*Journal of Fractional Calculus and Applications*
Vol. 9(2) July 2018, pp. 255-267.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

# ON HADAMARD $k$-FRACTIONAL INTEGRALS

SANA IQBAL, SHAHID MUBEEN AND MUHARREM TOMAR

ABSTRACT. In this paper, we introduced the Hadamard fractional integral in terms of a new parameter $k > 0$. We also proved some properties of this newly defined $k$-fractional integral. Some inequalities involving Hadamard $k$-fractional integral are also be proved.

## 1. INTRODUCTION

In mathematical analysis, the fractional calculus is a very helpful tool to perform differentiation and integration with the real number or complex number powers of the differential or integral operators. This subject has earned the attention of many researchers and mathematicians during last few decades (see [1, 2, 3, 7, 16, 17]). There is a large number of the fractional integral operators discussed in literature but because of their applications in many fields of sciences, the Riemann-Liouville fractional integral operator and Hadamard fractional integral operator have been studied extensively.

The Hadamard fractional integral operator was introduced by Hadamard [6]. It can be defined as follows:
Let $f \in L^1([a,b])$, the left and right sided Hadamard fractional integrals of order $\alpha \geq 0$ and $a > 0$ are defined respectively as

$$H^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}, \quad 0 < a < x \leq b \tag{1}$$

and

$$H^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t}, \quad 0 < a \leq x < b. \tag{2}$$

Because of the wide applications of above defined fractional integrals, many researchers extended their studies to derive more applications, properties and inequalities of Hölder, Minkowski, Hermite-Hadamard, Grüss and Ostrowski type involving left and right sided Hadamard fractional integrals for different types of functions (see [13, 14, 20, 21, 22, 24]).

2010 *Mathematics Subject Classification.* 26A33; 26D10; 26D15.
*Key words and phrases.* $k$-fractional integrals, Hadamard fractional integral, inequalities.
Submitted November 29, 2017.

In this paper, we extend the idea of different types of inequalities involving the Hadamard fractional integral in terms of a new parameter $k > 0$. For this, we introduce the $k$-analogue of Hadamard fractional integral with some properties. The theory of special $k$-functions was introduced about a decade ago when Diaz and Pariguan [5] defined the generalization of the classical gamma and beta functions in terms of a new parameter $k > 0$, called gamma and beta $k$-functions respectively.

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \; Re(\alpha) > 0.$$

and

$$B_k(\alpha, \beta) = \frac{1}{k} \int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1} dt, \quad Re(\alpha) > 0, Re(\beta) > 0. \tag{3}$$

This idea of generalization of special functions in terms of some new parameter fascinated many researchers and mathematicians. Several properties, identities and inequalities involving special $k$-functions were proved during past several years (see for instance [8, 9, 10, 11, 12, 18, 23]).

The functions $\Gamma_k$ defined on $\mathbb{R}^+$ and $B_k(x, y)$ on $(0, 1)$ hold the following four properties:

(1) $\Gamma_k(x + k) = x\,\Gamma_k(x)$;
(2) $\Gamma_k(k) = 1$;
(3) $\Gamma_k(x)$ is logarithmically convex;
(4) $B_k(x, y) = \dfrac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)}$.

For the first time, Mubeen and Habibullah [15] used this special $k$-functions theory in fractional calculus and introduced the $k$-fractional integral of the Riemann-Liouville type as

$$I^\alpha_{a,k} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t - x)^{\frac{\alpha}{k}-1} f(x) dx, \quad t \in [a, b],$$

where $\Gamma_k$ is the Euler gamma $k$-function.

Later, Romero et al. [19] introduced a new fractional operator called $k$-Riemann-Liouville fractional derivative by using gamma $k$-function. They also proved some properties of this newly defined fractional operator and found its relationship with Riemann-Liouville $k$-fractional integral.

In the subsequent section, we introduce a new fractional integration with parameter $k > 0$ which generalizes Hadamard fractional integrals. We also establish properties of semigroup for this integration. Finally we obtain some weighted Grüss type inequalities for new Hadamard $k$-fractional integral operator.

## 2. Main results

**Definition 1** For $k > 0$, let $f \in L^1([a, b])$, the left and right sided $k$-fractional integrals of order $\alpha \geq 0$ and $a > 0$ are defined respectively as

$$\mathcal{H}^\alpha_{a^+,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{\alpha}{k}-1} f(t) \frac{dt}{t}, \quad 0 < a < x \leq b \tag{4}$$

and

$$\mathcal{H}^{\alpha}_{b^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f(t)\frac{dt}{t}, \quad 0 < a \le x < b. \tag{5}$$

Throughout this paper we will use $\mathcal{H}^{\alpha}_{a,k}$ in place of $\mathcal{H}^{\alpha}_{a^+,k}$.

**Theorem 1** For $k > 0$, let $f$ be continuous on $[a, b]$ and $\alpha \ge 0$, $\beta \ge 0$. Then

$$\mathcal{H}^{\alpha}_{a,k}[\mathcal{H}^{\beta}_{a,k}f(x)] = \mathcal{H}^{\beta}_{a,k}[\mathcal{H}^{\alpha}_{a,k}f(x)] = \mathcal{H}^{\alpha+\beta}_{a,k}f(x). \tag{6}$$

*Proof.* By using relation (4) and Dirichlet's formula, we obtain

$$\begin{aligned}
\mathcal{H}^{\alpha}_{a,k}[\mathcal{H}^{\beta}_{a,k}f(x)] &= \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\mathcal{H}^{\beta}_{a,k}f(t)\frac{dt}{t}\\
&= \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\left(\frac{1}{k\Gamma_k(\beta)}\int_a^t \left(\ln\frac{t}{\xi}\right)^{\frac{\beta}{k}-1} f(\xi)\frac{d\xi}{\xi}\right)\frac{dt}{t}\\
&= \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x \frac{1}{\xi}f(\xi)\left[\int_\xi^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\xi}\right)^{\frac{\beta}{k}-1}\frac{dt}{t}\right]d\xi.
\end{aligned} \tag{7}$$

Now by using the change of variables $z = \ln\frac{t}{\xi}/\ln\frac{x}{\xi}$, we get

$$\begin{aligned}
\int_\xi^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\xi}\right)^{\frac{\beta}{k}-1}\frac{dt}{t} &= \left(\ln\frac{x}{\xi}\right)^{\frac{\alpha+\beta}{k}-1}\int_0^1 (1-z)^{\frac{\alpha}{k}-1}z^{\frac{\beta}{k}-1}dz\\
&= k\left(\ln\frac{x}{\xi}\right)^{\frac{\alpha+\beta}{k}-1}B_k(\alpha,\beta).
\end{aligned} \tag{8}$$

