega+\left(2a_2^2-a_3\right)\Psi_q^n(3,\lambda,\ell)\omega^2+...}\\&+\gamma\frac{1-[2]_q^2\Psi_q^n(2,\lambda,\ell)a_2\omega+[3]_q^2\left(2a_2^2-a_3\right)\Psi_q^n(3,\lambda,\ell)\omega^2+...}{1-[2]_q\Psi_q^n(2,\lambda,\ell)a_2\omega+[3]_q\left(2a_2^2-a_3\right)\Psi_q^n(3,\lambda,\ell)\omega^2+...}\\&=1+U_1(t)d_1\omega+\left(U_1(t)d_2+U_2(t)d_1{}^2\right)\omega^2+...\ .\end{aligned}\tag{25}$$

By equating the coefficients in (24)and (8) , we have

$$q(1+\gamma q)\Psi_q^n(2,\lambda,\ell)a_2=U_1(t)c_1,\tag{26}$$

$$\begin{aligned}&-q\left(1-\gamma+\gamma[2]_q^2\right)\left(\Psi_q^n(2,\lambda,\ell)\right)^2a_2^2+\left([3]_q-1\right)\left(1-\gamma+[3]_q\gamma\right)\Psi_q^n(3,\lambda,\ell)a_3\\&=\left(U_1(t)c_2+U_2(t)c_1^2\right)\end{aligned}\tag{27}$$

and

$$-q(1+\gamma q)\Psi_q^n(2,\lambda,\ell)a_2=U_1(t)d_1,\tag{28}$$

$$\begin{aligned}&-q\left(1-\gamma+\gamma[2]_q^2\right)\left(\Psi_q^n(2,\lambda,\ell)\right)^2a_2^2\\&+2\left([3]_q-1\right)\left(1-\gamma+\gamma[3]_q\right)\Psi_q^n(3,\lambda,\ell)a_2^2\\&-\left([3]_q-1\right)\left(1-\gamma+[3]_q\gamma\right)\Psi_q^n(3,\lambda,\ell)a_3\\&=\left(U_1(t)d_2+U_2(t)d_1{}^2\right).\end{aligned}\tag{29}$$

From (26) and (28) we have

$$c_1=-d_1,\tag{30}$$

and

$$2q^2(1+\gamma q)^2\left(\Psi_q^n(2,\lambda,\ell)\right)^2a_2^2=U_1^2(t)\left(c_1^2+d_1{}^2\right).\tag{31}$$

Also from (27)and (29) , we have

$$\begin{aligned}&-2q\left(1-\gamma+\gamma[2]_q^2\right)\left(\Psi_q^n(2,\lambda,\ell)\right)^2a_2^2+2\left([3]_q-1\right)\left(1-\gamma+\gamma[3]_q\right)a_2^2\Psi_q^n(3,\lambda,\ell)\\&=U_1(t)(c_2+d_2)+U_2(t)\left(c_1^2+d_1{}^2\right).\end{aligned}\tag{32}$$

By using (31) ,we get

$$\begin{aligned}&\begin{bmatrix}-2q\left(1-\gamma+\gamma[2]_q^2\right)\left(\Psi_q^n(2,\lambda,\ell)\right)^2a_2^2\\+2\left([3]_q-1\right)\left(1-\gamma+\gamma[3]_q\right)\Psi_q^n(3,\lambda,\ell)a_2^2\end{bmatrix}U_1^2(t)-\\&-2U_2(t)q^2(1+\gamma q)^2\left(\Psi_q^n(2,\lambda,\ell)\right)^2a_2^2\\&=U_1^3(t)(c_2+d_2),\end{aligned}\tag{33}$$

that is

$$a_2^2=\frac{U_1^3(t)\left(c_2+d_2\right)}{2\left[\begin{array}{c}-q\left(1-\gamma+\gamma\left[2\right]_q^2\right)\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2\\+\left(\left[3\right]_q-1\right)\left(1-\gamma+\gamma\left[3\right]_q\right)\Psi_q^n\left(3,\lambda,\ell\right)\end{array}\right]U_1^2\left(t\right)-\atop -U_2\left(t\right)q^2\left(1+\gamma q\right)^2\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2.} \tag{34}$$

Now, substitute (7) in (34) and using (31) , we have

$$|a_2|\leq\frac{4t\sqrt{t}}{\sqrt{2\left[\begin{array}{c}-q\left(1-\gamma+\gamma\left[2\right]_q^2\right)\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2\\+\left(\left[3\right]_q-1\right)\left(1-\gamma+\gamma\left[3\right]_q\right)\Psi_q^n\left(3,\lambda,\ell\right)\end{array}\right]4t^2\atop -\left(4t^2-1\right)q^2\left(1+\gamma q\right)^2\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2}}. \tag{35}$$

By subtracting (27) , from (29)and using (30) and (31) , we get

$$a_3=\frac{U_1^2(t)\left(c_1^2+{d_1}^2\right)}{2q^2\left(1+\gamma q\right)^2\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2}+\frac{U_1\left(t\right)\left(c_2-d_2\right)}{2\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}. \tag{36}$$

Applying (23)once again to the coefficients c_1, c_2, d_1, and d_2, then

$$|a_3|\leq\left(\frac{2t}{q\left(1+\gamma q\right)\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)}\right)^2+\frac{2t}{\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}. \tag{37}$$

Theorem 2.2. Let $E\left(\varkappa\right)\in T_{\gamma,t}^{q,n,\lambda,\ell}$ and $\tau\in\mathbb{C}$, then

$$\left|a_3-\tau a_2^2\right|\leq\begin{cases}\frac{2t}{\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}, 0\ \leq\left|\Delta\left(\tau\right)\right|\ \leq\ \frac{1}{2\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}\\ 4t\ \left|\Delta\left(\tau\right)\right|, \qquad \left|\Delta\left(\tau\right)\right|\ \geq\ \frac{1}{2\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}.\end{cases} \tag{38}$$

Proof. From (36)we have $a_3=\frac{U_1(t)(c_2-d_2)}{2\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}+a_2^2$.