By using (7) and (8) and relation (3) of beta $k$-function, we get

$$\begin{aligned}
\mathcal{H}^{\alpha}_{a,k}[\mathcal{H}^{\beta}_{a,k}f(x)] &= \frac{1}{k\Gamma_k(\alpha+\beta)}\int_a^x \left(\ln\frac{x}{\xi}\right)^{\frac{\alpha+\beta}{k}-1} f(\xi)\frac{d\xi}{\xi}\\
&= \mathcal{H}^{\alpha+\beta}_{a,k}f(x).
\end{aligned} \tag{9}$$

This completes the proof. □

**Theorem 2** Let $\alpha, \beta > 0$ and $k > 0$. Then following identity holds

$$\mathcal{H}^{\alpha}_{a,k}\left[\left(\ln\frac{x}{a}\right)^{\frac{\beta}{k}-1}\right] = \left(\ln\frac{x}{a}\right)^{\frac{\alpha+\beta}{k}-1}\frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} \tag{10}$$

*Proof.* By using (4) and change of variable $u = \dfrac{\ln(x/t)}{\ln(x/a)}$, $x \in (a, b]$,

$$\begin{aligned}
\mathcal{H}^{\alpha}_{a,k}\left[\left(\ln\frac{x}{a}\right)^{\frac{\beta}{k}-1}\right] &= \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{a}\right)^{\frac{\beta}{k}-1}\frac{dt}{t}\\
&= \left(\ln\frac{x}{a}\right)^{\frac{\alpha+\beta}{k}-1}\frac{1}{k\Gamma_k(\alpha)}\int_0^1 (1-u)^{\frac{\alpha}{k}-1}u^{\frac{\beta}{k}-1}du\\
&= \left(\ln\frac{x}{a}\right)^{\frac{\alpha+\beta}{k}-1}\frac{B_k(\alpha,\beta)}{\Gamma_k(\alpha)}
\end{aligned} \tag{11}$$

This completes the proof. □

**Corollary 1** Using definition of Hadamard $k$-fractional integral and relation (1), we can easily get

$$\mathcal{H}^{\alpha}_{a,k}(1) = \frac{(\ln(x/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}, \quad \alpha, k > 0\,. \tag{12}$$

The functions $f,\, g \subseteq \mathbb{R} \to \mathbb{R}$ are synchronous (asynchronous) on $[a,b]$, $a, b \in \mathbb{R}$ if

$$(f(x)-f(y))(g(x)-g(y)) \;\geq\; (\leq) 0 \;\; for\ all\, x,\ y\ \in [a,b].$$

**Theorem 3** Let $f$ and $g$ be two synchronous functions on $[a,b]$, then for all $\alpha, \beta > 0$ and $x > a$ the following inequalities for Hadamard $k$-fractional integral holds:

$$\mathcal{H}^{\alpha}_{a,k}[f(x)g(x)] \geq \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}\mathcal{H}^{\alpha}_{a,k}f(x)\mathcal{H}^{\alpha}_{a,k}g(x). \tag{13}$$

and

$$\begin{aligned} &\mathcal{H}^{\alpha}_{a,k}[f(x)g(x)]\frac{(\ln(x/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)} + \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}\mathcal{H}^{\beta}_{a,k}[f(x)g(x)] \\ \geq\;\; &\mathcal{H}^{\alpha}_{a,k}f(x)\mathcal{H}^{\beta}_{a,k}g(x) + \mathcal{H}^{\alpha}_{a,k}g(x)\mathcal{H}^{\beta}_{a,k}f(x). \end{aligned} \tag{14}$$

*Proof.* Since $f$ and $g$ are synchronous on $[a,b]$, therefore for all $\lambda, \eta \in [a,b]$, we have

$$[f(\eta)-f(\lambda)][g(\eta)-g(\lambda)] \geq 0, \tag{15}$$

that is

$$f(\eta)g(\eta) + f(\lambda)g(\lambda) \geq f(\lambda)g(\eta) + f(\eta)g(\lambda). \tag{16}$$

Multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\frac{1}{\eta}$ and integrating the resultant inequality with respect to $\eta$ over $(a,x)$, we get

$$\mathcal{H}^{\alpha}_{a,k}[f(x)g(x)] + \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}f(\lambda)g(\lambda) \geq g(\lambda)\mathcal{H}^{\alpha}_{a,k}f(x) + f(\lambda)\mathcal{H}^{\alpha}_{a,k}g(x). \tag{17}$$

Now multiplying the above inequality by $\frac{1}{k\Gamma_k(\alpha)}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{1}{\lambda}$ and integrating with respect to $\lambda$ over $(a,x)$, we obtain inequality (13).

To prove inequality (14), it is sufficient to multiply both sides of inequality (17) with $\frac{1}{k\Gamma_k(\beta)}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{1}{\lambda}$ and integrate the resultant inequality with respect to $\lambda$ over $(a,x)$. □

**Theorem 4** Let $f$ and $g$ be two synchronous functions on $[a,b]$ and $h \geq 0$, then for all $x > a$, $\alpha, \beta > 0$, the following inequality for Hadamard $k$-fractional integral holds:

$$\begin{aligned} &\mathcal{H}^{\alpha}_{a,k}(fgh(x))\frac{(\ln(x/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)} + \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}\mathcal{H}^{\beta}_{a,k}(fgh(x)) \\ \geq\;\; &\mathcal{H}^{\alpha}_{a,k}(fh(x))\mathcal{H}^{\beta}_{a,k}(g(x)) + \mathcal{H}^{\alpha}_{a,k}(gh(x))\mathcal{H}^{\beta}_{a,k}(f(x)) - \mathcal{H}^{\alpha}_{a,k}(h(x))\mathcal{H}^{\beta}_{a,k}(fg(x)) \\ &-\mathcal{H}^{\alpha}_{a,k}(fg(x))\mathcal{H}^{\beta}_{a,k}(h(x)) + \mathcal{H}^{\alpha}_{a,k}(f(x))\mathcal{H}^{\beta}_{a,k}(gh(x)) + \mathcal{H}^{\alpha}_{a,k}(g(x))\mathcal{H}^{\beta}_{a,k}(fh(x)) \end{aligned} \tag{18}$$

*Proof.* Since $f$ and $g$ are synchronous on $[a,b]$ and $h \geq 0$, therefore for all $\lambda, \eta \in [a,b]$, we have

$$[f(\eta) - f(\lambda)][g(\eta) - g(\lambda)][h(\eta) + h(\lambda)] \geq 0.$$

This implies

$$\begin{aligned} & f(\eta)g(\eta)h(\eta) + f(\lambda)g(\lambda)h(\lambda) \geq f(\lambda)g(\eta)h(\lambda) + f(\eta)g(\lambda)h(\lambda) - f(\eta)g(\eta)h(\lambda) \\ - \ & f(\lambda)g(\lambda)h(\eta) + f(\lambda)g(\eta)h(\eta) + f(\eta)g(\lambda)h(\eta) \end{aligned} \quad (19)$$

Multiplying both sides of the above inequality by $\frac{1}{k\Gamma_k(\alpha)}\left(\ln \frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\frac{1}{\eta}$ and integrating the resultant inequality with respect to $\eta$ over $(a,x)$, we get

$$\begin{aligned} & \mathcal{H}^{\alpha}_{a,k}(fgh(x)) + \frac{(\ln(x/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}(f(\lambda)g(\lambda)h(\lambda)) \\ \geq \ & \mathcal{H}^{\alpha}_{a,k}(fh(x))g(\lambda) + \mathcal{H}^{\alpha}_{a,k}(gh(x))f(\lambda) - \mathcal{H}^{\alpha}_{a,k}(h(x))f(\lambda)g(\lambda) \\ & -\mathcal{H}^{\alpha}_{a,k}(fg(x))h(\lambda) + \mathcal{H}^{\alpha}_{a,k}(f(x))g(\lambda)h(\lambda) + \mathcal{H}^{\alpha}_{a,k}(g(x))f(\lambda)h(\lambda). \end{aligned} \quad (20)$$

Now by multiplying both sides of above inequality with $\frac{1}{k\Gamma_k(\beta)}\left(\ln \frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{1}{\lambda}$ and integrate the resultant inequality with respect to $\lambda$ over $(a,x)$, we get the required inequality. □

**Theorem 5** Let $f$, $g$ and $h$ be three monotone functions defined on $[a,b]$ satisfying the inequality

$$[f(\eta) - f(\lambda)][g(\eta) - g(\lambda)][h(\eta) - h(\lambda)] \geq 0. \quad (21)$$

Then for all $\lambda, \eta \in [a,x]$, $x > a$ and $\alpha, \beta > 0$, the following inequality for Hadamard $k$-fractional integral holds:

$$\begin{aligned} & \mathcal{H}^{\alpha}_{a,k}(fgh(x))\frac{(\ln(x/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)} - \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}\mathcal{H}^{\beta}_{a,k}(fgh(x)) \\ \geq \ & \mathcal{H}^{\alpha}_{a,k}(fh(x))\mathcal{H}^{\beta}_{a,k}(g(x)) + \mathcal{H}^{\alpha}_{a,k}(gh(x))\mathcal{H}^{\beta}_{a,k}(f(x)) - \mathcal{H}^{\alpha}_{a,k}(h(x))\mathcal{H}^{\beta}_{a,k}(fg(x)) \\ & +\mathcal{H}^{\alpha}_{a,k}(fg(x))\mathcal{H}^{\beta}_{a,k}(h(x)) - \mathcal{H}^{\alpha}_{a,k}(f(x))\mathcal{H}^{\beta}_{a,k}(gh(x)) - \mathcal{H}^{\alpha}_{a,k}(g(x))\mathcal{H}^{\beta}_{a,k}(fh(x)) \end{aligned} \quad (22)$$

*Proof.* The proof of this theorem is similar to the proof of previous theorem. □

**Theorem 6** Let $f$ and $g$ be two functions on $[a,b]$, then for all $\alpha, \beta > 0$ and $x > a$ the following inequalities for Hadamard $k$-fractional integral holds:

(1) $\mathcal{H}^{\alpha}_{a,k}(f^2(x))\mathcal{H}^{\beta}_{a,k}(1) + \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}\mathcal{H}^{\beta}_{a,k}(g^2(x)) \geq 2\mathcal{H}^{\alpha}_{a,k}(f(x))\mathcal{H}^{\beta}_{a,k}(g(x))$

(2) $\mathcal{H}^{\alpha}_{a,k}(f^2(x))\mathcal{H}^{\beta}_{a,k}(1) + \frac{\Gamma_k(\alpha+k)}{(\ln(x/a))^{\frac{\alpha}{k}}}H^{\beta}_{a,k}(g^2(x)) \geq 2\mathcal{H}^{\alpha}_{a,k}(f(x))\mathcal{H}^{\beta}_{a,k}(g(x))$

*Proof.* (1) Since $[f(\eta) - g(\lambda)]^2 \geq 0$, then we have

$$f^2(\eta) + g^2(\lambda) \geq 2f(\eta)g(\lambda).$$

Multiplying both sides of above inequality with $\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\ln \frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln \frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{1}{\eta\lambda}$ and integrate the resultant inequality with respect to $\eta$ and $\lambda$ over $(a,x)$, we get the required inequality.

(2) Use the inequality $[f(\eta)g(\lambda) - f(\lambda)g(\eta)]^2 \geq 0$.

□

To obtain weighted Grüss type inequality with one parameter involving Hadamard $k$-fractional integral, we need following Lemma.