Using (34) , we have

$$a_3-\tau a_2^2=\frac{U_1\left(c_2-d_2\right)}{2\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}+\left(1-\tau\right)a_2^2,$$

$$a_3-\tau a_2^2=\frac{U_1\left(c_2-d_2\right)}{2\left(\left[3\right]_q-1\right)\left(1-\gamma+\left[3\right]_q\gamma\right)\Psi_q^n\left(3,\lambda,\ell\right)}$$

$$+\left(1-\tau\right)\left(\frac{U_1^3\left(c_2+d_2\right)}{2\left[\begin{array}{c}q\left(1-\gamma+\gamma\left[2\right]_q^2\right)\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2\\+\left(\left[3\right]_q-1\right)\left(1-\gamma+\gamma\left[3\right]_q\right)\Psi_q^n\left(3,\lambda,\ell\right)\end{array}\right]U_1^2-\atop -U_2q^2\left(1-\gamma+\gamma\left[2\right]_q\right)^2\left(\Psi_q^n\left(2,\lambda,\ell\right)\right)^2}\right),$$

by simple calculations we get

$$a_3 - \tau a_2^2 = U_1 \left(\Delta(\tau) + \frac{1}{2\left([3]_q - 1\right)\left(1-\gamma+[3]_q\gamma\right)\Psi_q^n(3,\lambda,\ell)} \right) c_2$$
$$+ U_1 \left(\Delta(\tau) - \frac{1}{2\left([3]_q - 1\right)\left(1-\gamma+[3]_q\gamma\right)\Psi_q^n(3,\lambda,\ell)} \right) d_2,$$

where

$$\Delta(\tau) = \frac{(1-\tau)U_1^2}{\begin{array}{c} 2\left[\begin{array}{c} -q\left(1-\gamma+\gamma[2]_q^2\right)\left(\Psi_q^n(2,\lambda,\ell)\right)^2 \\ +\left([3]_q-1\right)\left(1-\gamma+\gamma[3]_q\right)\Psi_q^n(3,\lambda,\ell) \end{array}\right] U_1^2 - \\ -U_2 q^2 (1+\gamma q)^2 \left(\Psi_q^n(2,\lambda,\ell)\right)^2 \end{array}}.$$

So, we conclude (38).

For $\tau = 1$, in Theorem 2.2, we have.

Corollary 2.1. If $E(\varkappa) \in T_{\gamma,t}^{q,n,\lambda,\ell}$, then

$$\left|a_3 - a_2^2\right| \le \frac{2t}{2\left([3]_q - 1\right)\left(1-\gamma+[3]_q\gamma\right)\Psi_q^n(3,\lambda,\ell)}. \tag{39}$$

For $\gamma = 1$, in Theorem 2.2, we have.

Corollary 2.2. If $E(\varkappa) \in T_t^{q,n,\lambda,\ell}$, then

$$\left|a_3 - \tau a_2^2\right| \le \left\{ \begin{cases} \frac{2t}{[3]_q\left([3]_q-1\right)\Psi_q^n(3,\lambda,\ell)}, & 0 \le |\Delta(\tau)| \le \frac{1}{2[3]_q\left([3]_q-1\right)+\Psi_q^n(3,\lambda,\ell)} \\ 4t\ \ |\Delta_1(\tau)|, & |\Delta(\tau)| \ge \frac{1}{2[3]_q\left([3]_q-1\right)\Psi_q^n(3,\lambda,\ell)}, \end{cases} \right. \tag{40}$$

where

$$\Delta_1(\tau) = \frac{(1-\tau)U_1^2}{2\left[\begin{array}{c} -q[2]_q^2\left(\Psi_q^n(2,\lambda,\ell)\right)^2 \\ +[3]_q\left([3]_q-1\right)\Psi_q^n(3,\lambda,\ell) \end{array}\right] U_1^2 - U_2[2]_q^2 q^2\left(\Psi_q^n(2,\lambda,\ell)\right)^2}. \tag{41}$$

Taking $q = 1^-$ in Theorem 2.1, we have

Corollary 2.3. If $E(\varkappa) \in T_{\gamma,t}^{n,\lambda,\ell}$, then

$$|a_2| \le \frac{4t\sqrt{t}}{\sqrt{\begin{array}{c} 2\left[-(1+3\gamma)\left(\Psi^n(2,\lambda,\ell)\right)^2 + 2(1+2\gamma)\Psi^n(3,\lambda,\ell)\right]4t^2 \\ -\left(4t^2-1\right)(1+\gamma)^2\left(\Psi^n(2,\lambda,\ell)\right)^2. \end{array}}} \tag{42}$$

and

$$|a_3| \le \left(\frac{2t}{(1+\gamma)\left(\Psi^n(2,\lambda,\ell)\right)}\right)^2 + \frac{2t}{2(1+2\gamma)\Psi^n(3,\lambda,\ell)}.$$

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