**Lemma 1** Let $w$ be an integrable function on $[a,b]$ with $\varphi < w(x) < \Phi$ and let $p$ be a positive function on $[a,b]$. Then for all $x, \alpha > 0$ we have

$$\begin{aligned} & \left[\mathcal{H}^{\alpha}_{a,k}p(x)\right]\left[\mathcal{H}^{\alpha}_{a,k}(pw^2(x))\right] - \left[\mathcal{H}^{\alpha}_{a,k}(pw(x))\right]^2 \\ = & \left[\Phi\mathcal{H}^{\alpha}_{a,k}p(x) - \mathcal{H}^{\alpha}_{a,k}(pw(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pw(x)) - \varphi\mathcal{H}^{\alpha}_{a,k}p(x)\right] \\ & -\mathcal{H}^{\alpha}_{a,k}p(x)\left[\mathcal{H}^{\alpha}_{a,k}\{(\Phi - w(x))(w(x)-\varphi)p(x)\}\right]. \end{aligned} \tag{23}$$

*Proof.* We use the following equality (see [4])

$$\begin{aligned} & [\Phi p(\lambda) - w(\lambda)p(\lambda)][p(\eta)w(\eta) - \varphi p(\eta)] + [\Phi p(\eta) - w(\eta)p(\eta)][p(\lambda)w(\lambda) - \varphi p(\lambda)] \\ & -p(\eta)p(\lambda)\left[\Phi - w(\eta)\right]\left[w(\eta)-\varphi\right] - p(\eta)p(\lambda)\left[\Phi - w(\lambda)\right]\left[w(\lambda)-\varphi\right] \\ = & \; p(\lambda)w^2(\eta)p(\eta) + p(\eta)w^2(\lambda)p(\lambda) - 2p(\lambda)w(\lambda)p(\eta)w(\eta). \end{aligned} \tag{24}$$

Multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\frac{1}{\eta}$ and integrate the resultant inequality with respect to $\eta$ over $(a,x)$, we get

$$\begin{aligned} & [\Phi p(\lambda) - w(\lambda)p(\lambda)]\left[\mathcal{H}^{\alpha}_{a,k}(pw(x)) - \varphi\mathcal{H}^{\alpha}_{a,k}p(x)\right] \\ & + \left[\Phi\mathcal{H}^{\alpha}_{a,k}p(x) - \mathcal{H}^{\alpha}_{a,k}(wp(x))\right][p(\lambda)w(\lambda) - \varphi p(\lambda)] \\ & -p(\lambda)\mathcal{H}^{\alpha}_{a,k}\{p(x)\left[\Phi - w(x)\right]\left[w(x)-\varphi\right]\} - p(\lambda)\left[\Phi - w(\lambda)\right]\left[w(\lambda)-\varphi\right]\mathcal{H}^{\alpha}_{a,k}p(x) \\ = & \; p(\lambda)\mathcal{H}^{\alpha}_{a,k}(w^2p(x)) + w^2(\lambda)p(\lambda)\mathcal{H}^{\alpha}_{a,k}p(x) - 2p(\lambda)w(\lambda)\mathcal{H}^{\alpha}_{a,k}(pw(x)). \end{aligned} \tag{25}$$

Now multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{1}{\lambda}$ and integrate the resultant inequality with respect to $\lambda$ over $(a,x)$, we get the required inequality. □

**Theorem 7** Let $f$ and $g$ be two integrable functions on $[a,b]$ with $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ and let $p$ be a positive function on $[a,b]$, then for all $x > a$, $\alpha > 0$, we have

$$\begin{aligned} & \left|\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\alpha}_{a,k}(pfg(x)) - \mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\alpha}_{a,k}(pg(x))\right| \\ \leq & \frac{\left[\mathcal{H}^{\alpha}_{a,k}p(x)\right]^2}{2}(\Phi-\varphi)(\Psi-\psi). \end{aligned} \tag{26}$$

*Proof.* First of all, we define

$$\begin{aligned} F(\eta,\lambda) &= [f(\eta)-f(\lambda)][g(\eta)-g(\lambda)] \\ &= f(\eta)g(\eta) + f(\lambda)g(\lambda) - f(\eta)g(\lambda) - f(\lambda)g(\eta), \; \eta,\lambda \in (a,x), \; a < x < b. \end{aligned} \tag{27}$$

Multiplying both sides of (27) with

$$\frac{1}{k^2\Gamma_k^2(\alpha)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{p(\eta)p(\lambda)}{\eta\lambda}.$$

and integrating the resultant inequality with respect to $\eta$ and $\lambda$ over $(a,x)$, we get

$$\begin{aligned} & \frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{p(\eta)p(\lambda)}{\eta\lambda}F(\eta,\lambda)d\eta d\lambda \\ = & \; 2\left[\mathcal{H}^{\alpha}_{a,k}p(x)\right]\left[\mathcal{H}^{\alpha}_{a,k}(pfg(x))\right] - 2\left[\mathcal{H}^{\alpha}_{a,k}(pf(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pg(x))\right]. \end{aligned} \tag{28}$$

Now applying the Cauchy-Schwartz inequality to the left hand side of the equality (28),
we obtain

$$\begin{aligned}
&\left(\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{p(\eta)p(\lambda)}{\eta\lambda}F(\eta,\lambda)d\eta d\lambda\right)^2\\
\leq\ & \frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{p(\eta)p(\lambda)}{\eta\lambda}[f(\eta)-f(\lambda)]^2d\eta d\lambda\\
&\times\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b(\ln\frac{x}{\eta})^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\alpha}{k}-1}\frac{p(\eta)p(\lambda)}{\eta\lambda}\left[g(\eta)-g(\lambda)\right]^2d\eta d\lambda\\
&=\left\{2\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pf^2(x))\right]-2\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]^2\right\}\\
&\quad\times\left\{2\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pg^2(x))\right]-2\left[\mathcal{H}_{a,k}^{\alpha}(pg(x))\right]^2\right\}. \qquad (29)
\end{aligned}$$

From (28) and (29), we can write the following inequality

$$\begin{aligned}
&\left\{2\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pfg(x))\right]-2\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]\left[\mathcal{H}_{a,k}^{\alpha}(pg(x))\right]\right\}^2\\
\leq\ &\left\{2\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pf^2(x))\right]-2\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]^2\right\}\\
&\times\left\{2\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pg^2(x))\right]-2\left[\mathcal{H}_{a,k}^{\alpha}(pg(x))\right]^2\right\}. \qquad (30)
\end{aligned}$$

If we apply (23) for $w=f$ and then $w=g$, we obtain the inequalities respectively:

$$\begin{aligned}
&\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pf^2(x))\right]-\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]^2\\
=\ &\left[\Phi\mathcal{H}_{a,k}^{\alpha}p(x)-\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))-\varphi H_{a,k}^{\alpha}p(x)\right]\\
&-\mathcal{H}_{a,k}^{\alpha}p(x)\left[\mathcal{H}_{a,k}^{\alpha}\{(\Phi-f(x))(f(x)-\varphi)p(x)\}\right] \qquad (31)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pg^2(x))\right]-\left[\mathcal{H}_{a,k}^{\alpha}(pg(x))\right]^2\\
=\ &\left[\Psi H_{a,k}^{\alpha}p(x)-H_{a,k}^{\alpha}(pg(x))\right]\left[H_{a,k}^{\alpha}(pg(x))-\psi\mathcal{H}_{a,k}^{\alpha}p(x)\right]\\
&-\mathcal{H}_{a,k}^{\alpha}p(x)\left[\mathcal{H}_{a,k}^{\alpha}\{(\Psi-g(x))(g(x)-\psi)p(x)\}\right]. \qquad (32)
\end{aligned}$$

Now since

$$-\mathcal{H}_{a,k}^{\alpha}p(x)\left[\mathcal{H}_{a,k}^{\alpha}\{(\Phi-f(x))(f(x)-\varphi)p(x)\}\right]\leq 0$$

and

$$-\mathcal{H}_{a,k}^{\alpha}p(x)\left[\mathcal{H}_{a,k}^{\alpha}\{(\Psi-g(x))(g(x)-\psi)p(x)\}\right]\leq 0,$$

then we have

$$\begin{aligned}
&\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pf^2(x))\right]-\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]^2\\
\leq\ &\left[\Phi\mathcal{H}_{a,k}^{\alpha}p(x)-\mathcal{H}_{a,k}^{\alpha}(pf(x))\right]\left[\mathcal{H}_{a,k}^{\alpha}(pf(x))-\varphi\mathcal{H}_{a,k}^{\alpha}p(x)\right] \qquad (33)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\mathcal{H}_{a,k}^{\alpha}p(x)\right]\left[\mathcal{H}_{a,k}^{\alpha}(pg^2(x))\right]-\left[\mathcal{H}_{a,k}^{\alpha}(pg(x))\right]^2\\
\leq\ &\left[\Psi\mathcal{H}_{a,k}^{\alpha}p(x)-\mathcal{H}_{a,k}^{\alpha}(pg(x))\right]\left[\mathcal{H}_{a,k}^{\alpha}(pg(x))-\psi\mathcal{H}_{a,k}^{\alpha}p(x)\right]. \qquad (34)
\end{aligned}$$

By using inequalities (30), (33) and (34), we get

$$\begin{aligned}&\left\{2\left[\mathcal{H}^{\alpha}_{a,k}p(x)\right]\left[\mathcal{H}^{\alpha}_{a,k}(pfg(x))\right]-2\left[\mathcal{H}^{\alpha}_{a,k}(pf(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pg(x))\right]\right\}^2\\&\leq 4\left\{\left[\Phi\mathcal{H}^{\alpha}_{a,k}p(x)-\mathcal{H}^{\alpha}_{a,k}(pf(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pf(x))-\varphi\mathcal{H}^{\alpha}_{a,k}p(x)\right]\right\}\\&\times\left\{\left[\Psi\mathcal{H}^{\alpha}_{a,k}p(x)-\mathcal{H}^{\alpha}_{a,k}(pg(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pg(x))-\psi\mathcal{H}^{\alpha}_{a,k}p(x)\right]\right\}.\end{aligned}\tag{35}$$

Since $2cd \leq (c+d)^2$, $c,d\in\mathbb{R}$, then it yields

$$2\left\{\left[\Phi\mathcal{H}^{\alpha}_{a,k}p(x)-\mathcal{H}^{\alpha}_{a,k}(pf(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pf(x))-\varphi\mathcal{H}^{\alpha}_{a,k}p(x)\right]\right\}\leq\left[(\Phi-\varphi)\mathcal{H}^{\alpha}_{a,k}p(x)\right]^2 \tag{36}$$

$$2\left\{\left[\Psi\mathcal{H}^{\alpha}_{a,k}p(x)-\mathcal{H}^{\alpha}_{a,k}(pg(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pg(x))-\psi\mathcal{H}^{\alpha}_{a,k}p(x)\right]\right\}\leq\left[(\Psi-\psi)\mathcal{H}^{\alpha}_{a,k}p(x)\right]^2. \tag{37}$$

The required inequality can be obtained by taking inequalities (35)–(37) into account. □

**Lemma 2** Let $f$ and $g$ be two integrable functions on $[a,b]$ with $\varphi < f(x) < \Phi$ and
$\psi < g(x) < \Psi$ and let $p$ and $q$ be two positive functions on $[a,b]$, then for all $\alpha,\beta > 0$ and
$x > a$, we have

$$\begin{aligned}&\Big\{\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qfg(x))+\mathcal{H}^{\beta}_{a,k}q(x)\mathcal{H}^{\alpha}_{a,k}(pfg(x))\\&-\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(qg(x))-\mathcal{H}^{\beta}_{a,k}(qf(x))\mathcal{H}^{\alpha}_{a,k}(pg(x))\Big\}^2\\\leq\ &\Big\{\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qf^2(x))+\mathcal{H}^{\alpha}_{a,k}(pf^2(x))\mathcal{H}^{\beta}_{a,k}q(x)-2\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(qf(x))\Big\}\\&\times\Big\{\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qg^2(x))+\mathcal{H}^{\alpha}_{a,k}(pg^2(x))\mathcal{H}^{\beta}_{a,k}q(x)-2\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\beta}_{a,k}(qg(x))\Big\}\end{aligned}\tag{38}$$

*Proof.* By using (27), we have

$$\begin{aligned}&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)q(\lambda)}{\eta\lambda}F(\eta,\lambda)d\eta d\lambda\\&=\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qfg(x))+\mathcal{H}^{\beta}_{a,k}q(x)\mathcal{H}^{\alpha}_{a,k}(pfg(x))\\&-\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(qg(x))-\mathcal{H}^{\beta}_{a,k}(qf(x))\mathcal{H}^{\alpha}_{a,k}(pg(x)).\end{aligned}\tag{39}$$

By using Cauchy-Schwartz inequality for double integrals in (39), we can write

$$\begin{aligned}&\left[\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)q(\lambda)}{\eta\lambda}[f(\eta)-f(\lambda)][g(\eta)-g(\lambda)]d\eta d\lambda\right]^2\\\leq\ &\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)q(\lambda)}{\eta\lambda}[f(\eta)-f(\lambda)]^2d\eta d\lambda\\&\times\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)q(\lambda)}{\eta\lambda}[g(\eta)-g(\lambda)]^2d\eta d\lambda\\=\ &\left[\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qf^2(x))+\mathcal{H}^{\alpha}_{a,k}(pf^2(x))\mathcal{H}^{\beta}_{a,k}q(x)-2\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(qf(x))\right]\\&\times\left[\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qg^2(x))+\mathcal{H}^{\alpha}_{a,k}(pg^2(x))\mathcal{H}^{\beta}_{a,k}q(x)-2\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\beta}_{a,k}(qg(x))\right].\end{aligned}\tag{40}$$

□

**Lemma 3** Let $w$ be an integrable function on $[a,b]$ with $\varphi < u(x) < \Phi$ and $p$ be a positive function on $[a,b]$. Then for all $\alpha,\beta > 0$ and $x > 0$, we have

$$\begin{aligned}
& \mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(pw^2(x)) + \mathcal{H}^{\alpha}_{a,k}(pw^2(x))\mathcal{H}^{\beta}_{a,k}p(x) - 2\mathcal{H}^{\alpha}_{a,k}(pw(x))\mathcal{H}^{\beta}_{a,k}(pw(x)) \\
= & \left[\Phi\mathcal{H}^{\beta}_{a,k}p(x) - \mathcal{H}^{\beta}_{a,k}(pw(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pw(x)) - \varphi\mathcal{H}^{\alpha}_{a,k}p(x)\right] \\
& \left[\Phi\mathcal{H}^{\alpha}_{a,k}p(x) - \mathcal{H}^{\alpha}_{a,k}(pw(x))\right]\left[\mathcal{H}^{\beta}_{a,k}(pw(x)) - \varphi\mathcal{H}^{\beta}_{a,k}p(x)\right] \\
& -\mathcal{H}^{\beta}_{a,k}p(x)\left[\mathcal{H}^{\alpha}_{a,k}\{(\Phi - w(x))(w(x)-\varphi)p(x)\}\right] \\
& -\mathcal{H}^{\alpha}_{a,k}p(x)\left[\mathcal{H}^{\beta}_{a,k}\{(\Phi - w(x))(w(x)-\varphi)p(x)\}\right]. \qquad (41)
\end{aligned}$$

*Proof.* Multiplying both sides of (24) with

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)p(\lambda)}{\eta\lambda}$$

and integrating the resultant inequality with respect to $\eta$ and $\lambda$ over $(a,x)$, we get the desired equality. □

**Theorem 8** Let $f$ and $g$ be two integrable functions on $[a,b]$ with $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ and let $p$ be a positive function on $[a,b]$, then for all $x > a$, $\alpha,\beta > 0$, we have

$$\begin{aligned}
& \Big\{\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(pfg(x)) + \mathcal{H}^{\alpha}_{a,k}(pfg(x))\mathcal{H}^{\beta}_{a,k}p(x) \\
& \quad -\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(pg(x)) - \mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\beta}_{a,k}(pf(x))\Big\}^2 \\
\leq & \Big\{\left[\Phi\mathcal{H}^{\alpha}_{a,k}p(x) - \mathcal{H}^{\alpha}_{a,k}(pf(x))\right]\left[\mathcal{H}^{\beta}_{a,k}(pf(x)) - \varphi\mathcal{H}^{\beta}_{a,k}p(x)\right] \\
& \quad + \left[\Phi\mathcal{H}^{\beta}_{a,k}p(x) - \mathcal{H}^{\beta}_{a,k}(pf(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pf(x)) - \varphi\mathcal{H}^{\alpha}_{a,k}p(x)\right]\Big\} \\
& \times\Big\{\left[\Psi\mathcal{H}^{\alpha}_{a,k}p(x) - \mathcal{H}^{\alpha}_{a,k}(pg(x))\right]\left[\mathcal{H}^{\beta}_{a,k}(pg(x)) - \psi\mathcal{H}^{\beta}_{a,k}p(x)\right] \\
& \quad + \left[\Psi\mathcal{H}^{\beta}_{a,k}p(x) - \mathcal{H}^{\beta}_{a,k}(pg(x))\right]\left[\mathcal{H}^{\alpha}_{a,k}(pg(x)) - \psi\mathcal{H}^{\alpha}_{a,k}p(x)\right]\Big\}. \qquad (42)
\end{aligned}$$

*Proof.* Since

$$[\Phi - f(\eta)][f(\eta) - \varphi] \geq 0$$

and

$$[\Psi - g(\eta)][g(\eta) - \psi] \geq 0.$$

Then we can write

$$-\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}[(\Phi - f(x))(f(x)-\varphi)] - \mathcal{H}^{\beta}_{a,k}p(x)\mathcal{H}^{\alpha}_{a,k}[(\Phi - f(x))(f(x)-\varphi)] \leq 0 \quad (43)$$

and

$$-\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}[(\Psi - g(x))(g(x)-\psi)] - \mathcal{H}^{\beta}_{a,k}p(x)\mathcal{H}^{\alpha}_{a,k}[(\Psi - g(x))(g(x)-\psi)] \leq 0. \quad (44)$$

By using lemma **??** twice for $w = f$ and $w = g$ and then by using (43) and (44), we obtain the required inequality. □

**Theorem 9** Let $f$ and $g$ be two integrable functions on $[a,b]$ satisfying $\varphi < f(x) < \Phi$,

$\psi < g(x) < \Psi$ $(\varphi, \Phi, \psi, \Psi \in \mathbb{R})$ and let $p$ and $q$ be two positive functions on $[a,b]$, then for all $\alpha, \beta > 0$ and $x > a$, we have

$$\mathcal{H}^{\alpha}_{a,k}(pfg(x))\mathcal{H}^{\beta}_{a,k}q(x) + \mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qfg(x)) - \mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(qg(x)) \\ -\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\beta}_{a,k}(qf(x)) \leq \mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}q(x)\left[(\Phi-\varphi)(\Psi-\psi)\right]. \tag{45}$$

*Proof.* Since $f$ and $g$ be two integrable functions defined on $[a,b]$ satisfying the conditions $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ on $[a,b]$, so for $\eta, \lambda \in [a,b]$, we can write

$$[f(\eta) - f(\lambda)][g(\eta) - g(\lambda)] \leq (\Phi - \varphi)(\Psi - \psi), \tag{46}$$

that is to say

$$f(\eta)g(\eta) + f(\lambda)g(\lambda) - f(\eta)g(\lambda) - f(\lambda)g(\eta) \leq (\Phi - \varphi)(\Psi - \psi). \tag{47}$$

Multiplying both sides of above inequality with

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)q(\lambda)}{\eta\lambda}$$

integrate the resultant inequality with respect to $\eta$ and $\lambda$ over $(a,x)$, we obtain the required inequality. □

If we use $\alpha = \beta$ in inequality (45), then we obtain the following inequality.

**Corollary 2** Let $f$ and $g$ be two integrable functions on $[a,b]$ satisfying $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ $(\varphi, \Phi, \psi, \Psi \in \mathbb{R})$ and let $p$ and $q$ be two positive functions on $[a,b]$, then for all $x > a$, $\alpha > 0$, we have

$$\mathcal{H}^{\alpha}_{a,k}(pfg(x))\mathcal{H}^{\alpha}_{a,k}q(x) + \mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\alpha}_{a,k}(qfg(x)) - \mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\alpha}_{a,k}(qg(x)) \\ -\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\alpha}_{a,k}(qf(x)) \leq \mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\alpha}_{a,k}q(x)\left[(\Phi-\varphi)(\Psi-\psi)\right]. \tag{48}$$

**Theorem 10** Let $f$ and $g$ be two integrable functions on $[a,b]$ satisfying

$$|f(\eta) - f(\lambda)| \leq |g(\eta) - g(\lambda)|, \quad \alpha > 0 \text{ and } \eta, \lambda \in [a,b]. \tag{49}$$

Also let $p$ and $q$ be two positive functions on $[a,b]$, then for all $\alpha, \beta > 0$ and $x > a$, we have

$$\begin{aligned} & \mathcal{H}^{\alpha}_{a,k}(pfg(x))\mathcal{H}^{\beta}_{a,k}q(x) + \mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\beta}_{a,k}(qfg(x)) \\ & \quad -\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\beta}_{a,k}(qg(x)) - \mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\beta}_{a,k}(qf(x)) \\ \leq\ & \mathcal{H}^{\alpha}_{a,k}(p(x))\mathcal{H}^{\beta}_{a,k}(qg^2(x)) + \mathcal{H}^{\alpha}_{a,k}(pg^2(x))\mathcal{H}^{\beta}_{a,k}q(x) \\ & \quad -2\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\beta}_{a,k}(qg(x)). \end{aligned} \tag{50}$$

*Proof.* Since for all $\eta, \lambda \in [a,b]$, $f$ and $g$ satisfy the condition (49), we can also write

$$[f(\eta) - f(\lambda)][g(\eta) - g(\lambda)] \leq [g(\eta) - g(\lambda)]^2. \tag{51}$$

Multiplying both sides of above inequality with

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\lambda}\right)^{\frac{\beta}{k}-1}\frac{p(\eta)q(\lambda)}{\eta\lambda}$$

integrate the resultant inequality with respect to $\eta$ and $\lambda$ over $(a,x)$, we obtain the required inequality. □

If we use $\alpha=\beta$ in inequality (50), then we obtain the following inequality.

**Corollary 3** Let $f$ and $g$ be two integrable functions on $[a,b]$ satisfying the condition (49). Also let $p$ and $q$ be two positive functions on $[a,b]$, then for all $x>a$, $\alpha>0$, we have

$$\begin{aligned} & \mathcal{H}^{\alpha}_{a,k}(pfg(x))\mathcal{H}^{\alpha}_{a,k}q(x)+\mathcal{H}^{\alpha}_{a,k}p(x)\mathcal{H}^{\alpha}_{a,k}(qfg(x)) \\ & \quad -\mathcal{H}^{\alpha}_{a,k}(pf(x))\mathcal{H}^{\alpha}_{a,k}(qg(x))-\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\alpha}_{a,k}(qf(x)) \\ \leq\; & \mathcal{H}^{\alpha}_{a,k}(p(x))\mathcal{H}^{\alpha}_{a,k}(qg^2(x))+H^{\alpha}_{a,k}(pg^2(x))H^{\alpha}_{a,k}q(x)-2\mathcal{H}^{\alpha}_{a,k}(pg(x))\mathcal{H}^{\alpha}_{a,k}(qg(x)). \end{aligned} \tag{52}$$

**Theorem 11** Let $r\geq 1$ and $f$ and $g$ be two positive functions on $[a,b]$ such that for all $x>a$ $0<\mathcal{H}^{\alpha}_{a,k}f^r,\mathcal{H}^{\alpha}_{a,k}g^r<\infty$. If

$$0<m\leq\frac{f(\eta)}{g(\eta)}\leq M<\infty,\quad \eta\in[a,b], \tag{53}$$

then for all $\alpha>0$

$$\begin{aligned} & \left[\mathcal{H}^{\alpha}_{a,k}(f^r(x))\right]^{\frac{1}{r}}+\left[\mathcal{H}^{\alpha}_{a,k}(g^r(x))\right]^{\frac{1}{r}} \\ \leq\; & \frac{1+M(m+2)}{(m+1)(M+1)}\left[\mathcal{H}^{\alpha}_{a,k}((f+g)^r(x))\right]^{\frac{1}{r}}. \end{aligned} \tag{54}$$

*Proof.* By using the condition (53) for all $x>a$ and $\eta\in[a,b]$, we have

$$\frac{1}{M}\leq\frac{g(\eta)}{f(\eta)}.$$

This implies

$$\left(\frac{1}{M}+1\right)^r\leq\left(\frac{g(\eta)}{f(\eta)}+1\right)^r.$$

This gives us

$$(M+1)^r f^r(\eta)\leq M^r(f+g)^r(\eta). \tag{55}$$

Similarly, we can have

$$(m+1)^r g^r(\eta)\leq(f+g)^r(\eta). \tag{56}$$

Multiplying both sides of inequalities (55) and (56) with $\frac{1}{k\Gamma_k(\alpha)}\left(\ln\frac{x}{\eta}\right)^{\frac{\alpha}{k}-1}\frac{1}{\eta}$ and integrating with respect to $\eta$ over $(a,x)$, we get the following inequalities respectively

$$\left[\mathcal{H}^{\alpha}_{a,k}(f^r(x))\right]^{\frac{1}{r}}\leq\frac{M}{M+1}\left[\mathcal{H}^{\alpha}_{a,k}((f+g)^r(x))\right]^{\frac{1}{r}} \tag{57}$$

and

$$\left[\mathcal{H}^{\alpha}_{a,k}(g^r(x))\right]^{\frac{1}{r}}\leq\frac{1}{m+1}\left[\mathcal{H}^{\alpha}_{a,k}((f+g)^r(x))\right]^{\frac{1}{r}}. \tag{58}$$

By adding inequalities (57) and (58), we get the desired inequality. □

**Theorem 12** Let $r\geq 1$ and $f$ and $g$ be two integrable functions on $[a,b]$ such that for all $x>a$ $0<\mathcal{H}^{\alpha}_{a,k}f^r,\mathcal{H}^{\alpha}_{a,k}g^r<\infty$. If the condition (53) is satisfied, then for all $\alpha>0$

$$\begin{aligned} & \left[\mathcal{H}^{\alpha}_{a,k}(f^r(x))\right]^{\frac{2}{r}}+\left[\mathcal{H}^{\alpha}_{a,k}(g^r(x))\right]^{\frac{2}{r}} \\ \geq\; & \left(\frac{(m+1)(M+1)}{M}-2\right)\left[\mathcal{H}^{\alpha}_{a,k}(f^r(x))\right]^{\frac{1}{r}}\left[\mathcal{H}^{\alpha}_{a,k}(g^r(x))\right]^{\frac{1}{r}}. \end{aligned} \tag{59}$$

*Proof.* Multiplying the inequalities (57) and (58), we obtain

$$\frac{(m+1)(M+1)}{M}\left[\mathcal{H}^{\alpha}_{a,k}(f^r(x))\right]^{\frac{1}{r}}\left[\mathcal{H}^{\alpha}_{a,k}(g^r(x))\right]^{\frac{1}{r}} \leq \left[\mathcal{H}^{\alpha}_{a,k}((f+g)^r(x))\right]^{\frac{2}{r}}. \quad (60)$$

By applying the Minkowski's integral inequality to the right hand side of above inequality, we obtain the required inequality. □

**Remark 1** All these properties and inequalities can be proved for right sided Hadamard $k$-fractional integral.

**Acknowledgement.** The authors would like to thank the anonymous referees for his/her comments that helped us to improve this article.

## References

[1] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Inequal. Pure Appl. Math., **10** (2009) Art 86.
[2] P.L. Chebyshev, *Sur les expressions approximatives des integrales definies par les autres prises entre les mmes limites*, Proc. Math. Soc. Charkov., **2** (1882) 93–98.
[3] Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci., **9** (2010) 493–497.
[4] Z. Dahmani, L. Tabharit, *On Weighted Grüss type Inequalities via Fractional Integration* , J. Adv. Res. Pure Math. (JARPM), **2** (2010) 31-38.
[5] R. Diaz and E. Pariguan, *On Hypergeometric Functions and Pochhammer k-symbol*, Divulgaciones Matemticas, **15** (2007) 179–192.
[6] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, Journal of pure and applied mathematics, **4**(8) (1892) 101?-186.
[7] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Res. Notes Math. Ser., Vol. 301, Longman, New York, 1994.
[8] C.G. Kokologiannaki, *Properties and Inequalities of Generalized k-gamma, beta and zeta Functions*, Int. J. Contemp. Math. Sciences, **5**(14) (2010) 653–660.
[9] C.G. Kokologiannaki and V. Karasniqi, *Some Properties of k-gamma Function*, Le Mathematics, **LXVIII** (2013) 13–22.
[10] C.G. Kokologiannaki and V.D. Sourla, *Bounds for k-gamma and k-beta functions*, Journal of Inequalities and Special Functions, **4**(3) (2013) 1–5.
[11] V. Krasniqi, *Inequalities and Monotonicity for the Ration of k-gamma Function*, Scientia Magna, **6** (2010) 40–45.
[12] V. Krasniqi, *A Limit for the k-gamma and k-beta Function*, Int. Math. Forum, **5** (2010) 1613–1617.
[13] S. Liu, J. Wang, *Hermite-Hadamard type Fractional Integral Inequalities for Geometric-Geometric Convex Functions*, Le Matematiche, **LXX** (2015) Fasc. I 3?-20.
[14] M. Z. Sarıkaya, E. Set, H. Yaldız and N. Ba?ak, *Hermite-?Hadamard?s inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, **57**(9) (2013) 2403–2407.
[15] S. Mubeen, G. M. Habibullah, *k-fractional integrals and application*, Int. J. Contemp. Math. Sci. **7** (2012) 89–94.
[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, London, 1999.
[17] H. A. Salem, A. El-Sayed, *Weak solution for fractional order integral equations in reflexive Banach spaces*, Mathematica Slovaca, **55**(2) (2005) 169–181.
[18] R. Suryanarayana and C. Rao, *Some new inequalities for the generalized $\epsilon-$gamma, beta and zeta Functions*, International Journal of Engineering Research and Technology (IJERT), **1**(9) (2012) 1–4.
[19] L. G. Romero, L. L. Luque, G. A. Dorrego and R. A. Cerutti, *On the k-Riemann–Liouville Fractional Derivative*, Int. J. Contemp. Math. Sciences, **8**(1) (2013) 41–51.
[20] E. Set, M. Tomar and M. Z. Sarıkaya, *On generalized Grüss type inequalities for k-fractional integrals*, Applied Mathematics and Computation, **269** (2015) 29-?34.
[21] J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite–Hadamard type inequalities and applications to special means*, Journal of Inequalities and Applications, **1** (2013) 1–15.

[22] J. Wang, J. Deng and M. Fečkan, *Exploring s-e-Condition and Applications to some Ostrowski Type Inequalities Via Hadamard Fractional Integrals*, Mathematica Slovaca, **64** (2014), No. 6, 1381-1396.

[23] J. Zhang and H.N. Shi, *Two double inequalities for $k$-gamma and $k$-Riemann zeta functions*, Journal of Inequalities and Applications, **1** (2014) 191.

[24] W. Sudsutad, S. K. Ntouyas and J. Tariboon, *Fractional Integral Inequalities via Hadamard?s Fractional Integral*, Absttarc and Applied Analysis. **Volume 2014**, Article ID 563096, 11 pages.

S. Iqbal
Department of Mathematics University of Sargodha, Sargodha, Pakistan
*E-mail address*: sna1310@gmail.com

S. Mubeen
Department of Mathematics University of Sargodha, Sargodha, Pakistan
*E-mail address*: smjhanda@gmail.com

M. Tomar
Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey
*E-mail address*: muharremtomar@gmail.com